Transformations of Ising Models

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The "star-triangle" and "decoration" transformations are generalized so as to apply to arbitrary mechanical systems coupled to the spins of a standard Ising net. This leads to exact solutions for further plane Ising nets and also for lattices in which the spins on alternate sites have a magnitude greater than $S=\frac{1}{2}$. A general class of antiferromagnetic Ising models is constructed; exact closed expressions can be derived for all the thermodynamic and magnetic properties of these models in an arbitrary magnetic field.

The magnetizations and susceptibilities of Ising nets in which different spins have different magnetic moments are investigated and a valuable relation between the susceptibilities of the honeycomb and triangular lattices is derived. It is shown how correlation functions involving a given spin can be expressed in terms of correlations involving the nearest-neighbor spins instead.

1. INTRODUCTION

THE critical point (Curie temperature) of the square-lattice Ising model of a ferromagnet was originally located by Kramers and Wannier[1] who noticed a symmetry property of the partition function in the absence of a magnetic field. Onsager showed how this symmetry arose because the square net is topologically "self-dual."[2] More generally the partition function in zero magnetic field for a given net is reciprocally related to the partition function (in zero field) of its dual net; thus the partition function of the honeycomb (plane hexagonal) net is derivable from that of the triangular net by the dual transformation and vice versa.

Onsager[3] further discovered a "star-triangle" transformation whereby a "star" consisting of a central spin coupled to three neighboring spins could be transformed into a triangle of three spins coupled to each other (see Fig. 1). This leads to a further connection between the triangular and honeycomb nets which enables their critical points to be located (see Wannier, reference 2).

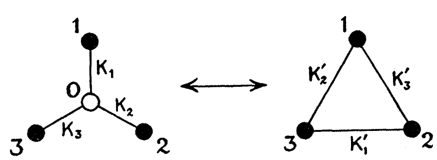

FIG. 1. The star-triangle transformation.

The star-triangle transformation is essentially algebraic and is not restricted to plane nets, whereas the dual transformation depends on a topological property of those nets that can be developed onto a two-dimensional manifold without any crossing bonds. Another algebraic transformation is the "decoration" or "iteration" transformation. This enables a central spin coupled to *two* neighboring spins to be replaced by a single bond connecting the two outer spins (see Fig. 2). This transformation has the advantage that it holds even with a magnetic field present. With its aid Naya[4] derived expressions for the spontaneous magnetizations of the honeycomb and Kagomé lattices from Potts' formula[5] for the magnetization of the triangular net. The transformation was also used by Syozi and Nakano[6] who discussed the magnetization of ferrimagnetic decorated lattices such as the decorated square net (Fig. 3) where spins on the bonds have different magnetic moments from those at the vertices of the net.

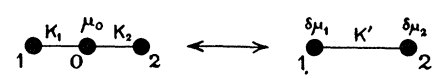

FIG. 2. The decoration or iteration transformation.

In this note we show how the two algebraic transformations of the Ising model—the star-triangle and the decoration transformation—can be considerably generalized. In fact, the central spin which is removed by these transformations may be replaced by an *arbitrary statistical mechanical system*. This system may consist of any number of "spins" and other entities which are coupled to the two, or three, outer spins and it may also depend on the magnetic field or other external variables.

These generalizations lead to exact solutions for a

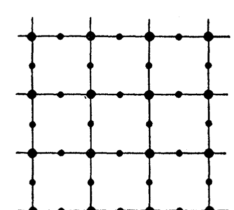

FIG. 3. The decorated square net. The spins on the bonds may have different magnetic moments from those on the vertices of the basic square net.

[1] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).
[2] G. H. Wannier, Revs. Modern Phys. **17**, 50 (1945).
[3] L. Onsager, Phys. Rev. **65**, 117 (1944).
[4] S. Naya, Progr. Theoret. Phys. Japan **11**, 53 (1954).
[5] R. B. Potts, Phys. Rev. **88**, 352 (1952).
[6] I. Syozi and H. Nakano, Progr. Theoret. Phys. Japan **13**, 69 (1955).

large number of new Ising models. Thus many further plane nets may be discussed; for example, the "expanded Kagomé lattice" which has the same coordination number as the honeycomb lattice but a different critical point. Certain models with spins greater than $\frac{1}{2}$ can also be solved; for example, the honeycomb lattice in which the spins or alternate sites have magnitude $\frac{1}{2}$ and S, where S is arbitrary [see Fig. 7(a)]. The dependence of the Curie temperature on S, the magnitude of the spin, has been determined for this and two similar lattices.

The generalized transformations also apply to finite clusters of spins. Thus, for example, the partition function of the truncated tetrahedron (four triangular and four hexagonal faces) can be derived easily from that of the tetrahedron. As shown by Domb and Sykes,[7] the study of such finite clusters is useful in obtaining exact series expansions for the susceptibilities of two- and three-dimensional Ising lattices.

By placing "complementary" spin systems on alternate bonds of simple Ising lattices one discovers a general class of *antiferromagnetic* Ising models. These models are especially interesting since they can be solved completely in the *presence of a magnetic field*, so yielding closed expressions for the susceptibility, the magnetization, etc. as functions of the field strength and the temperature.[8]

Finally we consider the transformation of expressions for the susceptibility and correlation functions of various lattices. Naya[4] and Syozi and Nakano[6] discussed the magnetization of certain loose-packed lattices (the honeycomb and the square net) in which the spins on alternate sites had *differing* magnetic moments. We show that for such lattices (including their three-dimensional counterparts like the simple cubic lattice), relations for the susceptibility can also be derived. Furthermore, if the susceptibility of the honeycomb is known, then the susceptibility of the triangular net may be derived from it. (This result is not, of course, included in the standard star-triangle relation between these lattices, since this only holds in zero magnetic field and does not apply to the magnetization or susceptibility.) At present no exact solutions for the susceptibility of any Ising lattice are known, but series expansions may be obtained by counting configurations. This is exceedingly laborious on the triangular lattice, but relatively easy on the honeycomb because of its loose-packed structure and low coordination number. Consequently the transformation theorem has proved very useful. With its aid, and using a special "cluster-expansion" theorem for the susceptibility,[9] Sykes has been able to derive and check the first twelve terms in the high-temperature expansion of the susceptibility of the triangular net. The theorem further enables one to calculate the susceptibility of the antiferromagnetic triangular lattice at temperatures below the corresponding ferromagnetic Curie point. This is valuable since the normal series for the triangular lattice diverges in this region and no "low-temperature" series can be derived because of the high degeneracy of the antiferromagnetic ground state.

The correlation functions of the triangular and honeycomb lattices are also related to one another. This is established by showing that on any lattice a correlation function involving a particular spin can be expressed as a linear function of correlations involving the nearest neighbors of the given spin instead.

[7] C. Domb and M. F. Sykes, Phil. Mag. 2, 733 (1957).

[8] Detailed discussion of the physical properties of these models is being reserved for another paper.

[9] M. F. Sykes and M. E. Fisher, Phys. Rev. Letters 1, 321 (1958).

2. BOND AND VERTEX DECORATION

We consider the generalization of the decoration or iteration process first since it is simpler than the star-triangle transformation. The Ising partition function Z_D of a decorated lattice containing bonds decorated with a single spin, as in Fig. 2, involves summations over the spin states of the decorating spins. These summations may be performed individually and before the summations over the states of the vertex spins. A summation over the two states $s_0=\pm1$ of a typical decorating spin s_0 will have the form

$$\Sigma_0=\sum_{s_0=\pm1}\exp(K_1s_0s_1+K_2s_0s_2+L_0s_0), \qquad (1)$$

where

$$K_1=J_{01}/kT,\quad K_2=J_{02}/kT,\quad L_0=\mu_0H/kT, \qquad (2)$$

J_{01} and J_{02} being the interaction energies between the spin s_0 and its neighbors s_1 and s_2, μ_0 being the magnetic moment of s_0, H being the magnetic field, and T the temperature. In the standard way we may now introduce a modified interaction parameter K' by

$$e^{4K'}=\frac{\cosh(K_1+K_2+L_0)\cosh(K_1+K_2-L_0)}{\cosh(K_1-K_2+L_0)\cosh(K_2-K_1+L_0)}, \qquad (3)$$

and increments $\delta\mu_1=kT\delta L_1$ and $\delta\mu_2=kT\delta L_2$ to the magnetic moments of spins 1 and 2 by

$$e^{4\delta L_1}=\frac{\cosh(K_1+K_2+L_0)\cosh(K_1-K_2+L_0)}{\cosh(K_1+K_2-L_0)\cosh(K_1-K_2-L_0)}, \qquad (4)$$

and

$$e^{4\delta L_2}=\frac{\cosh(K_1+K_2+L_0)\cosh(K_2-K_1+L_0)}{\cosh(K_1+K_2-L_0)\cosh(K_2-K_1-L_0)}. \qquad (5)$$

The sum Σ_0 may then be written as a simple exponential factor

$$\Sigma_0\equiv f\exp(K's_1s_2+\delta L_1s_1+\delta L_2s_2), \qquad (6)$$

where the identity holds for the $2^2=4$ possible joint spin states of spins 1 and 2 [i.e., $(s_1,s_2)=(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$]. The function f is independent of all spin variables and is defined by

$$f^4=16\cosh(K_1+K_2+L_0)\cosh(K_1+K_2-L_0)\times\cosh(K_1-K_2+L_0)\cosh(K_2-K_1+L_0). \quad (7)$$

The form of (6) corresponds to a direct single bond from spin 1 to spin 2 of energy $J_{12}=kTK'$, and implies that the magnetic moments μ_1 and μ_2 of the decorated lattice have been increased by $\delta\mu_1$ and $\delta\mu_2$, respectively. The partition function Z_D is thus reduced to a summation over the spins of the original undecorated lattice only, so that

$$Z_D(K,L)=f^{N_D}Z(K',L'), \quad (8)$$

where N_D is the number of decorated bonds and $Z(K',L')$ is the partition function of the original basic lattice but with suitably modified interactions K', and modified magnetic moments $\mu'=kTL'=\mu+\sum\delta\mu$, the sum $\sum\delta\mu$ including all the increments from the decorated bonds meeting at a single vertex. (If all the bonds are decorated symmetrically then $\delta\mu_1=\delta\mu_2=\delta\mu$ and $\sum\delta\mu=q\delta\mu$ where q is the coordination number of the basic lattice; in this case $N_D=Nq/2$, there being N sites on the basic lattice.)

To generalize the above transformation suppose that the central, decorating spin in Fig. 2 is replaced by an arbitrary mechanical system which interacts with the spins 1 and 2 and with the external magnetic field (and possibly other external influences). This system will have a set of energy levels E_i depending on the spins s_1 and s_2 and on the field H. The factor Σ_0 in the total partition function now becomes a sum over the internal energy states of the decorating system, namely

$$\Sigma_0=\sum_i\exp\{-(1/kT)E_i(s_1,s_2;H)\}. \quad (9)$$

This is essentially the partition function of the decorating system for fixed s_1 and s_2 and may be written

$$\psi(s_1,s_2;H)=\sum_i\exp\{-(1/kT)E_i(s_1,s_2;H)\}. \quad (10)$$

To express Σ_0 as a simple exponential factor as in Eq. (6) it is now only necessary to generalize the definitions of K', δL_1, δL_2, and f as follows:

$$e^{4K'}=\psi_{++}\psi_{--}/\psi_{+-}\psi_{-+}, \quad (11)$$

$$e^{4\delta L_1}=\psi_{++}\psi_{+-}/\psi_{--}\psi_{-+},\quad e^{4\delta L_2}=\psi_{++}\psi_{-+}/\psi_{--}\psi_{+-}, \quad (12)$$

and

$$f^4=\psi_{++}\psi_{--}\psi_{+-}\psi_{-+}, \quad (13)$$

where $\psi_{++}=\psi(+1,+1;H)$ and so on. With these definitions, (9) may be represented identically in the "single-bond" form (6).

Thus even when the bonds of the basic lattice are decorated with arbitrary systems the complete partition function Z_D of the resulting lattice may be derived, by (8), from the partition function Z of the original lattice. Evidently the transformation is in no way restricted to two-dimensional nets. It also holds in the presence of a magnetic field. When the decorating system is a single simple spin we have

$$\psi(s_1,s_2;H)=2\cosh(K_1s_1+K_2s_2+L_0),$$

and the general formulas (11), (12), and (13) reduce to those given previously. Other special cases will be discussed below

If the decorating system is only coupled to one spin or, in other words, if it is "hooked" onto a single vertex, the transformations simplify and may be written

$$e^{2\delta L}=\psi_+/\psi_-, \quad (14)$$

$$f^2=\psi_+\psi_-, \quad (15)$$

for each decorated vertex. No modification of the coupling energies between the basic lattice spins is necessary.

3. STAR-TRIANGLE TRANSFORMATION

To generalize the star-triangle transformation suppose that the central spin in the star of Fig. 1 represents an arbitrary mechanical system with energy states depending on the three neighboring spins. The total partition function for the resulting (star) lattice then includes the factor

$$\Sigma_0=\sum_i\exp\{-(1/kT)E_i(s_1,s_2,s_3)\}, \quad (16)$$

which we want to write in the simple form

$$\Sigma_0=f\exp(K_1's_2s_3+K_2's_3s_1+K_3's_1s_2), \quad (17)$$

corresponding to the triangle in Fig. 1. Now there are $2^3=8$ distinct joint spin states of the three neighboring spins, but there are only four independent parameters in (17), namely f, K_1', K_2', and K_3'. Consequently, unless certain restrictions are imposed, it is not possible to write (17) as an identity valid for all s_1, s_2, and s_3. At first sight it seems the situation might be improved by including the magnetic coupling terms δL_1s_1, δL_2s_2, and δL_3s_3 in the exponential of (17). This, however, yields only *three* extra parameters (δL_1, δL_2, and δL_3), whereas another *four* are required to ensure complete generality. Instead we assume that the partition function

$$\psi(s_1,s_2,s_3)=\sum_i\exp\{-(1/kT)E_i(s_1,s_2,s_3)\}, \quad (18)$$

for the central, "decorating" system is invariant under the operation of total spin inversion

$$s_1\to-s_1,\quad s_2\to-s_2,\quad s_3\to-s_3.$$

Essentially this means that there is no preferred direction for any of the spins in the decorating system. Equivalently one may suppose that the spins *in* the decorating system have no magnetic moment or that the external field does not act on them. With this restriction we have

$$\begin{aligned}\psi(+,+,+)&=\psi(-,-,-)=\psi_0,\\ \psi(-,+,+)&=\psi(+,-,-)=\psi_1,\\ \psi(+,-,+)&=\psi(-,+,-)=\psi_2,\\ \psi(+,+,-)&=\psi(-,-,+)=\psi_3.\end{aligned} \quad (19)$$

The transformed parameters may now be defined by

$$f^4=\psi_0\psi_1\psi_2\psi_3, \quad (20)$$

$$e^{4K_1'}=\psi_0\psi_1/\psi_2\psi_3, \quad e^{4K_2'}=\psi_0\psi_2/\psi_3\psi_1, \quad e^{4K_3'}=\psi_0\psi_3/\psi_1\psi_2, \quad (21)$$

and with these values (17) becomes an identity. The partition function Z_S of the "starred" lattice is then related to the partition function Z of the basic "triangle" lattice by

$$Z_S(K_j;L_S,L_T)=f^{N_S}Z(K_j';L_T), \quad (22)$$

provided

$$L_S=0, \quad (23)$$

where L_S and L_T are the magnetic parameters for the star-vertices and triangle-vertices, respectively, and N_S is the number of star-vertices.

When the decorating system is a single nonmagnetic spin we have

$$\psi(s_1,s_2,s_3)=2\cosh(K_1s_1+K_2s_2+K_3s_3),$$

which is symmetric under total spin inversion as required for the validity of the transformation. The formulas (20) and (21) then reduce to the standard forms given by Houtappel[10] in his discussion of the simple hexagonal and triangular lattices.

Although the generalized star-triangle transformation is restricted to lattices in which the decorating systems on the star-vertices have zero magnetic moments, it applies equally to finite clusters of spins and lattices in three or more dimensions as to plane nets.

4. FURTHER TRANSFORMATIONS

It is natural to enquire whether the arguments above can be extended to cover models in which each arbitrary decorating system interacts with four or more spins. That this, unfortunately, is not possible except in very special cases can be shown as follows. If the decorating system is coupled to n spins of the basic lattice then its partition function $\psi(s_1,\cdots,s_n)$ will assume 2^n independent values corresponding to the 2^n distinct joint spin states of the n lattice spins. If bonds between all pairs of the n spins are allowed, and if it is supposed that each spin can have a magnetic moment, then there are $\frac{1}{2}n(n-1)+n+1$ parameters, K_{ij}', δL_i, and f, available. These will be sufficient to represent an arbitrary decorating system only if the inequality

$$\tfrac{1}{2}n(n-1)+n+1\geqslant 2^n \quad (24)$$

is satisfied. The only solutions of this inequality are $n=1$ and $n=2$ and these correspond to the vertex and bond decoration processes discussed above.

On the other hand, if spin inversion symmetry, corresponding to zero magnetic moments, is imposed, the number of independent values of $\psi(s_1,\cdots,s_n)$ is reduced to $\frac{1}{2}2^n=2^{n-1}$. In this case, however, the parameters δL_i must be set equal to zero (to preserve the symmetry) and so the appropriate inequality is

$$\tfrac{1}{2}n(n-1)+1\geqslant 2^{n-1}. \quad (25)$$

The largest solution of this is $n=3$, corresponding to the generalized star-triangle transformation. Evidently any transformations with n greater than or equal to 4 will involve considerable further restrictions on the structure of the decorating system and are likely to be of little interest.

If the spins of the basic lattice have a magnitude S greater than $S=\frac{1}{2}$, as assumed above, the right-hand sides of the inequalities (24) and (25) become $(2S+1)^n$ and $\frac{1}{2}(2S+1)^n+\frac{1}{2}$, respectively. In no case can these inequalities be satisfied with $S\geqslant 1$ and $n\geqslant 1$.

5. ISING NETS DERIVABLE BY THE TRANSFORMATIONS

In this section we point out how the generalized transformations lead easily to exact solutions for a number of further Ising nets. As shown by Naya[4] the complete ($H\neq 0$) partition function of the Kagomé lattice may be derived from that of the honeycomb. All the bonds of the honeycomb lattice are decorated with magnetic spins leaving nonmagnetic spins on the original vertices [see Fig. 4(a)]. The nonmagnetic star-vertices are then transformed into triangles to yield the (fully magnetic) Kagomé lattice [see Fig. 4(a)]. The main interest in this lattice lies in the fact that it has the same coordination number, $q=4$, as the square net but possesses a different topological structure. This leads to a slightly different critical point and also modifies the other thermodynamic and magnetic properties.

A very similar lattice to the Kagomé lattice may be derived by decorating each bond of the honeycomb with a chain of *two* magnetic spins [see Fig. 4(b)]. The

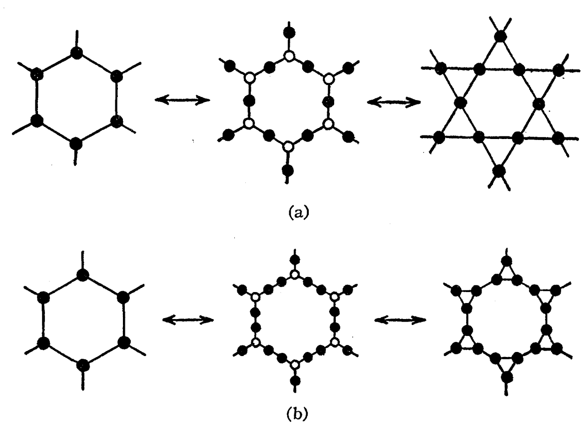

FIG. 4. (a) Transformation of the honeycomb lattice into the Kagomé lattice by means of the decoration and star-triangle processes. (b) Transformation of the honeycomb into the "expanded Kagomé lattice." The bonds of the honeycomb net are decorated with two spins in a row.

[10] R. M. F. Houtappel, Physica **16**, 425 (1950).

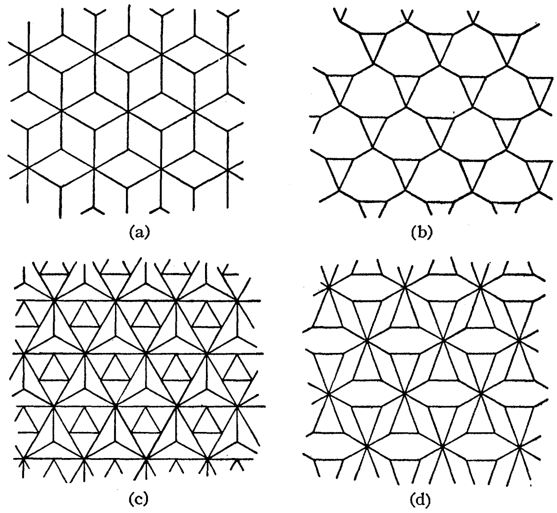

FIG. 5. Some Ising lattices of mixed coordination number: (a) the "diced" lattice with $q=3$ and 6, (b) a lattice of heptagons and triangles with $q=3$ and 4, (c) a lattice of triangles and quadrilaterals with $q=3$, 4, and 9, (d) a lattice of hexagons, quadrilaterals, and triangles with $q=3$ and 8.

appropriate transformations are derived from

$$\psi(s_1,s_2;H)=2e^K\cosh(K_1s_1+K_2s_2+2L)+2e^{-K}\cosh(K_1s_1-K_2s_2).$$

Application of the star-triangle transformation then leads to the "expanded Kagomé lattice" [shown on the right of Fig. 4(b)] which has the same coordination number as the honeycomb ($q=3$) but a different topological structure. The critical point of this new lattice is given by

$$\coth(J/kT_c)=1.67548 \quad \text{(expanded Kagomé, } q=3\text{)},$$

and the corresponding Curie temperature is lower than that of the honeycomb for which

$$\coth(J/kT_c)=3^{\frac{1}{2}}=1.73205 \quad \text{(honeycomb, } q=3\text{)}.$$

The relationship between the square and Kagomé lattices is similar; the corresponding results are

$$\coth(J/kT_c)=2.29663 \quad \text{(Kagomé, } q=4\text{)},$$

and

$$=1+2^{\frac{1}{2}}=2.41421 \quad \text{(square, } q=4\text{)}.$$

The dual lattice to the Kagomé lattice is the "diced lattice" [see Fig. 5(a)]. This is a lattice of mixed coordination number, $q=3$ and $q=6$, which may be derived from the triangular lattice by converting *all* the triangles into stars. Some other lattices with mixed coordination number which can be derived in a similar way are shown in Figs. 5(b), (c), and (d). In all these cases the partition functions derived from the triangular lattice apply only to zero field.

The partition functions and corresponding thermodynamic and magnetic properties of finite clusters of spins can always be written down in closed form. For all but the smallest clusters, however, this entails considerable labor. On the other hand, the thermodynamic and magnetic properties of a simple cluster such as the tetrahedron (four spins linked to one another) can easily be transformed to yield the corresponding properties of more complex clusters. Thus by the same sequence of transformations illustrated in Figs. 4(a) and 4(b), the properties of the octahedron and truncated tetrahedron may be derived from those of the tetrahedron, and those of the cuboctahedron and the truncated cube from those of the cube. The zero-field partition function of the cube may, in turn, be derived by applying the star-triangle transformation to all faces of the tetrahedron. The zero-field partition function of the rhombic dodecahedron can be derived from the octahedron in a similar way. These transformations are illustrated in Fig. 6.[11] As pointed out in the introduction, the study of such finite clusters is useful in calculating and checking the various "lattice constants" that arise in the derivation of series expansions for three-dimensional lattices, etc.[7]

The structures of possible decorating systems are not restricted to nearest-neighbor interactions or to topologies that that can be projected onto a plane without crossing bonds. Similarly the decorating systems need not contain only a finite number of spins. Thus, infinite "chains," "ladders," or "triangular tubes" may be attached to vertices, bonds, or triangles of a basic lattice. For the most part, however, these

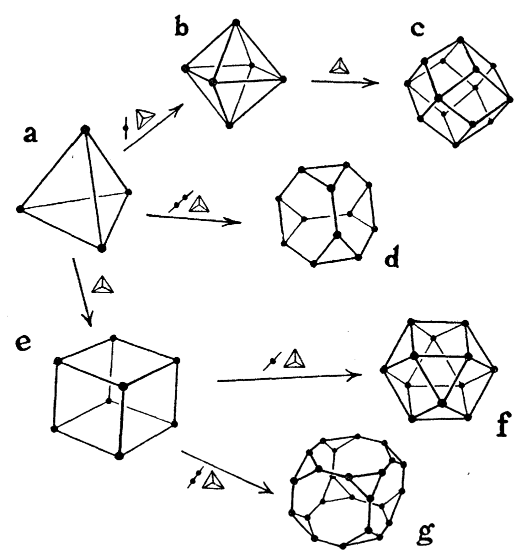

FIG. 6. Some finite clusters which can be derived from the tetrahedron by the decoration and star-triangle transformations: (a) tetrahedron, (b) octahedron, (c) rhombic dodecahedron, (d) truncated tetrahedron, (e) cube, (f) cuboctahedron, (g) truncated cube.

[11] *Note added in proof.*—I. Syozi has noted some of these transformations and describes further applications of the simple decoration process in *Statistics of Two-Dimensional Lattices II*, Rev. Kobe Univ. Merchantile Marine (Japan) 21 (1955).

extra possibilities seem to be of relatively little physical interest.

6. MODELS WITH SPIN GREATER THAN $\frac{1}{2}$

In the standard Ising model the magnitude of all spins is taken to be $S=\frac{1}{2}$ and the variables s_i can only assume the $2S+1=2$ values, -1 or $+1$. By using the generalized transformations, however, we may replace a fraction of the spins on certain lattices by "anomalous" spins of arbitrary magnitude. Thus for a spin of magnitude S interacting symmetrically with three standard spins of magnitude $\frac{1}{2}$, the transformations are derived from

$$\psi(s_1,s_2,s_3)=\sum_{s=-S}^{S}\exp\left\{K\frac{s}{S}(s_1+s_2+s_3)\right\}$$

$$=\sinh\left[\frac{2S+1}{2S}K(s_1+s_2+s_3)\right]\Big/\sinh\left[\frac{1}{2S}K(s_1+s_2+s_3)\right], \quad (26)$$

where the interaction energy $J=kTK$ has been normalized so as to equal one-half the maximum change in coupling energy between the "anomalous" spin and one standard spin. The energy parameter of the corresponding triangle formed from the three standard spins is given by

$$e^{4K'}=\{1+2\cosh[(2S+1)K/S]\}\Big/[1+2\cosh(K/S)]. \quad (27)$$

Using this formula (and the corresponding expression for f) we may derive the exact partition function (in zero field) of the lattice shown in Fig. 7(a). This is a honeycomb lattice in which spins on alternate sites have magnitude $\frac{1}{2}$ and a magnitude S which is arbitrary. It is derived from the standard triangular lattice by transforming alternate triangles. The same transformation applied to all the triangles yields the "alternate diced lattice" shown in Fig. 7(b). This has "anomalous" spins on all vertices of coordination number 3. Finally, from the Kagomé lattice we can derive a honeycomb lattice of "anomalous" spins in which all the bonds are decorated with standard ($S=\frac{1}{2}$) spins. This is shown in Fig. 7(c). The fractions of "anomalous" spins on these three lattices are:

$$N_s/N=\tfrac{1}{2}\ \text{(alternate honeycomb)}$$
$$=\tfrac{2}{3}\ \text{(alternate diced lattice)}$$
$$=\tfrac{2}{5}\ \text{(decorated honeycomb)}.$$

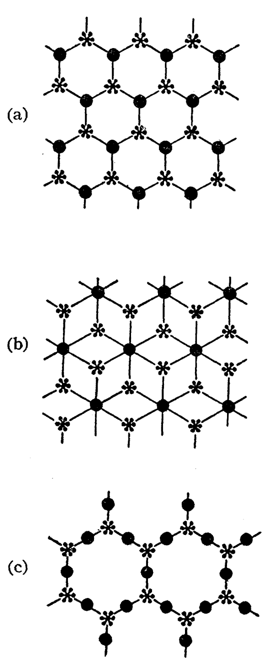

FIG. 7. Some soluble Ising lattices in which a fraction of the spins have an arbitrary magnitude S: (a) alternate honeycomb, (b) alternate diced lattice, (c) decorated honeycomb. The standard ($S=\frac{1}{2}$) spins are indicated by solid dots, while the "anomalous" (S arbitrary) spins are indicated by stars.

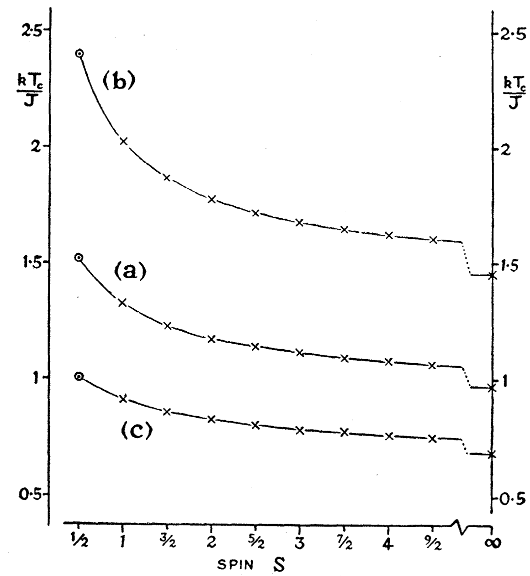

FIG. 8. The critical temperatures of the lattices of Fig. 7 as a function of S, the magnitude of the "anomalous" spins.

The critical temperatures of the three lattices may be calculated as functions of the spin S by solving (27) for K and using the known critical values of K' for the triangular and Kagomé lattices. The results are plotted in Fig. 8. As the magnitudes of the anomalous spins are increased from $S=\frac{1}{2}$ to $S=1$, the critical temperatures fall by 10 to 15%. As S increases further and tends to infinity the critical temperatures approach limiting values $T_c(\infty)$ which are about two thirds the values at $S=\frac{1}{2}$. The quantitative behavior is represented by

$$T_c(S)=[(2S+1)/2S]\{1-O(S^{-2})\}T_c(\infty). \quad (28)$$

The partition functions of the three lattices can be found readily even if direct bonds are introduced between all the standard spins (indicated by solid dots in Fig. 7). It is not possible, however, to introduce direct interactions between the anomalous spins or to eliminate all the standard spins from the networks.

It may be remarked at this point that the additive term $\ln f$ which appears in the expression derived from (8) or (22), for the free energy of a decorated lattice, is essentially the free energy of a decorated system averaged over all the possible states of the basic lattice spins to which it is coupled. Consequently this term can give rise to critical behavior only if the individual decorating systems exhibit critical behavior.

7. MAGNETIC SYMMETRIES OF THE BOND DECORATION PROCESS

The star-triangle process cannot be used to transform the magnetic properties of lattices (in particular, the spontaneous magnetization and the susceptibility) unless the star-vertices have zero magnetic moment as was the case in the transformation from honeycomb to Kagomé described above. The bond (and vertex) decoration process, on the other hand, may always be used. Furthermore this transformation has certain general symmetry properties with respect to the magnetic field which simplify its application and suggest new possibilities. These points will now be investigated.

Consider, for definiteness, a decorating system consisting of a finite number of spins s_i, of varying magnitudes S_i and magnetic moments μ_i, which interact with one another and with the two standard spins s_A and s_B at the ends of the decorated bond. The partition function of the system will be of the form

$$\psi(s_{\rm A},s_B;H)=\sum_{s_i=-S_i}^{S_i}\exp\left\{\sum_{i,j}K_{ij}s_is_j+s_{\rm A}\sum_i K_{i\rm A}s_i +s_B\sum_j K_{jB}s_j+L\sum_i \frac{\mu_i}{\mu}s_i\right\},\quad(29)$$

where the leading summation is over all possible spin states of the "internal" spins s_i. (μ is a merely a standard or reference magnetic moment.) Since the s_i are dummy variables which run symmetrically from $-S_i$ to $+S_i$ (by integral steps), the function $\psi(s_A,s_B;H)$ is always invariant under the operation of inversion of the internal spins:

$$s_i\to -s_i\quad\text{(internal spin inversion)}.$$

Now suppose that the magnetic field H acting on the decorated lattice is zero. In this case the argument of the exponential in (29) is invariant under the combined operations of internal spin inversion and external spin inversion:

$$s_A\to -s_A\quad\text{and}\quad s_B\to -s_B\ \text{(external spin inversion)}.$$

It follows that

$$\psi(s_A,s_B;0)=\psi(-s_A,-s_B;0),\quad(30)$$

and so

$$\psi_{++}(0)=\psi_{--}(0)\quad\text{and}\quad\psi_{+-}(0)=\psi_{-+}(0),\quad(31)$$

whence, by the transformation Eqs. (12),

$$\delta L_A(0)=\delta L_B(0)=0\quad(\text{all } K_{ij}\text{ and } T).\quad(32)$$

This means that the magnetic parameter L' for the undecorated (transformed) lattice vanishes identically at all temperatures; consequently the derivatives $\partial L'/\partial K$, $\partial^2L'/\partial K^2$, etc., also vanish. In other words zero (decorated) magnetic field transforms to zero (undecorated) magnetic field.

When the magnetic field H is not zero, the argument of the exponential in (29) is invariant under inversion of the internal and external spins combined with reversal of the magnetic field:

$$H\to -H,\quad L\to -L\quad\text{(field reversal)}.$$

Consequently

$$\psi(-s_A,-s_B;-H)=\psi(s_A,s_B;H),\quad(33)$$

and so

$$\begin{aligned}\psi_{++}(-H)&=\psi_{--}(H),\quad \psi_{--}(-H)=\psi_{++}(H),\\ \psi_{+-}(-H)&=\psi_{-+}(H),\quad \psi_{-+}(-H)=\psi_{+-}(H),\end{aligned}\quad(34)$$

whence, by the transformation Eqs. (12), (11), and (13),

$$\delta L_A(-H)=-\delta L_A(H),\quad \delta L_B(-H)=-\delta L_B(H),\quad(35)$$

$$K'(-H)=K'(H),\quad(36)$$

and

$$f(-H)=f(H).\quad(37)$$

Thus L' is an odd function of H (or L) whose *even*-order derivatives with respect to H (or L) vanish in zero field, but K' and f are even functions of H (or L) whose *odd*-order derivatives vanish in zero field. These symmetries considerably simplify the transformation equations for the zero-field properties of the lattices. On defining the reduced energy per vertex by

$$\mathfrak{U}(K)=-U(K)/NJ,\quad(38)$$

the reduced spontaneous magnetization per vertex by

$$\mathfrak{J}(K)=I(K)/N\mu,\quad(39)$$

and the "specific susceptibility" per vertex by

$$\xi(K)=kT\chi(K)/N\mu^2,\quad(40)$$

the transformation equations between the properties of the decorated lattice (denoted by the subscript D) per vertex of the basic lattice, and the properties of the basic lattice can be written, for zero field ($L=L'=0$),

$$\mathfrak{U}_D(K)=\frac{\partial K'}{\partial K}\mathfrak{U}(K')+\tfrac{1}{2}d\frac{\partial}{\partial K}\ln f,\quad(41)$$

$$\mathfrak{J}_D(K)=\frac{\partial L'}{\partial L}\mathfrak{J}(K'),\quad(42)$$

$$\xi_D(K)=\left(\frac{\partial L'}{\partial L}\right)^2\xi(K')+\frac{\partial^2K'}{\partial L^2}\mathfrak{U}(K')+\tfrac{1}{2}d\frac{\partial^2}{\partial L^2}\ln f.\quad(43)$$

The coefficient d denotes the number of decorated bonds meeting at each vertex of the basic lattice.

For any given decorating system ψ one can envisage a "complementary" system $\psi\dagger$ in which the signs of the magnetic moments of all the internal spins have been changed:

$$\psi \to \psi\dagger \quad \text{as} \quad \mu_i \to -\mu_i, \quad \text{all } i \text{ (moment reversal).}$$

The exponential in the formula for the complementary partition function $\psi\dagger$ is then invariant under internal spin inversion combined with a change of the signs of the interactions with the two external spins s_A and s_B. Thus the complementary system may equally well be regarded as derived from the original system by the operation

$$K_{iA} \to -K_{iA}, \quad K_{iB} \to -K_{iB}, \quad \text{all } i \text{ (external interaction reversal),}$$

no change being made in the sense of the magnetic moments. Now if the sign of the magnetic parameter L in the complementary partition function is altered, $\psi\dagger(L)$ is restored to $\psi(L)$, i.e.,

$$\psi\dagger(H) = \psi(-H), \tag{44}$$

whence, by (35), (36), and (37),

$$\delta L_A\dagger \equiv -\delta L_A, \quad \delta L_B\dagger \equiv -\delta L_B, \tag{45}$$

$$K\dagger' \equiv K', \tag{46}$$

$$f\dagger \equiv f, \tag{47}$$

where the identities hold for all H and all T. Thus the complementary system transforms into a bond with the same interaction K' as that derived from the original system. The factor f is also identical. The only difference between the two cases is the reversal of the signs of the increments to the magnetic moments of the two external spins A and B.

The foregoing results suggests a general model of an antiferromagnet. Consider a decorated lattice in which the spins on the vertices of the basic lattice have zero magnetic moment, and suppose, furthermore, that alternate bonds have been decorated with an arbitrary magnetic system and its complementary system. The arrangement is to be such that at each vertex the magnetic increments δL from the first set of systems are balanced by the increments $\delta L\dagger = -\delta L$ from the complementary systems. Consequently, when the decorated lattice is transformed, the magnetic parameter L' of the resultant undecorated lattice will be *identically zero*, the identity holding for all values of the original magnetic field H and temperature T. This means that the thermodynamic and magnetic properties of the decorated lattice in an *arbitrary magnetic field* can be derived from the properties of the original lattice in *zero magnetic field*.

The spins in a given system of such a decorated lattice are coupled to those in the neighboring complementary systems via a spin of the basic lattice. By the definition of a complementary system (in terms of reversed external interactions) the resultant interactions are negative so that the spins in the complementary set of systems tend to align antiparallel to their counterparts in the original set. Thus all such models will be antiferromagnetic and will show no spontaneous magnetization although long-range order will exist below a critical temperature. One of the simplest models of this type is the decorated square net shown in Fig. 9. The spins on the vertices of this square net have zero magnetic moment (open circles in the figure). The magnitude of the interaction energy is the same for all bonds, but is positive for the vertical bonds and negative for the horizontal bonds. The energy, specific heat, magnetization, and susceptibility as functions of the magnetic field and the temperature for this model can be derived readily from Onsager's expressions for the energy and specific heat of the simple square net.[3] The model displays some features of considerable physical interest but a detailed investigation and discussion is to be published separately.

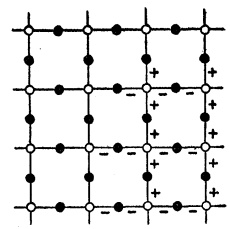

FIG. 9. A model of an antiferromagnet which is soluble in the presence of a magnetic field. Nonmagnetic spins are indicated by open circles. The interaction energy is positive for the vertical bonds but negative for the horizontal bonds.

8. MAGNETIC MOMENT TRANSFORMATIONS

Two Ising lattices with the same topological structure and the same interaction energies between their spins may differ in that corresponding spins on the two lattices have different magnetic moments. The zero-field thermodynamic properties of two such lattices will coincide but their magnetic properties will differ. In this section we study the relationship between the spontaneous magnetizations and susceptibilities of corresponding models of this type. The argument is mainly restricted to "loose-packed" lattices such as the two-dimensional square and honeycomb lattices, and the three-dimensional simple and body-centered cubic lattices. Although all the vertices of a loose-packed lattice are topologically identical, the lattice may be divided into two congruent sublattices A and B, such that all the nearest neighbors of an A vertex are B vertices and vice-versa. In other words a state of complete antiferromagnetic order is possible. It will be supposed that the spins of the A sublattice have magnetic moment μ_A while those of the B sublattice have a different moment μ_B which may be positive, negative, or zero.

A simple model of this type is the "semiferromagnetic" honeycomb lattice discussed by Naya,[4] in which alternate spins of a honeycomb lattice have zero magnetic moment, i.e., $\mu_A=\mu$, $\mu_B=0$. By employing the matrix representation of the Ising partition function for zero field and performing a perturbation calculation, Naya showed that the spontaneous magnetization per spin of the semiferromagnetic honeycomb was just half that of the normal ferromagnetic honeycomb lattice, i.e.,

$$I_{\text{Semiferro}}^{(0)}=\tfrac{1}{2}I_{\text{Ferro}}^{(0)}. \tag{48}$$

Using this result Naya was able to relate the magnetization of the normal honeycomb lattice to that of the triangular lattice.

In a similar way Syozi and Nakano[6] discussed a "ferr*i*magnetic square lattice" in which μ_A differed from μ_B and the interaction energy J was negative. By another somewhat lengthy matrix perturbation argument they showed that

$$I_{\text{Ferri}}^{(0)}=\tfrac{1}{2}(\mu_A-\mu_B)\mathcal{J}_{\text{Ferro}}^{(0)}, \tag{49}$$

where $\mathcal{J}_{\text{Ferro}}^{(0)}$ is the reduced spontaneous magnetization per spin of the normal ferromagnetic square lattice which has been given explicitly by Yang.[12]

The two results (48) and (49) are more or less obvious on intuitive grounds. They seem to be merely special cases of a general theorem which states that the spontaneous magnetization of *any* lattice of similar vertices (in two or more dimensions) which has different magnetic moments on different vertices, is given by

$$I_m^{(0)}=\bar{\mu}\mathcal{J}^{(0)}, \tag{50}$$

$$\bar{\mu}=\sum_{k=1}^{g}\gamma_k\mu_k\Big/\sum_{k=1}^{g}\gamma_k, \tag{51}$$

where the subscript m denotes the lattice with "mixed" magnetic moments μ_k $(k=1, 2, \cdots, g)$ occurring in proportions γ_k and were $\mathcal{J}^{(0)}$ is the reduced spontaneous magnetization of the corresponding standard ferromagnetic Ising lattice.

In their arguments Syozi and Nakano, and Naya used the "physical" definition of spontaneous magnetization as the limit

$$I^{(0)}=\lim_{H\to 0+}I(H), \tag{52}$$

where $I(H)$ is the thermodynamic mean magnetization in a field H. On the other hand, as shown for example in the review article by Newell and Montroll,[13] the spontaneous magnetization of a normal Ising lattice is related to the "long-range order" by

$$[\mathcal{J}^{(0)}]^2=[I^{(0)}/\mu]^2=\langle s_0s_\infty\rangle, \tag{53}$$

where

$$\langle s_0s_\infty\rangle=\lim_{h\to\infty}\langle s_0s_h\rangle,$$

$\langle s_0s_h\rangle$ being the mean (zero-field) correlation between the spins on sites 0 and h. This relation shows that the spontaneous magnetization is not really a "magnetic property" of the lattice but is essentially a reflection of the thermodynamic correlations or order. In the case where the magnetic moments are not all equal the relation (53) may be generalized to

$$[I_m^{(0)}]^2=\bar{\mu}^2\langle s_0s_\infty\rangle, \tag{54}$$

where $\bar{\mu}$ is the mean magnetic moment defined in (51). This result confirms the general theorem (50). Naya's formula (48) is recaptured by putting $g=2$, $\gamma_1=\gamma_2=\frac{1}{2}$, $\mu_1=\mu$, and $\mu_2=0$, while Syozi's and Nakano's relation (49) corresponds to $g=2$, $\gamma_1=\gamma_2=\frac{1}{2}$, $\mu_1=\mu_A$, and $\mu_2=-\mu_B$. The minus sign here takes account of the antiferromagnetic interactions (see the discussion below and in the previous section).

We now try to find a theorem similar to (50) for the susceptibilities of a standard and "mixed moments" lattice. The susceptibility per spin of a standard Ising lattice is derived from the partition function

$$Z=\sum_{s_i=\pm1}\exp\{K\sum_{ij}s_is_j+L\sum_i s_i\}, \tag{55}$$

by

$$\chi(H)=\frac{\mu^2}{NkT}\frac{\partial^2}{\partial L^2}\ln Z=\frac{\mu^2}{NkT}\left\{\frac{Z_{LL}}{Z}-\left(\frac{Z_L}{Z}\right)^2\right\}, \tag{56}$$

where the subscripts L denote differentiation with respect to L. The squared term $(Z_L/Z)^2$ is essentially the square of the magnetization. In the limit of zero field it must be replaced by the long-range order $\langle s_0s_\infty\rangle$. On performing the differentiations and letting H and L tend to zero, we obtain

$$NkT\chi=\textstyle\sum_h\sum_k\mu^2(\langle s_hs_k\rangle-\langle s_0s_\infty\rangle), \tag{57}$$

which expresses the susceptibility as a sum of all possible pair correlations $\langle s_hs_k\rangle$. When the magnetic moments vary from spin to spin, the formulas are easily generalized and yield

$$NkT\chi_m=\textstyle\sum_h\sum_k\mu_h\mu_k(\langle s_hs_k\rangle-\langle s_0s_\infty\rangle). \tag{58}$$

This expresses the susceptibility of a mixed lattice as a sum over the same pair correlations as in (57) but with the terms weighted by the factors $\mu_h\mu_k$.

For the case (denoted by a subscript l) of a loose-packed lattice with moments μ_A and μ_B on the two congruent sublattices, (58) becomes

$$NkT\chi_l=\tfrac{1}{2}(\mu_A^2+\mu_B^2)\Sigma_{AA}+6_A6_B\Sigma_{AB}, \tag{59}$$

where the sum Σ_{AA} is over all the correlation functions for which both spins are on the same sublattice, while Σ_{AB} consists of all those terms for which the two spins are on different sublattices. Now for the standard ferromagnetic lattice with interaction parameter K we have $\mu_A=\mu_B=\mu$ and so the total specific susceptibility

[12] C. N. Yang, Phys. Rev. **85**, 808 (1952).
[13] G. F. Newell and E. W. Montroll, Revs. Modern Phys. **25**, 378 (1953).

may be written

$$\xi(K)=(NkT/\mu^2)\chi(K)=\Sigma_{AA}+\Sigma_{AB}. \qquad (60)$$

On the other hand, the corresponding antiferromagnetic lattice with interaction parameter $-K$ is quite equivalent to the mixed lattice with $\mu_A=\mu=-\mu_B$ and interaction parameter $+K$. This equivalence follows from the invariance of the partition function (55) under change of sign of any of the dummy variables s_i. A reversal of the magnetic moments on one sublattice and a change in the sign of the interaction thus leaves the partition function unaltered since the changes can be exactly compensated by altering the signs of all the s_i on the one sublattice. By (59) the total specific susceptibility of the standard antiferromagnetic lattice is thus

$$\xi_{\text{Anti}}(K)=\xi(-K)=(NkT/\mu^2)\chi(-K) =\Sigma_{AA}-\Sigma_{AB}. \qquad (61)$$

Solving this equation with (60) for Σ_{AA} and Σ_{AB} and substituting in (59) shows that the susceptibility of a general "loose-packed mixed" lattice is related to the susceptibility of the corresponding standard lattice by

$$\chi_l=\tfrac{1}{2}(\mu_A{}^2+\mu_B{}^2)\chi_+ +\mu_A\mu_B\chi_-, \qquad (62)$$

where

$$\chi_+=(1/2\mu^2)[\chi(K)+\chi(-K)],$$

and $\qquad (63)$

$$\chi_-=(1/2\mu^2)[\chi(K)-\chi(-K)].$$

In the particular case of a semiferromagnetic lattice, for which one sublattice has no magnetic moments, (62) reduces to

$$\chi_S=\tfrac{1}{2}\mu^2\chi_+=\tfrac{1}{4}[\chi(K)+\chi(-K)]. \qquad (64)$$

These results may be seen in another way by writing the partition function in terms of the counting variable

$$v=\tanh K. \qquad (65)$$

In zero magnetic field

$$Z=(\cosh K)^{\frac{1}{2}Nq}\sum_{s_i=\pm1}\prod_{ij}(1+s_is_jv)$$

$$=(\cosh K)^{\frac{1}{2}Nq}\sum_{r=0}^{\infty}n(r)v^r, \qquad (66)$$

where $n(r)$ is the number of closed graphs of r bonds that can be formed on the lattice $[n(0)=1]$. No bond must be used more than once and the graphs must have no "odd vertices," i.e., an even number of bonds must meet at each point. For a loose-packed lattice it follows that the total number of bonds in any closed graph is even so that (66) is an expansion in even powers of v only. The first derivative Z_L has no expansion in v since the zero-field magnetization vanishes identically at high temperatures and v is a "high-temperature variable" which tends to zero as T becomes infinite.

For the second derivative we have

$$Z_{LL}=(\cosh K)^{\frac{1}{2}Nq}\sum_{s_i=\pm1}\sum_h\sum_k s_hs_k\prod_{ij}(1+s_is_jv)$$

$$=(\cosh K)^{\frac{1}{2}Nq}\sum_{r=0}^{\infty}g(r)v^r, \qquad (67)$$

where $g(r)$ is the number of graphs of r bonds which have *two odd vertices*. (The degenerate cases in which these odd vertices coincide, and cancel one another, must be included.) For a loose-packed lattice all those graphs in which the two odd vertices (corresponding to s_h and s_k) are on the same sublattice will have an even number of bonds. Conversely in all graphs with an odd number of bonds the two odd vertices will be on different sublattices. Thus the even powers of v in the expansion (67) correspond to "AA" correlations whilst the odd powers correspond to "AB" correlations. Since the expansion of Z has only even powers, the same decomposition holds for $Z_{LL}/Z=NkT\chi/\mu^2$. Consequently, if the susceptibility of a standard lattice is expanded in powers of v,

$$\frac{NkT}{\mu^2}\chi=\sum_{r=0}^{\infty}x(r)v^r, \qquad (68)$$

and compared with (60), we see at once that the even powers correspond to Σ_{AA} and the odd powers to Σ_{AB}. The general theorem (62) may thus be written

$$\chi_l=\frac{1}{NkT}\{\tfrac{1}{2}(\mu_A{}^2+\mu_B{}^2)\sum_{t=0}^{\infty}x(2t)v^{2t} +\mu_A\mu_B\sum_{t=0}^{\infty}x(2t+1)v^{2t+1}\}, \qquad (69)$$

which shows explicitly that if m terms of the standard susceptibility expansion (68) are known, say by counting configurations, then the same number of terms in the expansion of χ_l for a mixed lattice can be written down at once. In particular the series for the corresponding semiferromagnetic lattice consists of just the even terms of the standard expansion. It is worth noting that the theorem holds for three-dimensional lattices and finite clusters as well as for the usual plane lattices.

9. SUSCEPTIBILITIES OF THE HONEYCOMB AND TRIANGULAR LATTICES

The susceptibilities of the honeycomb and triangular lattices will now be related with the aid of the results of the previous section. We follow the method used by Naya[4] in deducing the spontaneous magnetization of the honeycomb from that of the triangular lattice. Firstly we notice that the star-triangle transformation may be used to connect the triangular lattice (subscript T) in the presence of a magnetic field with the *semi-*

ferromagnetic honeycomb lattice (*SH*) in the presence of a field. (The transformation is applied only to the nonmagnetic honeycomb vertices.) Thus

$$Z_{SH}(K,L)=2^{\frac{1}{2}N}(\cosh 3K \cosh^3 K)^{N/8} Z_T(K',L), \quad (70a)$$

where

$$e^{4K'}+1=2\cosh 2K, \quad (70b)$$

and where the honeycomb lattice has N spins and the triangular $\frac{1}{2}N$ spins. This leads to

$$\chi_{SH}(K)=\tfrac{1}{2}\chi_T(K'), \quad (71)$$

where $\chi_{SH}(K)$ and $\chi_T(K')$ are the zero-field susceptibilities *per spin* of the semiferromagnetic honeycomb and triangular lattices, respectively. We now use the theorems of the previous section to relate χ_{SH} to the susceptibility χ_H of the standard ferromagnetic honeycomb lattice (H). Since only series expansions for the susceptibilities are available at present, it is convenient to express the final results in terms of the two counting variables

$$v=\tanh K \quad \text{and} \quad w=\tanh K'. \quad (72)$$

We find that the susceptibility of the triangular lattice is given in terms of the honeycomb susceptibility by

$$\chi_T(w)=\tfrac{1}{2}[\chi_H(v)+\chi_H(-v)]=\frac{\mu^2}{NkT}\sum_{t=0}^{\infty} x_H(2t)v^{2t}, \quad (73)$$

where, in virtue of (70b),

$$\begin{aligned} v^2&=w(1+w)/(1+w^3)\\ &=w+w^2-w^4-w^5+w^7+w^8-w^{10}-w^{11}+\cdots, \end{aligned} \quad (74)$$

or

$$\begin{aligned} w&=\frac{1+v^2}{2v^2}\left\{1-\left[1-\frac{4v^4}{(1+v^2)^2}\right]^{\frac{1}{2}}\right\}\\ &=v^2-v^4+2v^6-4v^8+9v^{10}-21v^{12}\\ &\qquad +51v^{14}-127v^{16}+\cdots. \end{aligned} \quad (75)$$

Evidently the first $2m$ terms of the honeycomb expansion, or the first m even terms, are needed to calculate the first m terms (odd and even) of the triangular series. The relation between the susceptibilities of the two lattices is not symmetric since from the triangular expansion it is only possible to calculate half the terms in the honeycomb series. In contrast, from the first n terms of the susceptibility series for the Kagomé lattice one can calculate the first n terms (odd and even) of the honeycomb expansion and vice-versa.

As explained in the Introduction, the relation (73) is very useful in the actual calculation of the susceptibility series for the triangular lattice. The counting problems on the honeycomb are much easier than on the triangular lattice and twenty-four terms of the expansion for χ_H have been obtained. In this way the first twelve terms of the triangular susceptibility series have been derived.[14]

High-temperature expansions for the triangular lattice with *antiferromagnetic* interactions ($w<0$) only converge for temperatures above the critical temperature of the corresponding *ferromagnetic* lattice even though the energy, susceptibility, etc. of the antiferromagnetic lattice are analytic at all temperatures. The high-temperature series will not, therefore, yield numerical values for the susceptibility of the antiferromagnetic triangular lattice at low temperatures. Now, through the transformations (74) and (75), the antiferromagnetic triangular lattice corresponds to the honeycomb lattice with an imaginary energy parameter (negative v^2). The corresponding honeycomb series, however, converges for the whole temperature range of the antiferromagnetic triangular lattice since for $-1<w<0$ the modulus of v is always less than v_{critical}. Consequently it is possible to obtain numerical values of the susceptibility of the antiferromagnetic triangular lattice at temperatures below the ferromagnetic Curie point, by summing the terms of the corresponding honeycomb expansion and using the theorem (73) directly. This result is especially valuable since, owing to the high degeneracy of the ground state of the antiferromagnetic triangular lattice, it is not possible to obtain low-temperature expansions for this lattice.

If the susceptibility of the honeycomb (or series for it) were known as a function of the three interaction parameters K_1, K_2, and K_3, corresponding to the three lattice directions, then the use of the asymmetric star-triangle transformation would yield the triangular susceptibility (or series for it) as a function of the three parameters K_1', K_2', and K_3'. In this case the susceptibility of the square net may also be obtained by setting $K_3'=0$ or by letting K_3 become infinite. The asymmetric honeycomb lattice thus occupies a central position in that all the zero-field properties of the square, triangular, and Kagomé lattices can be derived from it by direct transformation.

The arguments used above may, of course, be applied to finite clusters. Thus the susceptibility of the tetrahedron can be derived from the even part of the susceptibility of the cube by a transformation similar to (73), (74), and (75).

10. TRANSFORMATION OF SPIN CORRELATION FUNCTIONS

The odd terms in the susceptibility series for the honeycomb (or other loose-packed lattice) correspond to spin pair correlations $\langle s_h s_k \rangle$ of type "*AB*" in which the two spins concerned are on different sublattices. The star-triangle transformation to the triangular lattice necessarily removes one of these spins and this is essentially the reason why the odd terms in χ_H cannot

[14] These terms and the details of the calculation are to be published by M. F. Sykes.

be derived from the triangular susceptibility expansion. In this section, however, we show that it is always possible to express an "AB" correlation in terms of "AA" correlations only. In this way the complete expression for the honeycomb susceptibility can be derived from the properties of the triangular lattice. It transpires that *higher-order* correlation functions of the triangular lattice, such as $\langle s_1s_2s_3s_4\rangle_T$, are involved so that although the relationship is of theoretical interest it is not actually very useful for practical calculations.

The theorem to be demonstrated shows that a correlation function involving a spin s_0 can be expressed as a linear combination of correlation functions involving the q spins, $s_1, s_2, \cdots s_q$, which are nearest neighbors to s_0. The mean value of the correlation product between spins $s_0, s_g, s_h, \cdots s_m$ on a net with general interactions K_{ij} is

$$\langle s_0s_gs_h\cdots s_m\rangle = \sum_{s_i=\pm1} s_0s_gs_h\cdots s_m \exp\{\sum_{ij} K_{ij}s_is_j\}/Z, \quad (76)$$

where Z is the partition function for the net. Consider the summation over $s_0=\pm1$ in the numerator of (76),

$$\Sigma_0(s_1,s_2,\cdots s_q) = \sum_{s_0=\pm1} s_0 \exp(K_1s_0s_1+K_2s_0s_2+\cdots+K_qs_0s_q), \quad (77)$$

which we may try to represent in the form

$$\Sigma_0(s_1,s_2,\cdots s_q) = \{\sum_{r=1}^{q} \alpha_r s_r + \sum_{t=1}^{\binom{q}{3}} \beta_t s_{a_t}s_{b_t}s_{c_t} + \sum_{u=1}^{\binom{q}{5}} \gamma_u s_{a_u}\cdots s_{e_u}+\cdots\} \times \sum_{s_0=\pm1} \exp(K_1s_0s_1+K_2s_0s_2+\cdots+K_qs_0s_q), \quad (78)$$

where the q coefficients α_r, the $\binom{q}{3}$ coefficients β_t, the $\binom{q}{5}$ coefficients γ_u, etc., are functions only of the q interaction parameters $K_1, K_2, \cdots, K_q$ between the spin s_0 and its nearest neighbors $s_1, s_2, \cdots, s_q$. In the products $s_{a_t}s_{b_t}s_{c_t}$, $s_{a_u}s_{b_u}s_{c_u}s_{d_u}s_{e_u}$, etc., no spin may appear more than once so that the longest product has q or $q-1$ factors, according as q is odd or even, and altogether there are

$$\binom{q}{1}+\binom{q}{3}+\binom{q}{5}+\cdots=2^{q-1} \quad (79)$$

distinct products with corresponding coefficients $\alpha_r, \beta_t, \gamma_u$, etc. Now there are 2^q distinct joint spin states of the nearest-neighbor spins and the identity of (77) and (78) must be enforced for all these states. The right-hand sides of (77) and (78) both change sign when the signs of all the spins $s_1, s_2, \cdots s_q$ are altered. Consequently we obtain only $\frac{1}{2}2^q=2^{q-1}$ independent equations between (77) and (78) which exactly determine the set of coefficients $\alpha_r, \beta_t, \gamma_u \cdots$ so ensuring that (78) is a faithful representation of (77). On inserting this transformation into (76), we get

$$\langle s_0s_gs_h\cdots s_m\rangle = \sum_{r=1}^{q} \alpha_r\langle s_rs_gs_h\cdots s_m\rangle + \sum_{t=1}^{\binom{q}{3}} \beta_t\langle s_{a_t}s_{b_t}s_{c_t}s_gs_h\cdots s_m\rangle+\cdots, \quad (80)$$

which expresses the correlation with s_0 as a sum of correlation functions involving the nearest-neighbor spins $s_1, s_2, \cdots s_q$ in place of s_0. If the coordination number q is greater than 2 the right-hand side of (80) will include correlation functions of higher order than the original correlation $\langle s_0s_gs_h\cdots s_m\rangle$.

In the case $q=3$, which is relevant to the honeycomb lattice, the explicit expression for the pair correlation is

$$\langle s_0s_g\rangle=\alpha_1\langle s_1s_g\rangle+\alpha_2\langle s_2s_g\rangle+\alpha_3\langle s_3s_g\rangle+\beta\langle s_1s_2s_3s_g\rangle, \quad (81)$$

where

$$\beta=\tfrac{1}{4}[\tanh(K_1+K_2+K_3)+\tanh(K_1-K_2-K_3) +\tanh(K_2-K_3-K_1)+\tanh(K_3-K_1-K_2)], \quad (82a)$$

and

$$\alpha_1=\tfrac{1}{4}[\tanh(K_1+K_2+K_3)+\tanh(K_1-K_2-K_3) -\tanh(K_2-K_3-K_1)-\tanh(K_3-K_1-K_2)], \quad (82b)$$

and α_2 and α_3 are derived by cyclic interchange of K_1, K_2, and K_3. If s_0 is a spin decorating a single bond, then $q=2$, and the formulas reduce to

$$\langle s_0s_g\rangle=\tfrac{1}{2}[\tanh(K_1+K_2)+\tanh(K_1-K_2)]\langle s_1s_g\rangle +\tfrac{1}{2}[\tanh(K_1+K_2)-\tanh(K_1-K_2)]\langle s_2s_g\rangle, \quad (83)$$

which involves only pair correlations.

The theorem (81) may now be used to change the "AB" correlations on the honeycomb lattice into "AA" correlations (the neighbors of a B-spin are all A-spins) and by the star-triangle transformation these can then be expressed as functions on the triangular lattice. In terms of the variables v and w, defined in (72), (74), and (75), we find that the complete susceptibility expression for the honeycomb can be written

$$\chi_H(v)=\left[1+\frac{3v(1+v^2)}{1+3v^2}\right]\chi_T(w)-\frac{\mu^2}{kT}\frac{2v^3}{1+3v^2}\theta_T(w), \quad (84)$$

where

$$\theta_T(w)=\sum_g \langle s_1s_2s_3s_g\rangle_T = [Z_T(w)]^{-1}\sum_{r=0}^{\infty} t(r)w^r, \quad (85)$$

and where $t(r)$ is the number of r-bond graphs on the triangular lattice with four odd vertices three of which occur at the vertices of a fixed triangle $(s_1s_2s_3)$. The function θ has no direct physical significance, although it should resemble somewhat the specific heat C and the temperature gradient of the susceptibility $\partial\chi/\partial T$, since both these functions have expansions in terms of restricted configurations with four odd vertices. The leading term in the expansion of θ_T is $w=v^2+\cdots$ so that the first two odd terms in $\chi_H(v)$ are, in fact, derivable from the triangular susceptibility alone.

It is obvious from the foregoing work that expressions similar to (84) will hold between the individual correlation functions of the triangular and honeycomb lattices. The triangular pair correlations will yield the honeycomb "AA" pair correlations, but the "AB" correlations will require the fourth-order correlation function $\langle s_1s_2s_3s_g\rangle_T$. A similar distinction will arise in the relations for the higher order honeycomb correlation functions.

11. CONCLUSIONS AND SUMMARY

We have shown how the decoration and star-triangle transformations may be generalized so as to apply to an arbitrary mechanical system coupled to two or three spins of a basic Ising net. In this way exact solutions are made available for many further plane Ising lattices, some of which have been illustrated. The transformations also ease the task of evaluating the thermodynamic and magnetic properties of some more complicated finite spin clusters. With the generalized transformations it becomes possible to introduce spins of arbitrary magnitude onto an Ising net. The critical points of three such lattices in which standard ($S=\frac{1}{2}$) and "anomalous" (S arbitrary) spins alternate have been evaluated as a function of S. For a fixed maximum interaction energy between spins, T_c falls and approaches a limiting value as S increases.

The symmetries of the decoration process with respect to the magnetic field have been investigated. The results simplify the application of the transformations and lead to the discovery of a general class of antiferromagnetic Ising models in which "complementary" spin systems interact with one another via nonmagnetic spins. All the thermodynamic and magnetic properties of these models can be evaluated exactly even in the presence of a magnetic field. (Detailed discussion of such models is postponed for a further publication.)

The work of Naya and of Syozi and Nakano on the spontaneous magnetizations of the semiferromagnetic honeycomb and the ferrimagnetic square net has been generalized for arbitrary lattices of mixed magnetic moments. Similar relations have been obtained for the susceptibilities of loose-packed lattices with different magnetic moments on the two sublattices. From these results a transformation connecting the susceptibilities of the triangular and honeycomb lattices has been derived. This is very useful in the numerical calculation of the triangular susceptibility series, since this can be deduced from that of the honeycomb which is much easier to evaluate. Finally it has been shown how a correlation function involving a particular spin can be expressed as a linear combination of correlation functions involving only the nearest neighbors of the given spin. This enables the individual correlation functions of the honeycomb and triangular lattices to be related to one another.

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