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Geometrical Significance of the Einstein-Maxwell Equations*

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Rainich geometries are analyzed in terms of the invariants associated with the Ricci vierbein of principal directions. At any point the four unit vectors of the vierbein pair off into two blades which contain the maxima and minima directions of mean curvature, respectively. The blades can "mesh" into smooth integral surfaces for certain electromagnetic fields. In general, neighboring blades are shown to be related by only two independent differential conditions.

I. INTRODUCTION

THE Einstein-Maxwell equations govern the gravitational behavior of classical electromagnetic radiation. Considerable interest attaches itself to electromagnetic fields which are non-null, in the sense that the square of the Maxwell stress tensor is greater than zero,[1]

$$\rho^2 \equiv \tfrac{1}{4} R^\rho{}_\sigma R^\sigma{}_\rho > 0. \tag{1}$$

When this non-null condition is satisfied, the whole content of source-free electrodynamics and Einstein gravitation theory is contained in the statement that the space-time geometry satisfies the following Rainich conditions[2,3]:

The Maxwell stress tensor has zero trace, or

$$R = 0. \tag{2}$$

The square of the stress tensor is proportional to the unit matrix, or

$$R^\mu{}_\sigma R^\sigma{}_\nu = \rho^2 \delta^\mu{}_\nu. \tag{3}$$

The electromagnetic energy density is positive definite, or

$$R_{00} > 0. \tag{4}$$

A certain vectorial combination of the Ricci curvature components and their first derivatives shall have zero curl; that is to say, this prescribed combination shall be expressible in the form of a gradient,

$$\frac{g_{\mu\sigma}\epsilon^{\sigma\rho\nu\lambda}R^\kappa{}_\rho R_{\kappa\nu;\lambda}}{4\rho^2\sqrt{(-g)}} = \alpha_{,\mu}, \tag{5}$$

where α is a scalar invariant, the "complexion" of the electromagnetic field.[3] The Levi-Civita symbol $\epsilon^{\sigma\rho\nu\lambda}$ is skew-symmetric in all pairs of indices, with $\epsilon^{0123} \equiv 1$.

Whenever the non-null condition (1) is satisfied, Rainich geometries (2)–(5) are entirely equivalent to Einstein's original description of gravitation and electromagnetic radiation. In this paper we seek to determine the geometrical meaning of Rainich's system of equations. With this purpose in mind, we proceed to reduce (2)–(5) to an equivalent, but more tractable, set of geometrical conditions.

II. RAINICH GEOMETRY IN TERMS OF THE RICCI VIERBEIN

The local canonical form of a non-null Rainich-Ricci tensor,[2,3] available to any point through an appropriate coordinate transformation, guarantees the existence of the Ricci vierbein of principal directions.[4] The sixteen vierbein components are established by the set of equations,

$$(R_{\mu\nu} - \rho_a g_{\mu\nu})\lambda_{a|}{}^\nu = 0, \quad [a = 0, 1, 2, 3], \tag{6}$$

$$g_{\mu\nu}\lambda_{a|}{}^\mu \lambda_{b|}{}^\nu \equiv \lambda_{a|\nu}\lambda_{b|}{}^\nu = e_a \delta_{ab}, \tag{7}$$

$$e_0 = -1, \quad e_1 = e_2 = e_3 = +1. \tag{8}$$

Relations (7) and (6) imply

$$g_{\mu\nu} = \sum_{a=0}^{3} e_a \lambda_{a|\mu}\lambda_{a|\nu}, \tag{9}$$

$$R_{\mu\nu} = \sum_{a=0}^{3} e_a \rho_a \lambda_{a|\mu}\lambda_{a|\nu}. \tag{10}$$

Substituting (10) into (3), we find the Ricci invariants

$$\rho_a = f_a \rho, \tag{11}$$

where each

$$f_a{}^2 = 1. \tag{12}$$

Then from (2) it follows that

$$\sum_{a=0}^{3} f_a = 0. \tag{13}$$

Without loss of generality we can satisfy (12) and (13) by taking

$$f_0 = f_1 = -1, \quad f_2 = f_3 = +1. \tag{14}$$

These values for f_a pair the vierbein legs into a negative and a positive blade (Fig. 1). The particular numerical

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[1] In the idealization for which space-time is treated as approximately flat, the electromagnetic form of (1) requires that $(\mathbf{E}^2 - \mathbf{H}^2)$ and $(\mathbf{E}\cdot\mathbf{H})$ do not both vanish. This non-null requirement is satisfied by a general superposition of waves traveling in various directions. An outstanding exception, however, is the case of a pure monochromatic wave traveling in a single direction. It is not known whether this null case has a rigorous correspondent which satisfies the Einstein-Maxwell equations.

[2] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[3] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

[4] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949), p. 113.

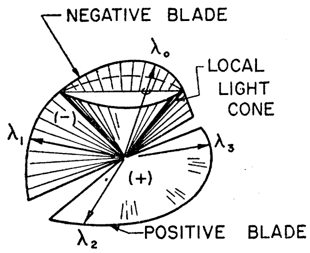

FIG. 1. The two blades of a Rainich geometry Ricci vierbein at a point.

assignment to f_0 in (14) fixes the sign of ρ according to (4),

$$\rho>0. \tag{15}$$

The Ricci tensor (10) with (11), (14), and (15),

$$R_{\mu\nu}=\rho \sum_{a=0}^{3} e_a f_a \lambda_{a|\mu}\lambda_{a|\nu}, \tag{16}$$

satisfies the first three requirements of a Rainich geometry.

A geometrical interpretation of the Ricci tensor (16) goes as follows. Let n^μ denote any unit vector which lies in the positive blade at a point. Then the mean curvature[4] of the space in the direction n^μ, defined by geometers as $R_{\mu\nu}n^\mu n^\nu$, is equal to $(+\rho)$. This is the maximum value attainable for all directions at the point. Furthermore, the minimum mean curvature $(-\rho)$ is associated with any unit vector which resides in the negative blade at a point.

Let us denote the intrinsic derivative in the direction of a vierbein leg by

$$\partial_a \equiv \lambda_{a|}{}^{\nu}\partial/\partial x^\nu. \tag{17}$$

The *structure coefficients* are introduced in the commutation relations

$$[\partial_a,\partial_b]\equiv \partial_a\partial_b-\partial_b\partial_a=\sum_{c=0}^{3} C_{cab}\partial_c. \tag{18}$$

In analogy to a hydrodynamic flow, these structure coefficients may be called "the projected vorticities" of the Ricci vierbein,

$$C_{cab}=-C_{cba}\equiv e_c(\lambda_{c|\mu,\nu}-\lambda_{c|\nu,\mu})\lambda_{a1}{}^{\mu}\lambda_{b1}{}^{\nu}. \tag{19}$$

With the aid of this intrinsic notation, Rainich's final condition Eq. (5) can be drastically simplified. Substituting (16) into (5) and performing a few obvious manipulations establishes the relation

$$\eta_a=\partial_a\alpha, \tag{20}$$

in which the η's are abbreviations for the structure coefficients,

$$\eta_0\equiv C_{123},\quad \eta_1\equiv C_{032}, \qquad \eta_2\equiv C_{310},\quad \eta_3\equiv C_{201}. \tag{21}$$

The integrability conditions equivalent to (20) are easily found with the help of (18):

$$\Omega_{ab}=0, \tag{22}$$

$$\Omega_{ab}\equiv \partial_a\eta_b-\partial_b\eta_a-\sum_{c=0}^{3} C_{cab}\eta_c. \tag{23}$$

Before looking at the geometrical content of (22) in the general case, we consider Rainich geometries for which the blades "mesh."

III. THE BLADES CAN "MESH"

Let us investigate the existence of two sets of integral surfaces which contain the blades at every point. The integral surfaces are defined by the intersections of pairs of σ hypersurfaces $[\sigma=\sigma(x^0,x^1,x^2,x^3)]$:

$$S_{(-)}=\left\{\begin{matrix}\sigma_0=\text{constant}\\ \sigma_1=\text{constant}\end{matrix}\right\}\text{such that}\left\{\begin{matrix}\partial_0\sigma_0=\partial_1\sigma_0=0\\ \partial_0\sigma_1=\partial_1\sigma_1=0,\end{matrix}\right. \tag{24}$$

and

$$S_{(+)}=\left\{\begin{matrix}\sigma_2=\text{constant}\\ \sigma_3=\text{constant}\end{matrix}\right\}\text{such that}\left\{\begin{matrix}\partial_2\sigma_2=\partial_3\sigma_2=0\\ \partial_2\sigma_3=\partial_3\sigma_3=0.\end{matrix}\right. \tag{25}$$

By virtue of (18) and (21), the integrability conditions which admit the existence of the negative-blade integral surfaces $S_{(-)}$ are

$$\eta_2=\eta_3=0. \tag{26}$$

Similarly, the necessary and sufficient conditions for the positive-blade integral surfaces $S_{(+)}$ are

$$\eta_0=\eta_1=0. \tag{27}$$

In view of (20), we have the theorem:

The negative-blade integral surfaces exist if and only if the normal to the α=constant hypersurface is contained in the negative blade at every point; similarly, the positive blade integral surfaces exist if and only if normal to the α=constant hypersurfaces is contained in the positive blade at every point. Finally, both sets of integral surfaces exist if and only if α is identically constant (Fig. 2).

The case of α identically constant, characteristic of the Reissner-Nordstrom solution[5,6] and most Rainich geometries presently known to us, is physically significant in that it describes an electromagnetic field which is purely electrical. This does not mean that the geometry is necessarily static, since $\alpha\equiv$constant is *not*

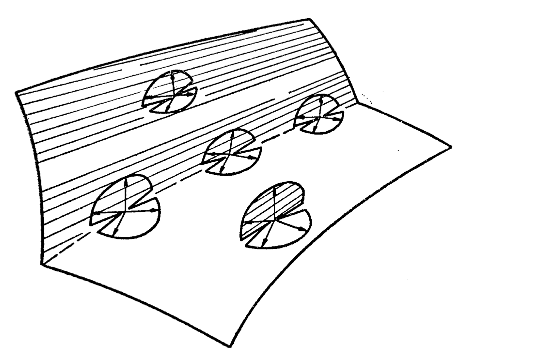

FIG. 2. The two systems of blades are, respectively, "meshed": α identically equals a constant.

[5] K. Reissner, Ann. Physik **50**, 106 (1916).
[6] L. Nordstrom, Proc. Amsterdam Acad. **20**, 1238 (1918).

sufficient to guarantee a group of motions in the direction of $\lambda_{0|\mu}$.

Notice that if the normal to the α= constant hypersurfaces is contained in the negative blade, not only do the negative-blade integral surfaces exist, but also the positive blades are contained in a set of hypersurfaces, namely, $\{\alpha=\text{constant}\}$. We are thus tempted to consider whether the general content of Rainich's fourth condition (20) amounts to the statement, "Two sets of hypersurfaces exist which contain the positive and negative blades, respectively." This possibility was studied and found to require more than the content of (20) or its equivalent equation, (22). Thus it does not allow electrodynamics to have its full known scope and has to be ruled out as a formulation of Rainich geometry. Misner and Wheeler[3] had previously rejected the much more stringent requirement, namely, that the blades shall *always* "mesh" into smooth integral surfaces of the form $S_{(+)}$ and $S_{(-)}$.

IV. THE CONTENT OF RAINICH'S FOURTH EQUATION

At first glance, the integrability conditions (22) comprise six independent conditions which must be satisfied by the geometry. However, since the Ricci vierbein is not uniquely determined, these conditions are not all independent. An internal gauge exists which can Lorentz-transform blade partners in the vierbein in a way which preserves (9), (16), and all relations which follow from them. Under the internal gauge transformations, the components of (23) within the blades (Ω_{01} and Ω_{23}) remain unchanged—they are gauge-invariant. The cross-components of (23) (Ω_{02}, Ω_{03}, Ω_{12}, Ω_{13}) suffer mixing with the two arbitrary internal gauge functions. The Rainich conditions (22) for the cross-components reduce to a pair of internal gauge conditions, and a pair of bonafide geometrical conditions. Hence, (22) states two and not four geometrical conditions on the cross components.

What is the nature of the two conditions in (22) which require the gauge-invariant components within the blades (Ω_{01} and Ω_{23}) to vanish? We shall prove that the conditions

$$\Omega_{01}=\Omega_{23}=0, \tag{28}$$

are trivially satisfied, as a consequence of the algebraic form of the Ricci tensor (16). Thus, the components of (22) within the blades express nothing that we do not already know about Rainich geometry.

The proof starts with the Poisson operator identity,

$$[\partial_c,[\partial_a,\partial_b]]+[\partial_a,[\partial_b,\partial_c]]+[\partial_b,[\partial_c,\partial_a]]\equiv 0. \tag{29}$$

We substitute (18) into this identity and find

$$\partial_c C_{eab}+\partial_a C_{ebc}+\partial_b C_{eca} + \sum_{d=0}^{3} (C_{dab}C_{ecd}+C_{dbc}C_{ead}+C_{dca}C_{ebd})\equiv 0. \tag{30}$$

Equating c and e in this expression and summing, we have

$$\Psi_{ab}\equiv\sum_{c=0}^{3} (\partial_c C_{cab}+\partial_a C_{cbc}+\partial_b C_{cca}) + \sum_{c,d=0}^{3} C_{ccd}C_{dab}\equiv 0. \tag{31}$$

Next, observe that the contracted Bianchi identities and (2) state that

$$R_{\mu}{}^{\sigma}{}_{;\sigma}=0. \tag{32}$$

Substituting (16) into this expression, we eventually obtain

$$\partial_a(\ln\rho)=2\xi_a, \tag{33}$$

where

$$\begin{aligned}\xi_0&\equiv C_{202}+C_{303},\\ \xi_1&\equiv C_{212}+C_{313},\\ \xi_2&\equiv C_{020}+C_{121},\\ \xi_3&\equiv C_{030}+C_{131}.\end{aligned} \tag{34}$$

The intrinsic curl of (33) is

$$\Phi_{ab}\equiv\partial_a\xi_b-\partial_b\xi_a-\sum_{c=0}^{3} C_{cab}\xi_c=0. \tag{35}$$

Now it is readily seen by referring to the definitions (23), (21), and (34), as well as the definition parts of (31) and (35), that

$$\begin{aligned}\Omega_{01}&=\Psi_{32}-\Phi_{32},\\ \Omega_{23}&=\Psi_{01}-\Phi_{01}.\end{aligned} \tag{36}$$

Hence, (31) and (35) prove the assertions made in (28). The Rainich conditions (22) within the blades are trivially satisfied.

The geometrical significance of the two gauge-free cross-components of (22) still remains a mystery. In view of (21) and (23), equation (22) expresses two independent relations between neighboring structure coefficients. At any point the structure coefficients can be regarded as structure constants for a Lie group. If the Riemannian manifold is considered to be a manifold of Lie groups, the two "mysterious" conditions in (22) may enjoy a clean geometrical interpretation in the language of fiber bundles.[7] This possibility seems worthy of investigation.

ACKNOWLEDGMENTS

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[7] N. Steenrod, *Topology of Fiber Bundles* (Princeton University Press, Princeton, 1951).