would not explain the regeneration of the short-lived θ_1^0. The only remaining explanation for the presence of θ_1^0's and Λ^0's in the chamber is the particle-mixture prediction. The observations appear consistent with this prediction. It is difficult to avoid the conclusion that a neutral K meson having essentially the properties predicted by Gell-Mann and Pais does indeed exist.

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Covariant Conservation Laws in General Relativity*

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A set of covariant conservation laws is constructed in the general theory of relativity. Their relationship to the generators of infinitesimal coordinate transformations is indicated. In a given coordinate system certain of these quantities may be naturally identified as energy and momentum. We can continue to recognize these conserved quantities in all coordinate systems due to the covariant character of the expressions.

1. INTRODUCTION

ATTEMPTS within the general theory of relativity to formulate a meaningful and unique expression for energy density have always proved to be inconclusive. The difficulties have been twofold: (a) there are many competing candidates for the energy density[1] (e.g., the Einstein canonical pseudotensor, the Landau-Lifshitz symmetric pseudotensor, and the infinity of Goldberg's expressions), and (b) none of these expressions possesses a simple transformation law. Even the total energy is no invariant. Two recent papers[2,3] have contributed to a clarification of the difficulties. In this paper we shall combine these recent advances to construct physically interesting and covariant conservation laws within the general theory of relativity.

The second section of this paper will briefly review the relevant results of the aforementioned papers of Møller and Bergmann. The third section will be devoted to the actual construction of the covariant conservation laws. The concluding section will briefly indicate the problems entailed by the identification of some of the conserved quantities with energy and momentum.

2. REVIEW OF RECENT PAPERS

Møller has shown how to remedy the most flagrant difficulty entailed by the lack of covariance of the Einstein pseudotensor, namely, the drastic alteration in the computed value for the energy density, and the total energy, caused by a mere renaming of the 3-space points by means of polar coordinates instead of by quasi-Galilean coordinates. By employing the von Freud expressions[4]

$$U_k{}^{[ij]}\equiv\frac{1}{(-g)^{\frac{1}{2}}}g_{km}[-g(g^{im}g^{jn}-g^{jm}g^{in})]_{,n}. \quad (2.1)$$

[Latin indices $=1, 2, 3, 4$; Greek indices $=1, 2, 3$; comma denotes ordinary partial differentiation; semicolon denotes covariant differentiation; $U_k{}^{[ij]}$ means $\frac{1}{2}(U_k{}^{ij}-U_k{}^{ji})$]; the "strongly" (i.e., identically) conserved Einstein pseudotensor may be written in the form

$$\frac{1}{2\kappa}\tau_k{}^i=\frac{1}{2\kappa}U_k{}^{[ij]}{}_{,j} \quad (2.2)$$

(κ is Einstein's gravitational constant). Møller observed that if we define the pseudotensor

$$\frac{1}{2\kappa}\mathcal{T}_i{}^k\equiv\frac{1}{2\kappa}(U_i{}^{[kl]}+V_i{}^{[kl]})_{,l}, \quad (2.3)$$

where

$$V_i{}^{[kl]}\equiv U_i{}^{[kl]}-\delta_i{}^kU_m{}^{[ml]}+\delta_i{}^lU_m{}^{[mk]}, \quad (2.4)$$

then this new quantity $(1/2\kappa)\mathcal{T}_i{}^k$ has the following desirable properties: (a) it is identically conserved;

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* Supported by the Air Force Office of Scientific Research.
[1] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).
[2] C. Møller, Ann. Phys. **4**, 347 (1958).
[3] P. G. Bergmann, Phys. Rev. **112**, 287 (1958).
[4] P. von Freud, Ann. Math. **40**, 417 (1939).

(b) for the total energy it yields the same value as the Einstein pseudotensor when the latter is computed in a quasi-Galilean coordinate system; (c) under coordinate transformations which do not involve the time, the energy density, $(1/2\kappa)\mathcal{T}_4{}^4$, behaves like a scalar density and the energy flux, $(1/2\kappa)\mathcal{T}_4{}^\alpha$, behaves like a vector density; (d) under linear transformations it behaves as a mixed second-order tensor. Conversely, if we demand of a pseudotensor properties (a), (b), (c), and (d), and further require that it not exceed the second order in the derivatives of the metric, then it is uniquely given by Eq. (2.3).

We recall that the Einstein pseudotensor may be thought of as the set of generators of the various infinitesimal coordinate transformations corresponding to the rigid parallel displacements of the coordinate origin.[3] We further observe that the fundamental conservation laws in physics are related to the invariance properties of the physical laws, and that general relativity is invariant under arbitrary curvilinear coordinate transformations. Consequently, Bergmann suggests[3] that we may expect a significant conserved quantity to correspond to each infinitesimal coordinate transformation.

If ξ^i is an arbitrary vector field which indicates an infinitesimal coordinate transformation, one readily finds for the general theory of relativity[3]

$$\mathcal{G}^{mn}\bar{\delta}g_{mn}+C^m{}_{,m}=0, \tag{2.5}$$

where

$$\bar{\delta}g_{mn}=-(\xi_{m;n}+\xi_{n;m}), \tag{2.6}$$

and

$$\mathcal{G}^{mn}\equiv(-g)^{\frac{1}{2}}G^{mn}=(-g)^{\frac{1}{2}}(R^{mn}-\tfrac{1}{2}g^{mn}R). \tag{2.7}$$

Thus, we see that C^i, which is defined by Eq. (2.5) up to an arbitrary curl field, is "weakly" conserved, that is, it is conserved modulo the field equations of the theory.

Combining Eqs. (2.5) and (2.6), we find that one possible expression for C^i is

$$C^i=2\xi^m\mathcal{G}_m{}^i. \tag{2.8}$$

By the addition of an appropriately chosen curl an alternative generating density can be found which is free of second derivatives, thus[3]:

$$\bar{C}^i\equiv C^i+(\xi^mU_m{}^{[in]})_{,n}=\xi^mt_m{}^i+\xi^m{}_{,n}U_m{}^{[in]} \tag{2.9}$$

where $t_m{}^i$ is the "weakly" conserved Einstein pseudotensor, and where we have used the identity

$$2\mathcal{G}_j{}^i=t_j{}^i-\tau_j{}^i. \tag{2.10}$$

By judicious choice of the vector field ξ^i, it can be shown that the infinity of conserved quantities referred to in the Introduction can each be obtained. [For example, taking ξ^i equal to a set of constants, Eq. (2.9) yields the conservation of the Einstein pseudotensor.] Thus each expression (Einstein, Landau-Lifshitz, etc.) represents a valid conservation law which is a generator of an appropriate (curvilinear) coordinate transformation.

3. COVARIANT CONSERVATION LAWS

The association of a conserved quantity with an infinitesimal coordinate transformation, in accordance with Eq. (2.9), is unique if we require that the generating density $\bar{C}^i$ be free of second-order derivatives of the metric. However, it is only necessary to require that the *generator* of a canonical transformation contain no second-order *time* derivatives. We therefore propose a different set of criteria for the establishment of a unique association between the conservation law and the infinitesimal coordinate transformations. We require, following Møller, that the resulting expression be *generally covariant*, and that for the special case of a rigid time-like translation the expression for the energy density reduce to that of Møller. We are therefore led to consider the expression

$$\bar{D}^i=(\xi^mU_m{}^{[il]}+\xi^mV_m{}^{[il]})_{,l}. \tag{3.1}$$

For the special case of $\xi=\delta_4{}^i$ this evidently reduces to $\mathcal{T}_4{}^i$; however, it is not yet generally covariant. If we express $U_m{}^{[ij]}$ and $V_m{}^{[ij]}$ by means of the metric tensor, we find that $\bar{D}^i$ may be written

$$\bar{D}^i=[2(-g)^{\frac{1}{2}}\xi^mg^{ip}g^{ln}(g_{mn,p}-g_{mp,n})]_{,l}. \tag{3.2}$$

Thus we see that if we add to $\bar{D}^i$ the curl field

$$W^i\equiv[2(-g)^{\frac{1}{2}}g^{ip}g^{ln}(\xi^m{}_{,p}g_{mn}-\xi^m{}_{,n}g_{mp})]_{,l}, \tag{3.3}$$

we obtain the identically conserved vector density

$$\begin{aligned}\mathcal{P}^i\equiv\bar{D}^i+W^i&=2[(-g)^{\frac{1}{2}}(\xi^{i;l}-\xi^{l;i})]_{,l}\\&=2(-g)^{\frac{1}{2}}(\xi^{i;l}-\xi^{l;i})_{;l}.\end{aligned} \tag{3.4}$$

We note that W^i is identically conserved, linear in $\xi^i{}_{,j}$, and vanishes identically when $\xi^i=\delta_4{}^i$. Thus $\mathcal{P}^i$ is precisely the unique covariant expression which we have been seeking.[5]

The "generalized" energy-flux vector

$$P^i(\xi)=\frac{1}{(-g)^{\frac{1}{2}}}\mathcal{P}^i=2(\xi^{i;l}-\xi^{l;i})_{;l} \tag{3.5}$$

satisfies the covariant conservation law

$$P^m{}_{;m}=0, \tag{3.6}$$

which readily assures the conservation of total "generalized" energy

$$P(\xi)\equiv\frac{1}{2\kappa}\int P^mdS_m=\frac{1}{2\kappa}\int\mathcal{P}^4d^3x. \tag{3.7}$$

As a consequence of Eq. (3.4) we find that for a spatially closed universe $P(\xi)=0$, for all (nonsingular) ξ^i. In general the integral in Eq. (3.7) may be converted into a boundary integral at spatial infinity. For a universe which is asymptotically Schwarzschild at spatial infinity

[5] Dr. R. Sachs informs me that he was led to a consideration of precisely this expression since it is the only covariant expression of lowest differential order, linear in an arbitrary vector field, which is identically conserved.

we readily find

$$P(\delta_4{}^i)=mc^2. \tag{3.8}$$

The discussion in this section has centered on the "strong" conservation law. The associated "weakly" conserved quantity is, by analogy with Eq. (2.9)

$$\mathcal{E}^i(\xi)=2\xi^m\mathcal{G}_m{}^i+\mathcal{P}^i(\xi). \tag{3.9}$$

In view of the fact that $\mathcal{E}^i(\xi)$ is the generating density for the infinitesimal coordinate transformations indicated by ξ^i, it is necessary to confirm that the generator of the infinitesimal canonical transformation

$$E(\xi)\equiv\int\mathcal{E}^4(\xi)d^3x \tag{3.10}$$

does not contain more than first-order derivatives in the time coordinate. From Eqs. (3.9) and (3.4) this is readily seen to be the case.

4. CONCLUSION

We have established a covariant conservation law associated with every infinitesimal coordinate transformation. The law is uniquely determined by the requirement that it coincide with the Møller expressions for the case of the rigid time translations. Since the rigid translations do not form an invariant subgroup of the group of general coordinate transformations, the identification of any of the quantities $P^i(\xi)$ as energy or momentum must be made separately in each coordinate system. We can continue to recognize the conserved quantity in other coordinate systems, but the identification with energy or momentum may no longer be as reasonable.

In view of the striking analogy, already noted by Møller,[2] which Eq. (3.5) has to the Maxwell equations, with ξ^i playing the role of the vector potential, we see that fields which differ from ξ^i by a gauge transformation yield precisely the same density distribution. Furthermore, vector fields which coincide near infinity necessarily yield, via Eq. (3.7), the same integral conservation laws. It may therefore become feasible to divide the vector fields into equivalence classes and to identify appropriate classes with energy and momentum. Alternatively, we have considered imposing certain natural covariant conditions on the vector fields which we choose to identify with energy and momentum. One such set of conditions might be that the density of energy flux should coincide with the momentum density. However, these investigations are still at an early stage.

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G-Conjugation and the Group-Space of the Proper Lorentz Group

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An earlier proposal of the author for the incorporation of isotopic spin into the foundations of the theory of spin ½ particles, based upon a consideration of the group-space of the proper Lorentz group, is here shown to require a certain definite conception of the essentially *spatiotemporal* character of all "internal" elementary particle phenomena such as isotopic spin. It is further shown that according to this proposal the particular spatiotemporal character of the 3-parameter group of isotopic spin rotations of a strongly interacting charge doublet is identical with that of another 3-parameter group of spinor transformations recently applied by Pauli to the neutrino. It thus follows that for leptons in general there should exist an analog of isotopic spin rotations, which must be expected to differ from the latter, however, in its physical interpretation. Through a similar generalization of the notion of *G*-conjugation, a fundamental criterion is then shown to be available for the classification of all spin ½ particles into the two families of leptons and baryons, respectively.

1

In a recent re-examination of the foundations of the theory of weak interactions, Pauli[1] has been led to consider the spinor transformation

$$\nu'=a\nu+b\gamma_5\nu^c, \tag{1}$$

where ν is the neutrino wave function and ν^c is its charge conjugate, and where a and b are complex parameters subject to the constraint

$$|a|^2+|b|^2=1. \tag{2}$$

Gürsey[2] has subsequently noted the existence of a certain formal correspondence between Pauli's spinor transformation (1) and the group of real Euclidean rotations. One of the objects of the present note is to point out that this correspondence is not, as would

[1] W. Pauli, Nuovo cimento 6, 204 (1957).

[2] F. Gürsey, Nuovo cimento 7, 411 (1958).