THE
PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 114, NO. 5 JUNE 1, 1959

Ion Confinement by Rotation in Magnetic Mirror Geometry*

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(Received January 6, 1959)

When a generally radial electric field is imposed upon a mirror magnetic field configuration, the resulting system has an increased ability to contain plasma between the mirror regions. The motion of the ions consists of a drift around the axis of symmetry upon which is superimposed a spiralling motion. The guiding center motion departs from the direction of magnetic flux surfaces. An adiabatic invariant is derived for the drift component of the motion, and an expression is derived for the mirror enhancement brought about by the **E**×**B** drift. The departure of the guiding center of a particle from a flux surface and the energy balance of its secular motion are calculated by using the adiabatic invariant.

I. INTRODUCTION

THE term "magnetic mirror configuration" is used to describe an axially symmetric magnetic field with lines of force tending toward the axis on each side of a central plane and constricting at two points along the axis. At these points the field intensity is a maximum, and between them it has a minimum. In this field configuration there are many orbits for charged particles which are bounded between the regions of maximum field, that is to say, between the magnetic mirrors. Typical orbits are spirals around the direction of the magnetic field. They are contained because as a particle moves toward stronger field regions, kinetic energy of motion along the lines of force is transferred to energy of motion around the lines. The possibility of confining high temperature plasmas in mirror systems has been studied extensively by Post and collaborators,[1] who reported results of their experimental and theoretical investigations at the 1958 Geneva Conference.[2]

In the present note we shall derive some properties of particle orbits in a magnetic mirror configuration with an added external electric field orthogonal to **B** as shown in Fig. 1. It will be shown that the leakage of particles through the mirrors is less than in a mirror configuration without an electric field.[3] At the 1958 Geneva Conference some results of experiments with such systems were presented.[4,5] In the following we shall call the arrangement an electrified magnetic mirror.

Below we discuss the single particle orbits in the electrified mirror configuration. A plasma drifts in the direction orthogonal to the electric and magnetic fields, in this case around the axis of symmetry. The particle motions of interest consist of the drift superimposed upon helices whose average direction slightly departs from that of the magnetic field. Starting from the exact single-particle Hamiltonian, an adiabatic invariant of the drift motion is derived, and an approximate analytical expression for the enhanced mirror confinement is given. The adiabatic invariant is used to calculate the guiding surfaces of the ions and the energy balance of their secular motion along the axis of the mirror configuration. Exact numerical calculations of the mirror enhancement are also given.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] R. F. Post, Bull. Am. Phys. Soc. Ser. II, **3**, 196 (1958).

[2] R. F. Post, "Summary of the UCRL Pyrotron Program," *Proceedings of The Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, September, 1958* (United Nations, New York, 1958).

[3] Although the authors arrived at their conclusions regarding this mirror enhancement produced by the **E**×**B** drift independently, the effect of centrifugal force in preventing particle escape out the ends of an electrified magnetic mirror was recognized by O. A. Anderson and W. R. Baker in 1956.

[4] Anderson *et al.*, "Study and Use of a Rotating Plasma," *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, September, 1958* (United Nations, New York, 1958).

[5] K. Boyer *et al.*, "Theoretical and Experimental Discussion of Ixion, A Possible Thermonuclear Device, *Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, September, 1958* (United Nations, New York, 1958).

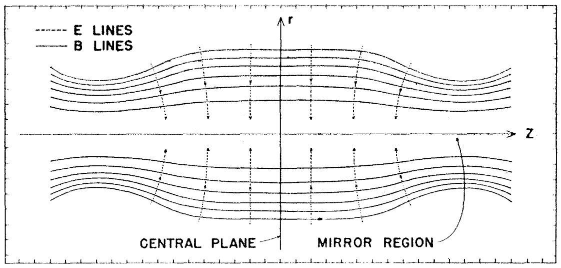

FIG. 1. Schematic diagram of an electrified magnetic mirror. The solid lines represent the magnetic field and the dashed lines normal to them the electric field.

II. PARTICLE ORBITS AND THE ADIABATIC INVARIANT

We consider the electrified magnetic mirror arrangement of Fig. 1. The electric field **E** is assumed radial at the central plane and everywhere normal to the **B** lines. Such an electric field of course requires a space charge distribution in the mirror region. This requirement can be satisfied experimentally because the electrons of a plasma can move easily in the direction of the magnetic field, tending to neutralize any component of **E** parallel to **B**. It is convenient to define the following angular velocities of a particle:

$$\omega_E = cE/Br, \tag{1}$$

$$\omega_c = eB_z/mc, \tag{2}$$

$$\omega_D = -\omega_c/2[1-(1-4\omega_E/\omega_c)^{\frac{1}{2}}], \tag{3}$$

where e is the charge of the particle and m its mass.[6] If the particle drifts about the axis with angular velocity ω_D there will be equilibrium between the centrifugal, electric, and Lorentz forces so that there is no net acceleration in the radial direction. In Eq. (3) the sign of the radical is chosen so that in the limit of weak electric fields ($\omega_E \ll \omega_c$), ω_D approaches $-\omega_E$, which is the angular velocity of the $\mathbf{E}\times\mathbf{B}$ drift.

The magnetic field, which has no θ component and which has axial symmetry, can be derived from a vector potential with only a θ component, A_θ. We define a flux function

$$\psi(r,z) = (er/c)A_\theta(r,z), \tag{4}$$

such that $2\pi(c/e)\psi$ is the flux enclosed in a circle of radius r. The potential energy of a particle is

$$V = e\Phi, \tag{5}$$

where Φ is the electrostatic potential function. Since ψ = constant describes **B** lines, and since **B** lines are equipotentials, it follows that

$$V = V(\psi). \tag{6}$$

The Hamiltonian of a particle in cylindrical coordinates under the influence of these fields is

$$H = p_z^2/2m + p_r^2/2\text{m} + (p_\theta-\psi)^2/2mr^2 + V, \tag{7}$$

[6] Gaussian units are used throughout.

where p_z, p_r, and p_θ are the canonical momenta

$$p_z = m\dot{z} \qquad p_r = m\dot{r} \qquad p_\theta = mr^2\dot{\theta} + \psi. \tag{8}$$

Since H is independent of θ, p_θ is a constant of motion and may be regarded as constant in (7) for a given particle. Then (7) may be regarded as the Hamiltonian of a particle moving in two Cartesian dimensions,[7] z and r, in an effective potential U given by the sum of the last two terms in (7).

We consider first the case with no electric field, $V=0$. For definiteness, we consider a positive particle and take ψ positive. A positive particle which does not encircle the axis[8] will have $\dot{\theta}=0$ at some point on its orbit. Equation (8) shows that such a particle has positive p_θ. Therefore, for a given value of z, $p_\theta-\psi$ will vanish at some radius, and the effective potential will be zero and have a minimum at this point. The locus in an r-z plane of these minima forms the bottom of a potential trough with the minimum following a **B** line. The two dimensional particle moves in this trough, oscillating between the two flux surfaces for which U equals the total energy. The bottom of the potential trough is the locus of the guiding center of the particle, while the oscillations represent the cyclotron motion about the guiding center.

If there is an applied electric field, the equipotential surfaces of U will be altered. For a given value of z, the minimum of U occurs where $\partial U/\partial r=0$. Evaluating $\partial U/\partial r$, one finds

$$-\frac{1}{mr^3}(p_\theta-\psi)^2 - \frac{1}{mr^2}(p_\theta-\psi)\frac{e}{c}rB_z - eE_r = 0. \tag{9}$$

Substituting expressions (1) and (2) for ω_E and ω_c and solving, one finds

$$p_\theta - \psi = -mr^2(\omega_c/2)[1-(1-4\omega_E/\omega_c)^{\frac{1}{2}}]. \tag{10}$$

If there were no electric field ($\omega_E=0$), this would give a minimum at $p_\theta-\psi=0$ as before. The right-hand side of Eq. (10) measures the amount that the guiding center of a particle of given p_θ departs from the flux surface $\psi(r,z)=p_\theta=$ constant.

Equations (10) and (3) state that for motion along the potential minimum,

$$p_\theta - \psi = mr^2\omega_D = L_D, \tag{11}$$

where L_D is the angular momentum of the particle drift motion under the combined action of centrifugal, electric, and Lorentz forces. Thus, for particles which oscillate but stay near the potential minimum, we may say that $L_D+\psi$ is an adiabatic invariant, which we shall call Λ. Since L_D and ψ are functions of position

[7] This method has been found previously to be convenient, by T. Northrop and collaborators, in discussing the problem of the nonadiabaticity of particle orbits in mirror magnetic field configurations.

[8] For particles which encircle the axis, the analysis is only slightly modified.

only, so is Λ, and a surface $\Lambda=$ constant thus determines the path of a particle's guiding center. We shall use the Λ invariant in Sec. IV to calculate the departure of the guiding center of a particle from a flux surface as well as the energy balance of its secular motion between the mirrors.

One can determine the direction of departure of the guiding center from a flux surface as follows: returning to Eq. (11), let us make the approximation of small E

$$L_D \approx -mrcE/B, \tag{12}$$

and take the case where E is negative. Then $p_\theta - \psi$ is positive at the potential minimum. Since on a flux surface, rE is roughly independent of z, and since B_z is larger in the mirror region than at the central plane, $p_\theta - \psi$ is less positive at the mirror than at the center. Therefore ψ must be greater in the mirror than at the center, or the guiding center moves outward across flux surfaces as it enters the mirror region.

III. EFFECT OF THE ELECTRIC FIELD ON THE LEAKAGE FROM THE MAGNETIC MIRROR

In order to calculate the leakage from the mirror in the presence of the $\mathbf{E}\times\mathbf{B}$ rotation, it is convenient to make a transformation. We define an equivalent flux function ψ' which includes the effect of the *radial* variation of V. Let us pick a given p_θ and a given z and define ψ' by requiring

$$(p_\theta-\psi)^2/2mr^2+V(r,z)=(p_v-\psi')^2/2mr^2+K(z). \tag{13}$$

Here $K(z)$ is taken to be the minimum value of the left hand side for the given value of z. This allows $p_\theta-\psi'$ to vanish at the minimum, and therefore allows ψ' to be a smooth, monotonic function of r, as it ought if it is to represent an effective flux function which still looks like that for a mirror device. Solving (13) for ψ', we find

$$\psi'=p_\theta \pm [(p_\theta-\psi)^2+2mr^2(V-K)]^{\frac{1}{2}}, \tag{14}$$

where the minus sign is taken for r inside the minimum and the plus sign outside.

The Hamiltonian (7) can now be rewritten,

$$H=p_z^2/2m+p_r^2/2m+(p_\theta-\psi')^2/2mr^2+K(z). \tag{15}$$

This is the Hamiltonian of a particle in a mirror magnetic field corresponding to ψ' with a longitudinal electric field whose potential is $K(z)$. There is no radial electric field, and hence the $\mathbf{E}\times\mathbf{B}$ rotation has been eliminated in the main. The rotation would be eliminated completely if the new electric field were parallel at all points to the new magnetic field. This condition would hold quite closely if we had found the minimum of the effective potential by taking its derivative in a direction normal to its trough instead of in the r direction. However, for a mirror field in which the B lines never make a large angle with the axis, we have eliminated most of the rotation.

The problem of the mirror reflection is now reduced to the simple one of finding the effect of the longitudinal electric field, or of the potential energy $K(z)$. We shall first determine $K(z)$ in the small E approximation which led to Eq. (12).

$K(z)$ has been defined as the minimum (in r) of the left hand side of Eq. (13). This minimum is near the point r_0 where $\psi=p_\theta$ [i.e., the bottom of the trough when $V(r,z)$ is neglected]. Let us expand the effective potential about this point, letting $x=r-r_0$. Then

$$(p_\theta-\psi)^2/2mr^2 \approx Ax^2,$$
$$V\approx Bx+C,$$

where A, B, and C are constants. V is a roughly logarithmic function of r, and we have kept constant and linear terms in its expansion. The constant C is the value of the potential V at the point where $\psi=p_\theta$, which is a B line, so that C is a constant independent of z also. A and B depend on z.

Now the minimum of Ax^2+Bx+C is at $x=-B/2A$ and the value at the minimum is (apart from the constant C, which can be neglected).

$$(Ax^2+Bx+C)_{\min}=-\frac{1}{4}\frac{B^2}{A}=-A(x_{\min})^2,$$

i.e., the minimum value is the negative of the quadratic term evaluated at the position of the minimum. Thus $K(z)$ is the negative of the term $(p_\theta-\psi)^2/2mr^2$ evaluated at the minimum. But at the minimum, $p_\theta-\psi$ is given by Eq. (11). Thus

$$K(z)=-\tfrac{1}{2}mv_D^2, \tag{16}$$

where $v_D=r\omega_D$ is the drift velocity evaluated on the same flux surface (of ψ') at each z. In the small E approximation

$$v_D \approx cE/B. \tag{17}$$

Now on a given flux surface,

$$B\sim 1/r^2,$$
$$E\sim 1/r,$$

so that

$$v_D(z)\sim r(z),\; K(z)\sim -r^2(z)\sim -1/B(z). \tag{18}$$

Thus the potential energy $K(z)$ is more negative in the center of the machine than in the mirror. The potential difference from center to mirror is such as to prevent particles from leaking. To determine the effect on the leakage, we use the fact that

$$w_\perp + w_{\parallel} + K(z) = \text{constant}$$

for the motion of a particle along a flux tube ($w_\perp$ and $w_{\parallel}$ are the perpendicular and parallel kinetic energies). Thus

$$w_\perp(c)+w_{\parallel}(c)+K(c)=w_\perp(m)+w_{\parallel}(m)+K(m),$$

where c refers to the center and m to the mirror. But, from the adiabatic theory

$$w_{\perp}(m)=w_{\perp}(c)R, \tag{19}$$

where the mirror ratio R is defined by

$$R=B(m)/B(c).$$

In order to pass through the mirror, a particle must have $w_{\parallel}(m)\geq 0$. Thus, for leakage

$$\begin{aligned} w_{\parallel}(c) &\geq w_{\perp}(c)(R-1)+K(m)-K(c) \\ &\geq w_{\perp}(c)(R-1)+\tfrac{1}{2}mv_D^2(c)(1-1/R). \end{aligned} \tag{20}$$

The last term gives the effect of the $K(z)$, and this term makes it more difficult for a particle to leak. If the particle motion is generated impulsively from rest, as it might be in practice, $\frac{1}{2}mv_D^2$ is about equal to $w_{\perp}+w_{\parallel}$).

In going from Eq. (17) to the final result (19) we have neglected the difference between the actual field B and the effective field connected with ψ'. This difference is a higher order effect.

IV. ENERGY BALANCE OF THE SECULAR MOTION

In this section we shall show how the result of Eq. (20) can be derived by considering the particle energy and the constraint imposed by the adiabatic invariant

$$\Lambda=L_D+\psi=\text{constant} \tag{21}$$

of Sec. II, without making the transformation of Sec. III. To this end it is convenient to split up the total energy W of the particle as follows:

$$W=w_{\parallel}+w_{\perp}+w_D+V. \tag{22}$$

Here $w_{\parallel}$ is the kinetic energy associated with the component of velocity parallel to $\mathbf{B}$, and V is the electrostatic potential energy. w_D is the kinetic energy associated with the drift motion about the axis

$$w_D=L_D^2/2mr^2=(\Lambda-\psi)^2/2mr^2, \tag{23}$$

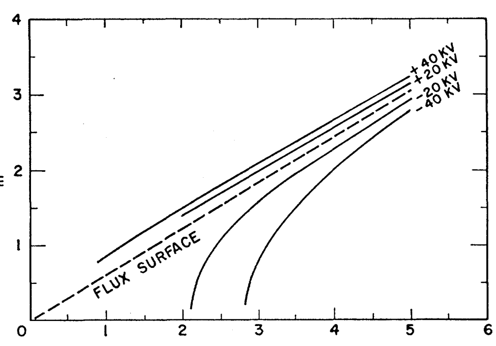

FIG. 2. Computed relationship between the radial position x_m (in units of the reference radius r_i) at the mirror and the corresponding position x_c at the central plane of the guiding center of a deuteron in an electrified magnetic mirror configuration. The solid curves are identified by the applied electrostatic potential at an outer radius of 12.6 cm. The dashed curve refers to the case of no electrostatic field.

where we have used the constraint imposed by the adiabatic invariant. Finally, $w_{\perp}$ is the kinetic energy associated with the cyclotron motion about the drifting guiding center. $w_{\perp}$ always obeys the adiabatic relation

$$w_{\perp}\sim B \tag{24}$$

regardless of whether the electric field is present or not. If the electric field is not present, the terms w_D and V disappear from Eq. (22); then the constancy of W together with Eq. (24) lead to the usual formula for the reflection of the particle by the mirror, namely Eq. (20) without the last term on the right. This term, as we shall see, comes from the terms w_D+V when the electric field is present.

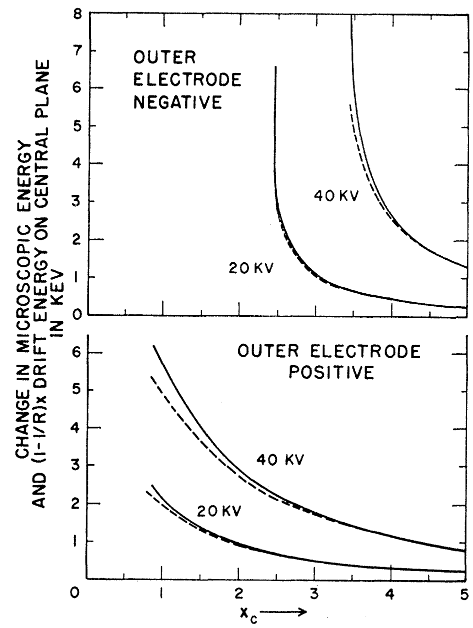

FIG. 3. Computed values of the change in macroscopic energy of a deuteron as it moves from the central plane into the mirror of an electrified magnetic mirror arrangement (solid curves). For comparison (1−1/R) times the drift kinetic energy at the central plane is also plotted against the radial position x_c at the central plane (dashed curves). The curves are labelled by the value of electrostatic potential applied at an outer radius of 12.6 cm.

According to Eq. (23), the differential of $w_{\perp}$ when the particle moves along a Λ surface is

$$\Delta w_D=-2w_D\Delta r/r-\omega_D\Delta\psi, \tag{25}$$

where ω_D is the drift angular velocity given by Eq. (3). In the approximation of small E used in Sec. III,

$$\omega_D=-\omega_E=V', \tag{26}$$

where the prime denotes differentiation with respect to ψ. The second term of the right-hand side of Eq. (25) is then just the negative of the change ΔV in the electrostatic potential energy of the particle. Hence

$$\Delta w_D+\Delta V=-2w_D\Delta r/r=-(mv_D^2/r)\Delta r. \tag{27}$$

This energy change comes from the work done against the centrifugal force associated with the **E×B** rotation, For a mirror ratio R not greatly different from unity in the small E approximation

$$-2\Delta r/r \approx \Delta B/B \approx (1-1/R). \quad (28)$$

When this approximation is used in Eq. (27), one sees that the change in (w_D+V) contributes exactly the last term in Eq. (20).

We can rewrite ΔV using Eq. (26) as follows:

$$\Delta V = V'\Delta\psi = \omega_E(\partial\psi/\partial r)_\Lambda \Delta r. \quad (29)$$

Using Eq. (26) in the expression (11) for L_D and differentiating Eq. (21), we find that

$$(\partial\psi/\partial r)_\Lambda = -(\partial L_D/\partial r)_\Lambda = -2L_D/r, \quad (30)$$

where additional terms involving ω_E/ω_c as a factor have been neglected. Therefore in the small E approximation

$$\Delta V = -4w_D\Delta r/r, \quad (31)$$

and Eq. (25) becomes

$$\Delta w_D = 2w_D\Delta r/r. \quad (32)$$

Thus the drift kinetic energy decreases as the particle moves into the mirror ($\Delta r/r$ negative). The increased mirror containment comes from the fact that the increase in electrostatic potential energy as the particle crosses flux surfaces is approximately twice this decrease. In experiments on electrified mirrors one would expect to see a smaller plasma drift velocity in the mirror than at the corresponding radius on the central plane.

Numerical calculations were carried out to find the changes Δw_D and ΔV as well as the departure of the guiding center of a deuteron from flux surfaces in an electric and magnetic field configuration approximating that of the Los Alamos experiments.[5] Using the exact expression (3) for the drift angular velocity, the energies w_D and V were computed at the central plane and the mirror on a Λ surface for an assumed electrostatic potential distribution.

$$\Phi = s\ln[\psi/\psi_c(r_i)], \quad (33)$$

where $\psi_c(r_i)$ is the flux function on the central plane at some reference radius r_i, chosen as 2.86 cm and s is a constant. For simplicity the magnetic field was taken to be uniform in both the central plane and the mirror with respective values of 5.00 and 13.25 kilogauss ($R=2.65$). Potentials of $+20$, -20, $+40$, and -40 kilovolts were assumed to be applied at $r=12.6$ cm in the central plane.

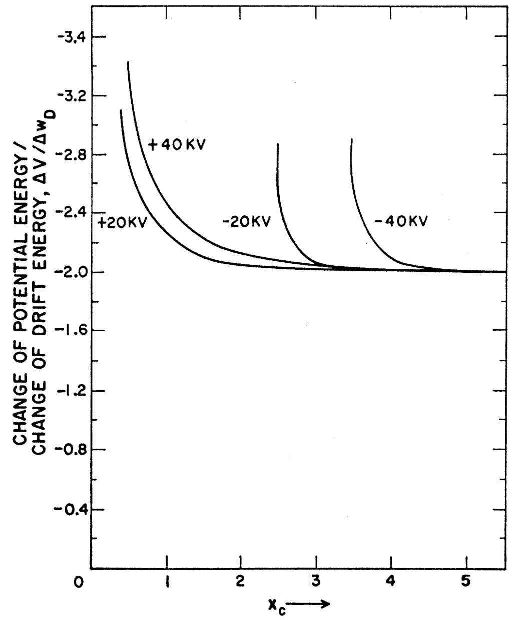

FIG. 4. Computed values of the ratio of the change of potential energy to the change of drift kinetic energy of a deuteron as it moves from the central plane to the mirror of an electrified mirror machine *versus* the radial position x_c at the central plane. The curves are labelled by the values of electrostatic potential applied at an outer radius of 12.6 cm.

In Fig. 2 is plotted the radial mirror position $x_m = r_m/r_i$ of a deutron against its corresponding central-plane position $x_c = r_c/r_i$. The straight line through the origin corresponds to a flux surface on which $x_m = x_c/R^{\frac{1}{2}}$.

In Fig. 3 the solid curves represent the variation of the mirror enhancement energy $(\Delta V+\Delta w_D)$ with x_c. The dashed curves show for comparison the drift energy at the central plane multiplied by $1-1/R$. For large values of x_c the two sets of curves coincide, in agreement with Eqs. (20) and (27). However, for small values of x_c the centrifugal effects cause a separation of the two sets of curves.

The ratio $\Delta V/\Delta w_D$ is plotted against x_c in Fig. 4. For large values of x_c the ratio approaches the value -2, in agreement with Eqs. (31) and (32).

ACKNOWLEDGMENTS

We acknowledge with pleasure some stimulating discussions of this subject with Keith Boyer and James L. Tuck.