interactions. If we impose this internal symmetry property on the interaction (6.1) and assume the experimentally well established hypothesis of a two-component neutrino, then we see immediately that the only possible form for O_i is γ_μ, i.e., we have a (V,A) interaction

$$L=\bar{p}\gamma_\mu(a+b\gamma_5)n\bar{e}\gamma_\mu(1+\gamma_5)\nu+\text{H.c.} \qquad (6.3)$$

Here the symmetry principle *allows* parity violation but fixes the form of the interaction.

One may also consider the question of additional approximate symmetries in weak interactions valid for both weak and strong interactions. It seems that, besides the charge symmetry just discussed, the weak interaction Lagrangian also shows a symmetry between the electron and the muon.[34] This obviously has nothing to do with charge symmetry. It is tempting to relate this symmetry to one of the generalized charge symmetries of the baryons. For instance, if we relate the symmetry $e\rightleftarrows\mu$ to the S_2 charge symmetry (α), this would imply that the weak Lagrangian is invariant under the simultaneous interchange of e with μ and of N_1 with N_2. We note that a Lagrangian with just this property has been recently proposed by Feynman.[16] Thus the extension of generalized charge symmetries (unlike continuous groups like isospin rotations) to leptons might not be devoid of meaning. The consequences of such symmetry principles in weak interactions will be discussed in a different paper. All we would like to say at present is that this attitude is consistent with the idea of a hierarchy of approximate symmetries in elementary particle interactions, with the weak interactions having lower symmetries than the strong interactions which exhibit all the universal symmetries and also have additional symmetries.

[34] The possibility that this symmetry has more than formal significance has recently been suggested by M. Goldhaber, Phys. Rev. Letters 1, 467 (1958).

ACKNOWLEDGMENTS

We would like to thank Professor A. Pais for many stimulating discussions on symmetries of strong interactions and for his encouragement. We thank Dr. J. Sakurai for discussions and communications regarding his results. We would also like to thank Dr. R. Behrends, Dr. J. Bernstein, Dr. P. Kabir, and Dr. S. Weinberg for interesting comments.

One of the authors (F.G.) gratefully acknowledges financial support from the International Cooperation Administration and a grant from the National Science Foundation made available through the Institute for Advanced Study. He would also like to express his sincere thanks to Professor R. J. Oppenheimer for the kind hospitality extended to him at the Institute.

PHYSICAL REVIEW VOLUME 114, NUMBER 4 MAY 15, 1959

Dispersion Relations for *p*-*n* Scattering*

J. HAMILTON†
Brookhaven National Laboratory, Upton, New York, and Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received December 31, 1958)

The application of dispersion relations to low-energy *p*-*n* scattering is examined. It is shown that Khuri's dispersion relation can be extended to include tensor forces, but serious difficulties appear on attempting to include exchange forces. The application of the relativistic field theory dispersion relations to low-energy scattering is made by using the effective range formula. The spurious poles of the S-matrix are related to the two- and three-pion terms in the unphysical region contribution for forward scattering.

I. INTRODUCTION

We shall examine in detail the application of dispersion relations to low-energy *p*-*n* scattering. First we consider the dispersion relations for the Schrödinger equation which were derived by Khuri[1] for an ordinary central potential. If these relations are to be of value for *p*-*n* scattering, they must be extended to include (a) tensor forces, or spin-orbital $[(\mathbf{L}\cdot\mathbf{S})]$ forces; (b) exchange forces.

In Sec. 2 we show that the extension to include tensor forces is straightforward. The result is that each element $\langle S|M|S'\rangle$ of the scattering matrix M obeys an uncoupled dispersion relation; in each case the inhomogeneous term is the first Born approximation. It is also shown that these dispersion relations are in agreement with the nonrelativistic limit of the field theoretic dispersion relations of Goldberger, Nambu, and Oehme.[2] When we say that these sets of relations are in agreement with each other, we mean that the contributions from the physical region and from the deuteron state are identical. The Born approximation in the first set is to be equated to the one-pion, two-pion, . . . etc., contributions from the unphysical region in the second set.

The extension to a spin-orbital force of the type $f(r)(\mathbf{L}\cdot\mathbf{S})$ has not been examined. It would appear that this presents somewhat harder mathematical problems because of the differential operators in $\mathbf{L}$.

* This work was supported in part by the Office of Naval Research and the U. S. Atomic Energy Commission.
† Permanent address: Christ's College, Cambridge, England.
[1] N. N. Khuri, Phys. Rev. 107, 1148 (1957).

[2] Goldberger, Nambu, and Oehme, Ann. Phys. (N.Y.) 2, 226 (1957).

In Sec. 3 we examine the possibility of including exchange (central) forces in Khuri's analysis of the scattering solutions of the Schrödinger equation. Here there are considerable difficulties, and it is not clear that the usual fixed momentum transfer type of dispersion relations exist. This is not merely a case of requiring mathematical vigor where intuition, or simple considerations, would suggest that the relations do exist. In fact, it is easy to see that one part of the scattering amplitude has much simpler analytic properties for backward scattering than for forward scattering. This difficulty may be related to the acausal nature of the exchange potential in the static limit.

In the field theoretic relativistic dispersion relations[2] there is no comparable difficulty, presumably because in field theory all interactions propagate causally. It is not clear why the field theoretic dispersion relations have a simple nonrelativistic form, whereas such simple relations have not been derived from the Schrödinger equation when exchange forces are present.

Our inability to find such relations means that we do not know how to relate the Fourier transform of the wave mechanical potential V to the unphysical region contributions in the field theory dispersion relations. The best we can do at present is to say that the sum of the unphysical region terms (apart from the deuteron) defines a potential "V." However, it is not clear what is the relation of "V" to the static potential used in the Schrödinger equation.

In Sec. 4 we examine another way of relating the field theoretic dispersion relations to more familiar quantities. We show how the one-, two-, and three-pion contributions in the unphysical region are related to the parameters appearing in the effective range formulas for low-energy triplet and singlet state p-n scattering. There is qualitative agreement between the positions of the poles of the S-matrix for s-wave scattering and the energies at which the above unphysical region terms occur. A quantitative comparison must await the completion of calculations of the two-pion and (possibly) the three-pion contributions.

The dispersion integral in the field theoretic relation we have used, appears to diverge at high energies. In Sec. 5 we discuss briefly the relation between such dispersion integrals and hard core potentials.

II. KHURI'S RELATION AND ITS GENERALIZATION

Khuri[1] examined the analytic properties of the scattering solutions of the Schrödinger equation for an ordinary central potential $V_0(r)$:

$$\nabla^2\psi(\mathbf{x})+k^2\psi(\mathbf{x})=V_0(r)\psi(\mathbf{x}). \tag{1}$$

These solutions have the asymptotic form

$$\psi(\mathbf{x})\simeq e^{i\mathbf{k}\cdot\mathbf{x}}+(f(k,\tau)/r)e^{ikr}, \tag{2}$$

where τ is the momentum transfer. Provided $V_0(r)$ is finite,[3] and for large r obeys $|V_0(r)|<e^{-\alpha r}$ (where α is some positive constant), it can be shown that, for $\tau<2\alpha$, $f(k,\tau)$ is a regular function of k in the upper half plane $\mathrm{Im}k\geqslant 0$ except for simple poles at the bound states $k_j=+i\kappa_j$ ($\kappa_j>0$). With the same conditions, Khuri also shows that $f(k,\tau)$ is uniformly bounded for $|k|\rightarrow\infty$ in $\mathrm{Im}k\geqslant 0$, and

$$\left[f(k,\tau)+\frac{1}{4\pi}\int V_0(r)\exp(i\boldsymbol{\tau}\cdot\mathbf{x})d^3x\right]\rightarrow 0,$$

as $\mathrm{Re}k\rightarrow\infty$ in $\mathrm{Im}k\geqslant 0$. Integrating $f(k')/(k'-k)$ (real k) over the contour consisting of the real axis and the infinite upper semicircle gives a dispersion relation for fixed $\tau<2\alpha$. Regarding f as a function of $E=k^2$ and τ, this can be written

$$\mathrm{Re}f(E,\tau)=-\frac{1}{4\pi}\bar{V}_0(\tau)+\frac{1}{\pi}P\int\frac{\mathrm{Im}f(E',\tau)}{E'-E}dE'+\sum_j\frac{R_j(\tau)}{E-E_j}, \tag{3}$$

where

$$-\frac{1}{4\pi}\bar{V}_0(\tau)=-\frac{1}{4\pi}\int\exp(i\boldsymbol{\tau}\cdot\mathbf{x})\bar{V}_0(r)d^3x \tag{4}$$

is the first Born approximation for scattering from $\mathbf{k}$ into $\mathbf{k}'$ ($\boldsymbol{\tau}=\mathbf{k}-\mathbf{k}'$). Also, $E_j=-\kappa_j^2$ are the bound-state energies and $R_j(\tau)$ is the residue of $f(E,\tau)$ regarded as a function of E at $E=E_j$.

The Residues

As an example of how the residues are determined we consider an s-wave bound state $E_1=-\kappa_1^2$. Suppose its wave function is $g(-i\kappa_1,r)/r$, normalized so that $g(-i\kappa_1,r)\rightarrow e^{-\kappa_1 r}$ as $r\rightarrow\infty$. The partial wave expansion gives

$$f(k,\tau)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l(k)}-1)P_l(1-\tau^2/2k^2), \tag{5}$$

where $S_l(k)=\exp[2i\delta_l(k)]$ is the S-matrix for the partial wave l. For s-waves, $S_0(k)=g(k,0)/g(-k,0)$, and the residue of $S_0(k)$ at $k=i\kappa_1$ (as a function of k) is[4]

$$-i\Big/\int_0^\infty[g(-i\kappa_1,r)]^2dr.$$

Using (3) the residue $R_1(\tau)$ is therefore given by

$$[R_1(\tau)]^{-1}=-\int_0^\infty[g(-i\kappa_1,r)]^2dr. \tag{6}$$

For p-n scattering the only bound state is the deuteron

[3] We do not state the sharpest form of Khuri's conditions on V_0.

[4] R. Jost, Helv. Phys. Acta **20**, 256 (1947); R. Jost and W. Kohn, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **27**, No. 9 (1953).

and the right-hand side of (6) is the well-known deuteron wave-function normalization factor.[5]

The Tensor Force

The scalar dispersion relation (3) proved by Khuri will now be generalized to include tensor forces. Suppose the interaction is of the form

$$V=V_0(r)+W(r)S_{12},$$

where $S_{12}=3(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{x})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{x})r^{-2}-(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})$. $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ are the spin matrices for the two nucleons. The state vectors for the total spin are $|S=0\rangle$; $|S=1,S_3\rangle$, $(S_3=1,0,-1)$ where S_3 is the component of total spin along the initial direction $\mathbf{k}$. A convenient notation for these state vectors is $|S\rangle$ where S denotes any one of (0,0), (1,1), (1,0), (1, −1).

The p-n wave function can be written

$$\psi=\sum_s\psi_s|S\rangle,$$

here $\psi_s=\langle S|\psi\rangle$ is a function of $\mathbf{x}$ only. These four functions ψ_s obey the coupled Schrödinger equations

$$\nabla^2\psi_s+k^2\psi_s=\sum_{s'}\langle S|V|S'\rangle\psi_{s'}. \tag{7}$$

By time reversal invariance $\langle S|V|S'\rangle$ is a real function; it is symmetric in S, S'. We assume $|W(r)|<e^{-\alpha r}$ fo $r\to\infty$, and look for a solution of (7) having the asymptotic form

$$\psi_{s'(s)}\simeq\delta_{s's}e^{i\mathbf{k}\cdot\mathbf{x}}+(e^{ikr}/r)\langle S'|M|S\rangle, \tag{8}$$

where $|S\rangle$ is the initial spin state.

The scattering matrix element $\langle S'|M|S\rangle$ contains a factor $\exp[i(S_3-S_3')\varphi]$; φ is the azimuthal angle measured about $\mathbf{k}$. Also, $\langle S'|M|S\rangle$ can be regarded as a function of k, τ and $(k^2-\frac{1}{4}\tau^2)^{\frac{1}{2}}$; in other words it is a function of k, $\cos\theta$ and $\sin\theta$ where θ is the center-of-mass scattering angle $(\theta=2\theta_{\rm lab})$.

Using the matrix kernel

$$K_{s's''}(\mathbf{x},\mathbf{y})=-\langle S'|V(\mathbf{y})|S''\rangle e^{ik|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|,$$

Eq. (7) can be replaced by the coupled integral equations

$$\psi_{s''(s)}(\mathbf{x})=\delta_{s's}e^{i\mathbf{k}\cdot\mathbf{x}}+\sum_{s''}\int K_{s's''}(\mathbf{x},\mathbf{y})\psi_{s''(s)}(\mathbf{y})d^3y. \tag{9}$$

Iterating once gives a (matrix) integral equation having bounded kernels and bounded inhomogeneous terms. The Fredholm solution can readily be written down, and the same method as Khuri used will show that $\langle S'|M|S\rangle$ is a regular function of k in $\text{Im}k\geqslant 0$ except for simple poles at the bound states $k_j=+i\kappa_j$ $(\kappa_j>0)$, provided we consider momentum transfer $\tau<2\alpha$. Further $\langle S'|M|S\rangle$ is uniformly bounded as $|k|\to\infty$ in $\text{Im}k\geqslant 0$, and also

$$\left[\langle S'|M(k,\tau)|S\rangle+\frac{1}{4\pi}\int\langle S'|V|S\rangle\exp(i\boldsymbol{\tau}\cdot\mathbf{x})d^3x\right]\to 0,$$

as $\text{Re}k\to\infty$ in $\text{Im}k\geqslant 0$. We can therefore derive dispersion relations if we know the crossing properties of $\langle S'|M|S\rangle$.

From now on we drop the factor $\exp[i(S_3-S_3')\varphi]$ from the elements of the scattering matrix. For $|S_3-S_3'|=0$ or 2, the amplitude $\langle S'|M|S\rangle$ is now a function of k and $\cos\theta=1-\tau^2/2k^2$. It is easy to see (by looking at the general form of the Fredholm expansion terms) that in these cases

$$\langle S'|M(k,\tau)|S\rangle^*=\langle S'|M(-k^*,\tau)|S\rangle.$$

For $|S_3-S_3'|=1$ the scattering amplitude is also linearly dependent on $\sin\theta=\tau(k^2-\frac{1}{4}\tau^2)^{\frac{1}{2}}/k^2$. The complex k plane has a cut from $-\frac{1}{2}$ to $+\frac{1}{2}$, and, on crossing, $\sin\theta$ will change sign. It follows that the general form of the crossing relation is

$$\langle S'|M(k,\tau)|S\rangle^*=(-1)^{(S_3'-S_3)}\langle S'|M(-k^*,\tau)|S\rangle.$$

Hence for $|S_3'-S_3|=0$ or 2, the dispersion relation is

$$\text{Re}\langle S'|M(E,\tau)|S\rangle=-\frac{1}{4\pi}\int\langle S'|V|S\rangle\exp(i\boldsymbol{\tau}\cdot\mathbf{x})d^3x$$
$$+\frac{1}{\pi}P\int_0^\infty dE'\frac{\text{Im}\langle S'|M(E',\tau)|S\rangle}{E'-E}+\frac{\langle S'|R_D(\tau)|S\rangle}{E-E_D}, \tag{10}$$

(we have only considered one bound state—the deuteron state with energy $E_D=-k_D^2$). The residue $\langle S'|R_D(\tau)|S\rangle$ is evaluated in a similar way to that of Goldberger *et al.*[2]

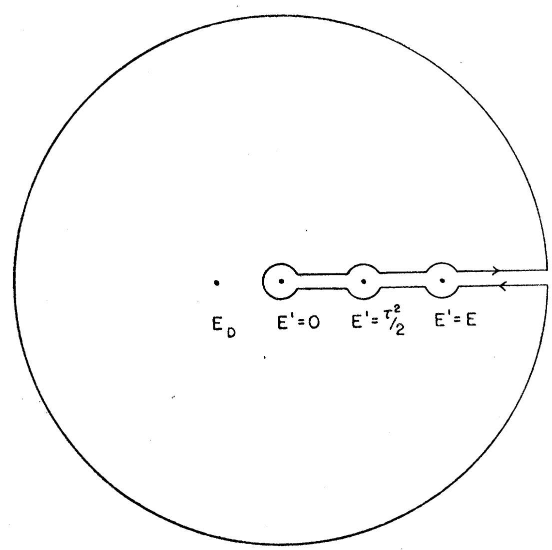

FIG. 1. The contour in the complex E'-plane for integrating Eq. (14).

[5] See, for example, H. A. Bethe and C. Longmire, Phys. Rev. **77**, 647 (1949).

For $|S_3-S_3'|=1$ there is a similar dispersion relation for $\langle S'|M(E,\tau)|S\rangle(E-\frac{1}{4}\tau^2)^{-\frac{1}{2}}$.

Relation to the Field Theoretic Formulas

The relation between these dispersion relations and the nonrelativistic limit of the field theory p-n relations derived by Goldberger, Nambu, and Oehme[2] should be examined. They write the scattering matrix in the form (we give the nonrelativistic limit)

$$M=\alpha+\beta\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}+i\gamma(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\mathbf{n} +\delta(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{m})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{m})+\epsilon(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{l})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{l}), \quad (11)$$

where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are unit vectors in the directions $\mathbf{k}+\mathbf{k}'$, $\mathbf{k}-\mathbf{k}'$, $\mathbf{k}\times\mathbf{k}'$. The coefficients α, β, γ, δ, ϵ are functions of k, $\cos\theta$ and $\sin\theta$. The relation between these coefficients and the scattering matrix elements is (see reference 2, Appendix)

$$\begin{aligned}
4\alpha&=2\langle 1,1|M|1,1\rangle+\langle 1,0|M|1,0\rangle+\langle 0|M|0\rangle,\\
4\beta&=-2\langle 1,1|M|1,-1\rangle+\langle 1,0|M|1,0\rangle-\langle 0|M|0\rangle,\\
4\gamma&=\sqrt{2}\langle 1,1|M|1,0\rangle-\sqrt{2}\langle 1,0|M|1,1\rangle,\\
4\delta&=(1-\sec\theta)\langle 1,1|M|1,1\rangle\\
&\quad+(1+\sec\theta)\langle 1,1|M|1,-1\rangle \qquad (12)\\
&\quad+\sec\theta\langle 1,0|M|1,0\rangle-\langle 0|M|0\rangle,\\
4\epsilon&=(1+\sec\theta)\langle 1,1|M|1,1\rangle\\
&\quad+(1-\sec\theta)\langle 1,1|M|1,-1\rangle\\
&\quad-\sec\theta\langle 1,0|M|1,0\rangle-\langle 0|M|0\rangle.
\end{aligned}$$

In the nonrelativistic limit $\alpha(E,\tau)$, $\beta(E,\tau)$, $\delta(E,\tau)$, $\epsilon(E,\tau)$ *separately* obey dispersion relations of the typical form

$$\mathrm{Re}\alpha(E,\tau)=(\mathrm{U.P.})+\frac{\alpha_D}{E-E_D}+\frac{1}{\pi}P\int_0^\infty dE'\frac{\mathrm{Im}\alpha(E',\tau)}{E'-E}. \quad (13)$$

Here α_D, $\beta_D\cdots$ are the residues at the deuteron pole $E=E_D=-\kappa_D^2$. Also (U.P.) denotes the unphysical region contributions coming from one-pion, two-pion, . . . etc., terms and from anti-proton-neutron scattering. [In (13) as in (3) and (10) there is another unphysical contribution arising from the range $0<E'<\frac{1}{4}\tau^2$ in the integral.] The coefficient γ obeys a dispersion relation obtained from (13) on replacing $\alpha(E',\tau)$ by $\gamma(E',\tau)(E'-\frac{1}{4}\tau^2)^{-\frac{1}{2}}$ everywhere.

The functions $\langle S'|M|S\rangle\sec\theta=\langle S'|M|S\rangle 2k^2/(2k^2-\tau^2)$ have the same analytic and boundedness properties as $\langle S'|M|S\rangle$ in $\mathrm{Im}k\geqslant 0$, except for extra poles on the real axis at $k=\pm\tau/\sqrt{2}$. On integrating the function

$$\frac{\langle S'|M(E',\tau)|S\rangle\, 2E'}{(E'-E)(2E'-\tau^2)},$$

around the contour in the E'-plane which is shown in Fig. 1, we see that for $|S_3-S_3'|=0$ or 2, we have the dispersion relation

$$\sec\theta\left\{\mathrm{Re}\langle S'|M(E,\tau)|S\rangle+\frac{1}{4\pi}\int\langle S'|V|S\rangle\exp(i\boldsymbol{\tau}\cdot\mathbf{x})d^3x\right\}$$
$$=\frac{1}{\pi}P\int_0^\infty dE'\frac{\mathrm{Im}\langle S'|M(E',\tau)|S\rangle}{E'-E}\frac{E'}{E'-\frac{1}{2}\tau^2}+\frac{\langle S'|R_D(\tau)|S\rangle}{E-E_D}\frac{E_D}{E_D-\frac{1}{2}\tau^2}-\mathrm{Re}\langle S'|M(E'=\tfrac{1}{2}\tau^2,\,\tau)|S\rangle\sec\theta(\tau^2/2E). \quad (14)$$

The last term in (14) arises from the extra pole at $E'=\frac{1}{2}\tau^2$.

If we substitute the dispersion relations (10) and (14) into (12) in order to derive dispersion relations for δ and ϵ, we obtain the typical relation

$$\mathrm{Re}\delta(E,\tau)=-\frac{1}{4\pi}\bar{V}_\delta+\frac{\delta_D}{E-E_D}+\frac{1}{\pi}P\int_0^\infty dE'\frac{\mathrm{Im}\delta(E',\tau)}{E'-E}, \quad (15)$$

where $-(1/4\pi)\bar{V}_\delta$ is the first Born approximation. This takes the place of the term (U.P.) in the field theory relations [see (13)]. The last term in (14) does not appear in (15). This is because when we use (12) to collect the correct linear combination of terms $\langle S'|M|S\rangle$ to give δ, the sum of these terms $\langle S'|M(E'=\frac{1}{2}\tau^2,\,\tau)|S\rangle$ will vanish. There is a general relation[6] between the elements of the scattering matrix,[7]

$$\langle 11|M|11\rangle-\langle 11|M|1,-1\rangle-\langle 10|M|10\rangle =\sqrt{2}\cot\theta\{\langle 10|M|11\rangle+\langle 11|M|10\rangle\}. \quad (16)$$

The left side of this equation is exactly the coefficient of $\sec\theta$ in the expressions for δ and ϵ in (12). Also, $E'=\frac{1}{2}\tau^2$ gives $\cos\theta'=0$; hence the left side of (16) vanishes for $E'=\frac{1}{2}\tau^2$.

It is now clear that our relations (10) and (13) agree with the nonrelativistic limit of the field theory relations (13). It is also clear that the relations for $\langle S'|M(E,\tau)|S\rangle\times(E-\frac{1}{4}\tau^2)^{-\frac{1}{2}}$ for $|S_3-S_3'|=1$ and the relations for $\gamma(E,\tau)(E-\frac{1}{4}\tau^2)^{-\frac{1}{2}}$ will agree. In each case the terms (U.P.) in the field theory relations are replaced by the first Born approximation.

III. EXCHANGE FORCES

The Fredholm method does not lead to simple dispersion relations when the potential is of the form

$$V=V_0(r)+V_M(r)P_x,$$

[6] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).
[7] As usual we drop all factors $e^{\pm i\phi}$, $e^{\pm 2i\phi}$.

where P_x is the space exchange operator. $V_0(r)$ and $V_M(r)$ are assumed to be finite central potentials, both of them obeying the condition $|V(r)|<e^{-\alpha r}$ as $r\rightarrow\infty$ for some fixed positive α. The scattering integral equation

$$\psi(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}+\int K(\mathbf{x},\mathbf{y})\psi(\mathbf{y})d^3y \tag{17}$$

has the kernel

$$K(\mathbf{x},\mathbf{y})=-(1/4\pi)V_0(y)e^{ik|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| -(1/4\pi)V_M(y)e^{ik|\mathbf{x}+\mathbf{y}|}/|\mathbf{x}+\mathbf{y}|. \tag{18}$$

The first few terms in the scattering amplitude derived from the Fredholm solution are [see reference 1, Eqs. (16) and (17)]

$$\begin{aligned}f(\mathbf{k}',\mathbf{k})=&-\frac{1}{4\pi}\int e^{-i\mathbf{k}'\cdot\mathbf{y}}V_0(y)e^{i\mathbf{k}\cdot\mathbf{y}}d^3y\\ &-\frac{1}{4\pi}\int e^{i\mathbf{k}'\cdot\mathbf{y}}V_M(y)e^{i\mathbf{k}\cdot\mathbf{y}}d^3y\\ &+\frac{1}{(4\pi)^2}\int e^{-i\mathbf{k}'\cdot\mathbf{y}}V_0(y)\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}V_0(x)e^{i\mathbf{k}\cdot\mathbf{x}}d^3xd^3y\\ &+\frac{1}{(4\pi)^2}\int e^{i\mathbf{k}'\cdot\mathbf{y}}V_M(y)\frac{e^{ik|\mathbf{x}+\mathbf{y}|}}{|\mathbf{x}+\mathbf{y}|}V_M(x)e^{i\mathbf{k}\cdot\mathbf{x}}d^3xd^3y\\ &+\frac{1}{(4\pi)^2}\int e^{-i\mathbf{k}'\cdot\mathbf{y}}V_0(y)\frac{e^{ik|\mathbf{x}+\mathbf{y}|}}{|\mathbf{x}+\mathbf{y}|}V_M(x)e^{i\mathbf{k}\cdot\mathbf{x}}d^3xd^3y\\ &+\frac{1}{(4\pi)^2}\int e^{i\mathbf{k}'\cdot\mathbf{y}}V_M(y)\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}V_0(y)e^{i\mathbf{k}\cdot\mathbf{x}}d^3xd^3y\\ &+\cdots.\end{aligned} \tag{18a}$$

The remaining terms in the complete Fredholm solution can readily be written down.

Those integrals in the numerator of the solution which contain either no factor V_M or an *even number* of factors V_M have the same boundedness properties in $\mathrm{Im}k\geqslant 0$ (for fixed $\tau<2\alpha$) as we saw in the ordinary force case. The proof follows Khuri's method closely. The basic step is the bound for the iterated kernel

$$K_2(\mathbf{x},\mathbf{y})=\int K(\mathbf{x},\mathbf{z})K(\mathbf{z},\mathbf{y})d^3z.$$

We can write

$$K_2(\mathbf{x},\mathbf{y})=\{A_{00}(\mathbf{x},\mathbf{y})V_0(y)+A_{M0}(\mathbf{x},\mathbf{y})V_0(y) +A_{0M}(\mathbf{x},\mathbf{y})V_M(y)+A_{MM}(\mathbf{x},\mathbf{y})V_M(y)\}1/y,$$

where A_{00} comes from the term in K_2 containing V_0V_0, etc. The triangular inequality

$$|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|\geqslant|\mathbf{x}-\mathbf{y}|$$

can be used to prove $|A_{00}(\mathbf{x},\mathbf{y})|<N\exp(-\kappa|\mathbf{x}-\mathbf{y}|)$ where N is a constant and $\kappa\equiv\mathrm{Im}k\geqslant 0$. Similarly the inequality

$$|\mathbf{x}+\mathbf{z}|+|\mathbf{z}-\mathbf{y}|\geqslant|\mathbf{x}+\mathbf{y}|$$

shows that both $|A_{0M}(\mathbf{x},\mathbf{y})|$ and $|A_{M0}(\mathbf{x},\mathbf{y})|$ are bounded in $\mathrm{Im}k\geqslant 0$ by a constant multiple of $\exp(-\kappa|\mathbf{x}+\mathbf{y}|)$. Finally $|A_{MM}(\mathbf{x},\mathbf{y})|<N'\exp(-\kappa|\mathbf{x}-\mathbf{y}|)$ where N' is a constant.

Using these results it is easy to show that the Fredholm denominator $\Delta(k^2)$ is regular in $\mathrm{Im}k\geqslant 0$ and $\Delta(k^2)\rightarrow 1$ as $|k|\rightarrow\infty$. Again the zeros of $\Delta(k^2)$ give the bound states ($k_j=+i\kappa_j$, $\kappa_j>0$). The analytic properties and the boundedness of the terms in the Fredholm numerator which have an *even* number of factors V_M follow readily. The contribution of these terms to the scattering amplitude is either of the form (a)

$$\int\exp(-i\mathbf{k}'\cdot\mathbf{z})V(z)M(\mathbf{z},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d^3yd^3z,$$

where $|M(\mathbf{z},\mathbf{y})|<N[V(y)/|\mathbf{y}|]\exp(-\kappa|\mathbf{z}-\mathbf{y}|)$ and N is a constant; or it is of the form (b)

$$\int\exp(i\mathbf{k}'\cdot\mathbf{z})V(z)\bar{M}(\mathbf{z},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d^3yd^3z,$$

where $|\bar{M}(\mathbf{z},\mathbf{y})|<\bar{N}[V(y)/|\mathbf{y}|]\exp(-\kappa|\mathbf{z}+\mathbf{y}|)$ and $\bar{N}$ is a constant. Here V is written for either V_0 or V_M. Khuri's methods will now show the required regularity and boundedness for $\tau<2\alpha$. {It may be useful to notice that the fourth integral in (18a) gives some idea of the behavior. The exponentials appearing can be written

$$\exp[ik|\mathbf{x}+\mathbf{y}|+i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})]\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{y}].$$

For constant τ this is bounded in $\mathrm{Im}k\geqslant 0$.}

Even and the Odd Dispersion Relations

Now we divide the scattering amplitude $f(k',k)$ into two parts

$$f(\mathbf{k}',\mathbf{k})=f_{\text{even}}(\mathbf{k}',\mathbf{k})+f_{\text{odd}}(\mathbf{k}',\mathbf{k}), \tag{19}$$

where f_{even}, f_{odd} arise from Fredholm numerator terms having respectively an even or odd number of factors V_M. It follows that f_{even} obeys the dispersion relation

$$\mathrm{Re}f_{\text{even}}(E,\tau)=-\frac{1}{4\pi}\int V_0(x)\exp(i\boldsymbol{\tau}\cdot\mathbf{x})d^3x +\frac{1}{\pi}P\int_0^\infty dE'\frac{\mathrm{Im}f_{\text{even}}(E',\tau)}{E'-E}+\sum_j\frac{R_j^{\text{even}}(\tau)}{E-E_j}, \tag{20}$$

where $R_j^{\text{even}}(\tau)$ is the residue of $f_{\text{even}}(E,\tau)$ at the bound state $E=E_j$. Eq. (20) holds for $\tau<2\alpha$.

The behavior of $f_{\text{odd}}(k',k)$ is very different. This can be seen from the fifth and sixth terms in (14), whose exponential factors are, respectively,

$$\exp[ik|\mathbf{x}-\mathbf{y}|+i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))]\exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{y}]$$

and

$$\exp[ik|\mathbf{x}+\mathbf{y}|+i(\mathbf{k}\cdot(\mathbf{x}+\mathbf{y}))]\exp[-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{y}].$$

These are not bound in $\mathrm{Im}k\geqslant 0$ for constant τ. However they are bound if we keep τ' fixed where $\boldsymbol{\tau}'=\mathbf{k}+\mathbf{k}'$. It is straightforward to examine the boundedness and analytic properties of all the terms in $f_{\mathrm{odd}}(\mathbf{k}',\mathbf{k})$ for fixed τ'. Then it appears that for fixed $\tau'<2\alpha$, $f_{\mathrm{odd}}(\mathbf{k}',\mathbf{k})$ obeys a dispersion relation whose inhomogeneous term is

$$-\frac{1}{4\pi}\int V_M(x)e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}d^3x.$$

Such a dispersion relation is associated with backward scattering.

Thus we see that if exchange forces are present the method of Khuri cannot be used to prove that the scattering amplitude $f(\mathbf{k}',\mathbf{k})$ of Eq. (19) obeys a simple dispersion relation for constant momentum transfer. Indeed the form of the first few terms in f_{odd} [see Eq. (14)] suggests that *in general* there may not be such a relation.

Discussion

Physically the results are perhaps not unexpected. The scattering solution of Eq. (1) in a certain way involved causality. The integral equation form of the solution shows that a disturbance $\psi(\mathbf{y})$ at $\mathbf{y}$ gives rise to outward propagating wavelets of the form

$$(e^{ik|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|)V_0(y)\psi(\mathbf{y}).$$

For an exchange potential the outgoing wavelets are

$$(e^{ik|\mathbf{x}+\mathbf{y}|}/|\mathbf{x}+\mathbf{y}|)V_M(y)\psi(\mathbf{y});$$

these wavelets can get ahead of the incident wave and thereby appear to violate causality. This suggests that there may be some difficulty in deriving dispersion relations in the case of exchange forces; it does not prove that such relations do not exist.[8]

These difficulties do not occur in the field theory case; there we have the nonrelativistic dispersion relations (13) for fixed momentum transfer τ.[9] This is not surprising, because the field theory description of the p-n interaction, even when it deals with exchange processes, is causal. The present difficulties could suggest the inadequacy of the static potential for p-n interactions.

If we cannot find a (fixed momentum transfer) dispersion relation for $f(k,\tau)$ when exchange forces occur, we *cannot* use the field theory relations (13) to describe the static potential in terms of the unphysical region (one-pion, two-pion, $\cdots$) terms (U.P.). Even if we were to conjecture that $f(k,\tau)$ did satisfy a dispersion relation of the form (3), we would not know what was the correct inhomogeneous term to replace $\bar{V}_0(\tau)$.[10]

Finally we notice a related difficulty which occurs in discussing solutions of (1) for p-p scattering without exchange forces. The scattering amplitude $f(\mathbf{k}',\mathbf{k})$ arises from the incident plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$. Therefore the actual scattering amplitude must, by Pauli's principle, be

$$f_{pp}(\mathbf{k}',\mathbf{k})=\frac{1}{\sqrt{2}}\{f(\mathbf{k}',\mathbf{k})\pm f(-\mathbf{k}',\mathbf{k})\},$$

where $+$ and $-$ occur in the singlet and triplet spin states. $f(\mathbf{k}',\mathbf{k})$ obeys the dispersion relation (3) with $|\mathbf{k}-\mathbf{k}'|$ constant, but $f(-\mathbf{k}',\mathbf{k})$ obeys a relation which differs from (3) in that $|\mathbf{k}+\mathbf{k}'|$ is constant. Again we cannot, for example, give a simple dispersion relation for forward p-p scattering.

IV. FIELD THEORY RELATIONS AT LOW ENERGIES

We examine the field theory relation for forward p-n scattering. The variable we consider is $D(E)$ where $\frac{1}{4}D(E)$ is the forward scattering amplitude for an unpolarized beam. In the notation of Goldberger *et al.*[11] $D=4\alpha(1+E/M)^{-\frac{1}{2}}$ where $E=k^2$ and k is the relative momentum in the center-of-mass system.[12] Where units are not explicitly stated we use the system with $\hbar=c=1$ and nucleon mass $M=1$; this gives units of energy $=940$ Mev, length $=2.11\times10^{-14}$ cm, area $=0.45$ mb.

The dispersion relation is[11]

$$\begin{aligned}(1+E/M)^{\frac{1}{2}}\,\mathrm{Re}D(E)&\\ =\frac{2\Gamma_\alpha(0)}{E_D-E}+\frac{f^2}{4\pi}\frac{1}{E-E_\mu}&-\frac{4}{\pi}\int_{(-\infty)}^{(+E_{2\mu})}dE'\frac{\mathrm{Im}\alpha_{n\bar{p}}(E')}{(-E')+E}\\ &+\frac{1}{\pi^2}P\int_0^\infty dE'([E'(1+E'/M)]^{\frac{1}{2}}\frac{\sigma_{np}(E')}{E'-E}.\qquad(21)\end{aligned}$$

In the last integral the optical theorem has been used to express $\mathrm{Im}D(E')$ in terms of the total cross section $\sigma_{np}(E')$. The first term on the right is the deuteron contribution, where $E_D=-\kappa_D^2=-2.25$ Mev; the residue is

$$2\Gamma_\alpha(0)=6\kappa_D/(1-\kappa_D r_T),$$

(r_T is the effective range of the ^3S state). The next term is the single pion pole contribution at center-of-mass energy $E_\mu=-\mu^2/4M=-5.15$ Mev where $\mu=$ pion mass. $(f^2/4\pi)$ is the renormalized coupling constant in Heaviside units ($f^2/4\pi\simeq0.08$). The rest of unphysical region (the continuum) and the antiparticle scattering contri-

[8] It may be that for certain forms of exchange potential we could use an analytic continuation of $f(k,\tau)$ from the real axis to the region $\mathrm{Im}k\leqslant 0$.

[9] Of course there may be some argument about the validity of the nonrelativistic limit which is used.

[10] S. Matsuyama and H. Miyazawa [Progr. Theoret. Phys. (Kyoto) **19**, 517 (1958)] suggest using a perturbation expansion in terms of V to determine V from the field theory dispersion relations.

[11] See reference 2, Eq. (6.16).

[12] In the nonrelativistic limit E is the energy in the c.m. system, and $E_{\mathrm{lab}}=2E$.

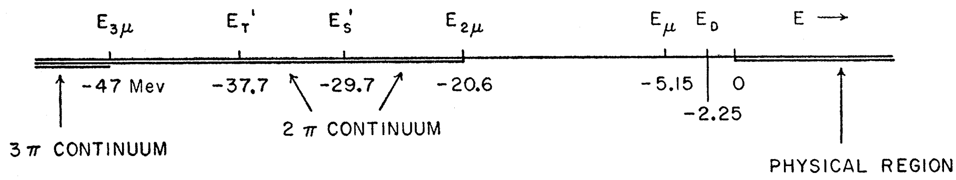

FIG. 2. The unphysical region (energies in center-of-mass system).

bution is in the term $\mathrm{Im}\alpha_{n\bar{p}}(E')$ in the first integral. Clearly for low-energy scattering (small E) the most important contributions[13] will come from small $(-E')$. Such terms are the two-pion annihilation, which occurs for $E'<E_{2\mu}=-\mu^2/M=-20.6$ Mev, and the three-pion annihilation which occurs for $E'<E_{3\mu}=-9\mu^2/4M=-47$ Mev. The unphysical region is shown in Fig. 2.

Effective Range Formulas

At low energies [say less than 20 Mev (lab)] we can write

$$\mathrm{Re}D(E)=(1/2k)\sin 2\delta_S+(3/2k)\sin 2\delta_T, \quad (22)$$

where δ_S and δ_T are the singlet and triplet s-wave phase shifts. These phase shifts are given accurately by the effective range formula

$$\begin{aligned} k\cot\delta_S&=-1/a_S+\tfrac{1}{2}r_Sk^2,\\ k\cot\delta_T&=-1/a_T+\tfrac{1}{2}r_Tk^2, \end{aligned} \quad (23)$$

where, in our units,

$$a_S=-112,\quad r_S=11.8,$$
$$a_T=25.4,\quad r_T=8.05.$$

Substituting (23) into (22) we get an accurate expression for $\mathrm{Re}D(E)$ at low energies. It is easy to check that the P-wave phase shifts give a very small correction to $\mathrm{Re}D(E)$ at these energies. This is done by *assuming* the P-wave phase shifts vary like k^3 and remembering that at 40 Mev the largest ^3P phase shift is about 11° and the ^1P phase shift is about −15°.

Before substituting (22) on the left of (21) we make one subtraction. This is because the dispersion relation as it stands probably does not have a convergent integral over σ_{np}. In any case, it is very useful to subtract once; in this way we can use the relation at low energy without having to know the values of σ_{np} at high energies particularly accurately. The subtracted relation is

$$\frac{1}{E}\left\{\left(1+\frac{E}{M}\right)^{\frac{1}{2}}\mathrm{Re}D(E)-D(0)\right\}=\frac{f^2}{4\pi}\frac{1}{E_\mu(E-E_\mu)}+\frac{2\Gamma_\alpha(0)}{E_D(E_D-E)}+\frac{4}{\pi}\int_{(-\infty)}^{E_{2\mu}}dE'\,\frac{\mathrm{Im}\alpha_{n\bar{p}}(E')}{(-E')[E+(-E')]}+\frac{1}{\pi^2}P\int_0^\infty dE'\left(\frac{M+E'}{E'}\right)^{\frac{1}{2}}\frac{\sigma_{np}(E')}{E'-E}. \quad (24)$$

On the left of (24), for low energies we substitute

$$\frac{1}{E}\left(\frac{1}{2k}\sin 2\delta_S+a_S\right)=a_S\frac{E+(2/r_S^2)(2-r_S/a_S)}{E^2+E(4/r_S^2)(1-r_S/a_S)+4/r_S^2a_S^2}, \quad (25)$$

and we use a similar expression for the triplet term. Again we can verify that the P-wave terms which should be added to (25) are unimportant. In the triplet case they are of relative order E (E is measured in the above units) and in the singlet case they are smaller.

In the low-energy range [say, $0<E<10$ Mev (c.m.)] the function on the right of (25) differs at the most by about 10% from the value we would obtain on putting $r_S=0$. In the triplet case we have a larger ratio (r_T/a_T) and the corresponding difference for some low-energy values of E is about 25%. We expect therefore that insofar as our results involve the effective range r, they will be appreciably more accurate for the triplet than the singlet case. We shall see also that the triplet effective range plays an important part in the deuteron contribution.

Evaluation of the Dispersion Relation

We now examine the various terms on the right of (24). Taking $(f^2/4\pi)\simeq 0.08$, the single pion gives

$$-14.6/(E+5.15\text{ Mev}), \quad (26)$$

where the numerator is in the above units. At low energies an antiproton and a neutron can only produce a single pion if they are in the singlet state. Hence the single pion term (26) contributes mainly to the singlet state scattering. (This is not true after a subtraction.)

For energies of a few Mev the term (26) contributes about 15% of the total singlet term (25); the proportion is much less at very low energies. Using Eq. (21) we can make an estimate of the contribution of the single pion term to the total singlet *cross section* at low energies; it gives a few percent of the whole. It therefore appears a reasonable, if rough, first approximation to ignore the single pion term.[14]

The second term on the right of (24) is the deuteron

[13] More precisely, the contributions which vary most rapidly with E.

[14] In a later paper we shall discuss the single pion term.

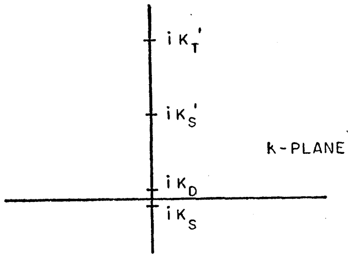

FIG. 3. The poles of the s-wave matrix $S(k)$ with $\kappa_D=(1/r_T)[1-(1-2r_T/a_T)^{\frac{1}{2}}]$, $K_T'=(1/r_T)[1+(1-2r_T/a_T)^{\frac{1}{2}}]$; $K_S=-(1/r_S)\times[(1-2r_S/a_S)^{\frac{1}{2}}-1]$, $K_S'=(1/r_S)[1+(1-2r_S/a_S)^{\frac{1}{2}}]$.

contribution. It is

$$\frac{6}{\kappa_D(1-\kappa_D r_T)}\frac{1}{E+2.25\text{ Mev}}=\frac{6}{\kappa_D(1-2r_T/a_T)^{\frac{1}{2}}}\frac{1}{E+2.25\text{ Mev}}=\frac{202}{E+2.25\text{ Mev}}. \quad (27)$$

(If we were to put $r_T=0$ in this large term, the error produced would be great—the correct normalization of the deuteron wave function is very important here.) The third term on the right of (24) contains the remaining unphysical region contributions. We shall now use Eq. (24) to find this term (or at least its low-energy part).

The dispersion integral is written

$$\frac{1}{\pi^2}P\int_0^\infty dE'\left(\frac{M+E'}{E'}\right)^{\frac{1}{2}}\frac{\sigma_{np}(E')}{E'-E}=\frac{1}{\pi^2}P\int_0^\infty\frac{dE'}{E'^{\frac{1}{2}}}\frac{\sigma_0(E')}{E'-E}+\frac{1}{\pi^2}P\int_0^\infty dE'\left(\frac{M+E'}{E'}\right)^{\frac{1}{2}}\frac{\bar{\sigma}(E')}{E'-E}, \quad (28)$$

where

$$\bar{\sigma}(E)=\sigma_{np}(E)-(1+E/M)^{-\frac{1}{2}}\sigma_0(E)$$

and

$$\sigma_0(E)=\frac{\pi}{E+(1/a_S-\frac{1}{2}Er_S)^2}+\frac{3\pi}{E+(1/a_T-\frac{1}{2}Er_T)^2}. \quad (29)$$

We notice that although the effective range formula (23) only applies to s-waves and is only valid [in form (23)] for low energies, the function $\sigma_0(E)$ is a good approximation to the total p-n cross section σ_{np} up to 50 Mev (lab). At 47 Mev the error is 4% and at 94 Mev it is 15%. It is clear that for energies up to 100 Mev (lab) $\bar{\sigma}(E)$ is small and we shall neglect any contributions from the last integral in (28) from the region of small E'.

The first integral on the right of (28) can be evaluated algebraically. This gives

$$-3a_T\frac{E+(2/r_T^2)(2-r_T/a_T)}{E^2+E(4/r_T^2)(1-r_T/a_T)+4/a_T^2r_T^2}+\frac{a_S}{(1-2r_S/a_S)^{\frac{1}{2}}}\frac{E+(2/r_S^2)(2-3r_S/a_S)}{E^2+E(4/r_S^2)(1-r_S/a_S)+4/r_S^2a_S^2}.$$

The Poles of $S(k)$

Substituting all these results in (24) gives

$$-\frac{f^2}{4\pi}\frac{1}{(-E_\mu)(E-E_\mu)}+\frac{4}{\pi}\int_{(-\infty)}^{E_{2\mu}}dE'\frac{\text{Im}\alpha_{n\bar{p}}(E')}{(-E')(E-E')}$$
$$=-\frac{2}{(1-2r_S/a_S)^{\frac{1}{2}}}\frac{1}{K_S'}\frac{1}{E+K_S'^2}-\frac{6}{(1-2r_T/a_T)^{\frac{1}{2}}}\frac{1}{K_T'}\frac{1}{E+K_T'^2}+\frac{1}{\pi^2}P\int_0^\infty dE''\left(\frac{M+E''}{E''}\right)^{\frac{1}{2}}\frac{\bar{\sigma}(E'')}{E''-E}. \quad (30)$$

Here

$$K_S'=(1/r_S)\{1+(1-2r_S/a_S)^{\frac{1}{2}}\},$$
$$K_T'=(1/r_T)\{1+(1-2r_T/a_T)^{\frac{1}{2}}\},$$

so $K_S'^2=29.7$ Mev, $K_T'^2=37.7$ Mev.

The first two terms on the right of (30) can be written

$$-12.5/(E+29.7\text{ Mev})-49.5/(E+37.7\text{ Mev}). \quad (31)$$

The integral over (E'') on the right of (30) can only give appreciable contributions for large E''. The present analysis is only valid for low-energy E (≤ 10 Mev) and for such E the integral over $\bar{\sigma}(E'')$ is effectively a constant. We shall discuss this constant in Sec. 5 below. In the unphysical region integral over $\text{Im}\alpha_{n\bar{p}}(E')$, we should expect (for small E) that the contributions from $E'\leq -50$ Mev behave as a constant.[15]

The first two terms on the right of (30) have a very simple form. They are in fact the residues of the singlet and triplet s-wave S-matrices $S(k)$ at the spurious poles $k=iK_S'$, $k=iK_T'$, respectively. [The poles of $S(k)$ are shown in Fig. 3.] The reason for this simple result is that we have effectively used a dispersion relation for $S(k)$. This is because $\sigma_0(E)$ is such a good approximation to σ_{np} even for high energies, and because the σ_0 term is separated out in (28). In this dispersion relation for $S(k)$ all the poles in $\text{Im}k\geq 0$ will contribute. It is well known that these "spurious" poles at $k=iK_S'$, $k=iK_T'$ do not correspond to actual bound states of the p-n system. A pole at $k=+iK$ need only give a bound state if $(1/K)$ is greater than the range of the p-n force[16];

[15] Our analysis is not sufficiently accurate to detect any variation in these terms.
[16] R. Jost and W. Kohn, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 27, No. 9 (1953).

this is true for the deuteron pole $k=i\kappa_D$ but not for the others.

Comparison with Experiment

We compare the left and right sides of (30) considering only the one-, two- and three-pion annihilation terms. The angular momentum and parity section rules show that 2π-annihilation cannot occur in a singlet neutron antiproton state. Both the singlet and the triplet states can give 3π annihilation.[17] Hence in (31) we expect the triplet term $(-49.5)/(E+37.7$ Mev) to give an estimate of all the 2π contribution, together with some 1π and 3π contribution. The energy -37.7 Mev is well within the 2π contribution (see Fig. 2). To get an accurate comparison it is at least necessary to make a good calculation of the 2π contribution in the field theory case.[18]

In (31) the singlet term $(-12.5/(E+29.7$ Mev) has to give the rest of the single pion term $(-14.6)/(E+5.15$ Mev) as well as the remainder of the 3π contribution. Since the 3π contribution can hardly be a rapidly varying function of E (for small E), the agreement here cannot be very good. However, we should remember that (a) these singlet terms are small, (b) for the reasons mentioned above (small r_S/a_S, etc.) we do not expect the singlet-state calculation to be particularly accurate.

A further source of inaccuracy should be noted. In our analysis we have separated the singlet and the triplet terms throughout. Although $\sigma_0(E)$ [Eq. (29)] is a reasonably good approximation for σ_{np} even at energies above 50 Mev (lab), the individual singlet and triplet cross section show larger relative errors. For example, a phase-shift analysis at 95 Mev[19] shows that the triplet part of σ_{np} is much smaller and the singlet part is much larger than is indicated by (29). However, the effect on our results would only be important if this type of error were appreciable for energies less than 50 Mev (lab).

V. "THE HARD CORE"

Under this heading we consider the last term in (30). There is some evidence that even at very high energies ($\sim$50 Bev lab) the total p-n cross section is about the same size as it is at a few Bev. If this is so, the integral over $\bar{\sigma}$ in (30) will not converge. This gives rise to an infinite constant on the right of (30) which should be removed by a further subtraction in the dispersion relation. As we have only considered the variation of certain terms with E (for small E) the analysis given above should not be affected by this infinity. (There may also be an infinity on the left of (30), coming from the integral of $\mathrm{Im}\alpha_{n\bar{p}}$ over the unphysical region.)

Here we use the simple dispersion relation (3) to suggest how an infinite term coming from the high-energy cross-section values can be regarded as a hard core effect. First, consider a finite potential repulsive sphere,

$$V(r)=V_0,\quad r<a$$
$$=0,\quad r>a,$$

where $V_0>0$. (We only examine the ordinary force case.) At zero energy ($k=0$) this behaves like an impenetrable sphere, so the zero-energy scattering amplitude is $f(0)=-a$. The forward scattering dispersion relation for $E=0$ is

$$-a+\tfrac{1}{3}V_0a^3=\frac{1}{2\pi^2}\int_0^\infty dk\,\sigma(k), \tag{32}$$

where $\sigma(k)$ is the total cross section.

If V_0 is large (i.e., $V_0a^2\gg 1$), in the energy range $V_0\gg k^2\gg 1/a^2$ we have

$$\sigma(k)\simeq 2\pi a^2. \tag{33}$$

For $k^2>V_0$ we use the Born approximation

$$\sigma(k)=\tfrac{1}{2}\pi(V_0a^2)^2/k^2. \tag{34}$$

Rough agreement can be obtained in (32) by using (33) for $k<k'$ and (34) for $k>k'$. Here k'^2 is the energy $k'^2=V_0^2a^2/4$ at which (33) and (34) are equal.

Letting $V_0\rightarrow\infty$, the high-energy cross section becomes $2\pi a^2$ and we see how both sides of (32) become infinite. For a nonrelativistic scattering problem Eq. (32) could thus be used to subtract out the hard-core effect.

ACKNOWLEDGMENT

I am much indebted to Professor M. L. Goldberger for several valuable discussions.

[17] See for example, H. A. Bethe and J. Hamilton, Nuovo cimento 4, 1 (1956).

[18] Professor M. L. Goldberger tells me that such a calculation is under way.

[19] R. N. J. Phillips, Proc. Phys. Soc. (London) A70, 721 (1957).