a formal point that for these energies the limit $\Delta E \to 0$ should be taken after the integration.

For $\Delta E/\epsilon \sim 1$, we may expand Eq. (3) in powers of αC and obtain[3]

$$b(\Delta E) = 1 - \alpha C \ln(\epsilon/\Delta E) + \cdots.$$

Clearly then $b(\Delta E)$ can be appreciably different from unity for possible energies and energy resolution. This is in contrast to the case[4] in which $\Delta E = \epsilon$ and the lowest order contribution and all lograthmic terms drop out. For $\epsilon \neq \Delta E$, substantial parts of the infrared contribution may be included simply in $b(\Delta E)$, permitting a rapid convergence of σ_n.

It is likely that Coulomb scattering and Møller scattering in the Bev region will soon be possible, the latter requiring intersecting beams of electrons. The value of $\ln(\Delta E/E)$ in these experiments is likely to be such that the accuracy of the conventional perturbation treatment to lowest order in the radiative corrections will not be adequate. The modification described above will include the larger part of the higher order corrections (the infrared contribution), and extend the region for which higher order perturbation calculations are unnecessary. An analysis of high-energy scattering using Eq. (1) is being prepared by Arthur Shaw, to whom I am grateful for a discussion of the above remarks.

PHYSICAL REVIEW VOLUME 113, NUMBER 2 JANUARY 15, 1959

Radiative Meson-Nucleon Scattering. II*

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(Received September 11, 1958)

A discussion is given of the corrections which must be made to the results of a previous calculation of radiative meson-nucleon scattering, in which the fixed-source model was used. Curves are given which show the spectrum and angular distribution of gamma rays emitted when positive mesons of energies 130 Mev, 175 Mev, and 220 Mev are scattered by protons.

I. INTRODUCTION

In a previous paper[1] the author has discussed briefly the possibility of using the bremsstrahlung emitted in meson-nucleon scattering to obtain information about meson-nucleon interactions. It was there shown that the bremsstrahlung matrix element could be expressed in terms of the data obtainable from other experiments, provided a model was used in which only P-wave mesons interacted with a fixed nucleon, no information about such quantities as "bare" coupling constants, the cutoff, etc., being needed. This suggests that bremsstrahlung will give information about what may be considered as recoil corrections to the fixed source theory (assuming that the fixed source model is a reasonable first approximation to the correct theory). The object of the present paper is to discuss, although in a rather qualitative way, such corrections, and to give the results of some numerical calculations, with the hope these results will be of use in discussing possible experiments.

It is important to keep in mind the following characteristics of the bremsstrahlung: the intensity of gamma radiation is a very rapidly decreasing function of the gamma-ray energy, and the intensity and angular distribution of the soft gamma radiation is completely determined by the scattering cross section, so only redundant information can be obtained from this relatively intense soft radiation. Thus, in order to provide significant information about meson-nucleon interactions, the intensity of high-energy gamma radiation must be measured very accurately, and in the presence of a much more intense low-energy radiation. For these reasons, it has seemed to the writer that it would be useful, for those who might be contemplating or attempting to perform bremsstrahlung experiments, to have an estimate of the expected spectrum, as an aid in searching for the recoil corrections and any unexpected effects. We shall give results for π^+-p scattering.

In the limit $K \to 0$, the matrix element can be written down exactly in terms of the scattering matrix element, with all recoil effects of course being included automatically. We shall use the results of I as an aid in extrapolating the matrix element to finite photon energies. This extrapolation involves a certain amount of guesswork; determination of the correct extrapolation is left to experiment. The starting point of the present work is therefore the result of I—Eq. (I52)—to which the reader is referred. Our first problem will be to modify (I52) in such a way that it will be *correct* when $K \to 0$. It has recently been shown[2] that not

* This work was supported, in part, by an Alfred P. Sloan Foundation Research Fellowship.
† Alfred P. Sloan Foundation Fellow.
[1] R. E. Cutkosky, Phys. Rev. **199**, 209 (1958); hereafter referred to as I.

[2] F. E. Low, Phys. Rev. **110**, 974 (1958). From this work the limit of β as $K \to 0$ may be calculated.

only the K^{-1} term, but also the term in the cross section which is constant when $K\rightarrow 0$, may be expressed in terms of the elastic scattering matrix element. We have not availed ourselves of these results, since we are especially interested in rather large values of K.

II. CORRECTIONS TO THE STATIC-NUCLEON THEORY

The corrections which must be introduced to make the static-nucleon matrix element correspond to the result of an exact theory may be supposed to be of three types: (I) corrections to the current, (II) corrections to the meson-nucleon interaction, (III) trivial kinematical corrections, for center-of-mass motion, etc.

The principle correcton to the current involves taking into account the convective current of the nucleon, which we do only in a nonrelativistic approximation: we add $\mathbf{j}_n{}^c=e\mathbf{v}$ to the magnetic moment of the physical proton, where $\mathbf{v}$ is the velocity of the proton. The radiation from this part of the current interferes destructively with the radiation from the pion current in π^+-p scattering, reducing appreciably the amount of radiation, especially in the backward direction. We shall not consider the corrections to the contribution of the proton current which arise from the finite momentum transferred to the proton, since at the energies in which we are interested these are minor corrections to a term which is itself not large.

The interaction current which is added to the fixed-source model to make it gauge invariant is supposed to simulate the contribution of antiprotons, strange particles, etc., to the current of a proper theory. It does not represent this current exactly, except in certain nonphysical limits (as in the Kroll-Ruderman theorem). Experiments on photoproduction indicate that the contribution of the true current to positive meson emission near threshold is substantially less that is suggested by the fixed-source model. This suggests that we should include in M^1 (the contribution of the interaction current to the bremsstrahlung matrix element) a factor β, which should not be strongly energy dependent, but which should decrease the magnitude of M^1 in the case of π^+-p scattering, perhaps by as much as 20% near the resonance energy. This factor β would have a different value when negative mesons are scattered by protons. A measurement of the very-high-energy part of the bremsstrahlung would essentially be a measurement of the magnitude of this factor β.[2]

The quasi-classical part of the contribution of the meson current is given by Eq. (I25), which we rewrite here using a slightly different notation:

$$M^{(2)}=\frac{2ie\epsilon_{3\alpha\nu}\mathbf{q}\cdot\boldsymbol{\varepsilon}(\psi_0,\ U_{\mathbf{q}+\mathbf{K},\nu}\psi_{\mathbf{p}\beta}{}^{(+)})}{(4K\omega_q)^{\frac{1}{2}}(\omega_p{}^2-\omega_{\mathbf{q}+\mathbf{K}}{}^2+i\epsilon)}+\frac{2ie\epsilon_{3\mu\beta}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\psi_{\mathbf{q}\alpha}{}^{(-)},\ U_{\mathbf{K}-\mathbf{p},\mu}\psi_0)}{(4K\omega_p)^{\frac{1}{2}}(\omega_q{}^2-\omega_{\mathbf{p}-\mathbf{K}}{}^2+i\epsilon)}. \quad (1)$$

In I the symbol $V_p{}^*=(2\omega_p)^{-\frac{1}{2}}U_p$ was used. In the convariant theory, a similar term will arise, and will contain a modified meson current and modified meson propagation functions, as well as the appropriately modified scattering matrix element. We shall actually use Eq. (1) as it stands, with a minor correction for the motion of the center of mass, but it is desirable to have some idea of the qualitative differences that might arise in the covariant theory.

An insight into the modifications to $M^{(2)}$ can be obtained by ignoring all differences between bare mesons and physical mesons, but otherwise attempting to use a relativistic theory. By so doing, one merely repeats the derivation of I, except that the momentum end energy of the nucleon appear in the result for $M^{(2)}$:

$$M^{(2)}=\frac{2ie\epsilon_{3\alpha\nu}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(4K\omega_q)^{\frac{1}{2}}\Delta_q}(\psi_{(-\mathbf{q}-\mathbf{K})},\ U_{\mathbf{q}+\mathbf{K},\nu}\psi_{(-\mathbf{p})\mathbf{p}\beta}{}^{(+)})+\frac{2ie\epsilon_{3\mu\beta}\mathbf{p}\cdot\boldsymbol{\varepsilon}}{(4K\omega_p)^{\frac{1}{2}}\Delta_p}(\psi_{(-\mathbf{q}-\mathbf{K})-\mathbf{q}\alpha}{}^{(-)},\ U_{\mathbf{K}-\mathbf{p},\mu}\psi_{(-\mathbf{p})}), \quad (2)$$

where

$$\Delta_q=(\omega_p+E_p-E_{\mathbf{q}+\mathbf{K}})^2-\omega_{\mathbf{q}+\mathbf{K}}{}^2=2K\omega_q-2\mathbf{K}\cdot\mathbf{q}.$$

In (2) a notation is used in which $\psi_{(-\mathbf{q}-\mathbf{K})}$ and $\psi_{(-\mathbf{p})\mathbf{p}\beta}{}^{(+)}$ are states in which the nucleon has momenta $-\mathbf{q}-\mathbf{K}$ and $-\mathbf{p}$, respectively. The matrix elements appearing in (2) could be identified with the scattering matrix for meson-nucleon scattering, if the momenta were such that energy were conserved. However, unless $K=0$, Eq. (2) contains the scattering matrix *off* the energy shell.

An examination of the way in which the matrix element can depend on the independent covariants that can be formed from the momenta in the general case suggests that one appropriate way to discuss the extrapolation off the energy shell is as follows: We express the matrix element (in the c.m. system) in the form

$$T_{rp}=\frac{(\psi_{(-\mathbf{r})},U_{\mathbf{r}}\psi_{(-\mathbf{p})\,\mathbf{p}}{}^{(+)})}{(2\omega_rE_rE_p)^{\frac{1}{2}}}=\sum_{lJ}\frac{p^lr^lA_{lJ}(\boldsymbol{\theta})T_{lJ}(\omega_p{}^*,\Delta)}{(4\omega_p\omega_rE_pE_r)^{\frac{1}{2}}}, \quad (3)$$

where $A_{lJ}(\boldsymbol{\theta})$ is a matrix in the spin space of the nucleon and $\boldsymbol{\theta}$ denotes the scattering direction, while l and J are the orbital and total angular momenta and $\omega_p{}^*=\omega_p+E_p-M$, (the isotopic spin is suppressed from the notation). The factor r^l gives the effect of the centrifugal barrier on the virtual meson; all nontrivial features of the extrapolation off of the energy shell appear through the dependence on the variable $\Delta=(\omega_p+E_p-E_r)^2-\omega_r{}^2$. For P-wave scattering, using the static model, and interpreting the result as holding in the c.m. system of the meson and the nucleon, is equivalent to using Eq. (3) but ignoring the dependence on Δ. We shall also extrapolate the S-wave part part of the matrix by ignoring the Δ dependence.

The form of the corrections to $M^{(2)}$ suggests several remarks. First, the success of the static model suggests that for P-states, the Δ dependence will not be important. However, for S- and D-states, the variation of the matrix element with Δ may be quite strong. Second, the factors Δ_q^{-1} and Δ_p^{-1} in the two terms of $M^{(2)}$ are responsible for the characteristic features of bremsstrahlung—the large amount of soft gamma radiation, and the strong forward peaking of the radiation from relativistic particles. Therefore, the measurement of the corrections to $M^{(2)}$, which depend on Δ_q or Δ_p being large, will require looking for an isotropic, or nearly isotropic, component of the high-energy gamma radiation.

While the effects we have discussed above do not exhaust the corrections that must be made in the static model, we may expect that they are the most important effects. The term $M^{(2)}$ does not need a further correction for the effect of the physical structure of the π meson, as has been pointed out by Low.[2]

III. KINEMATICAL EFFECTS

The relation between the matrix element T_{qp} and the scattering cross section is

$$d\sigma_{\text{scat}}=\sum_{\text{spins}}(2\pi)^{-2}\gamma_p{}^2|T_{qp}|^2d\Omega_q, \tag{4}$$

where $\gamma_p{}^{-1}=1+\omega_p/M$. Therefore we redefine the quantity h_p by including the statistical factor: $h_p=(1+\omega_p/M)p^{-3}\sin\delta_p$. The scattering matrix element is then

$$T_{\mathbf{q}\alpha,\,\mathbf{p}\beta}=-4\pi(4\omega_q\omega_p)^{-\frac{1}{2}}[h_pe^{i\delta_p}(3\mathbf{p}\cdot\mathbf{q}-\mathbf{q}\cdot\boldsymbol{\sigma}\mathbf{p}\cdot\boldsymbol{\sigma})-a_3e^{i\alpha_p}](\delta_{\alpha\beta}-\tfrac{1}{3}\tau_\alpha\tau_\beta), \tag{5}$$

where we include, in addition to scattering in the $(\frac{3}{2},\frac{3}{2})$ state, only an approximation to scattering in the S-state. The phase shift in the $T=\frac{3}{2}$ S state is α_p, and a^3 is $(1+1/M)$ times the scattering length.

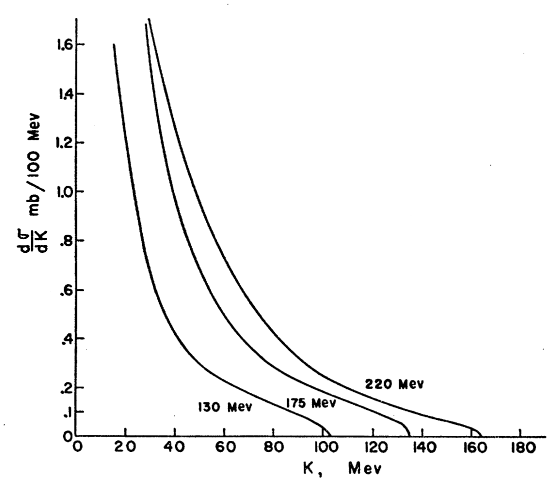

FIG. 1. Spectrum of gamma rays, in the c.m. system, emitted in the radiative scattering of positive mesons with initial laboratory energies of 130, 175, and 220 Mev by protons.

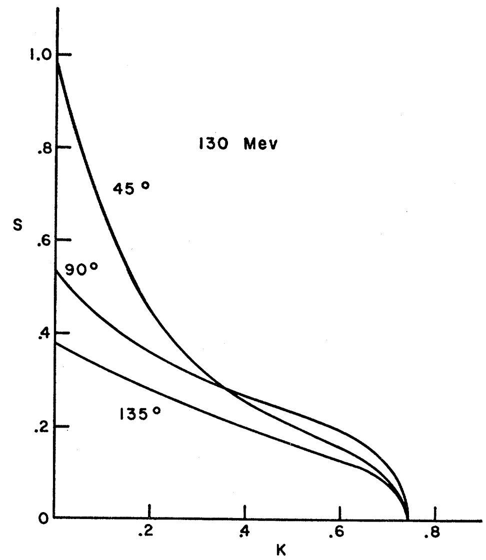

FIG. 2. The function $S(K,\theta)$ which gives the angular and spectral distribution of gamma rays in the c.m. system at the laboratory energy 130 Mev. For these curves it was assumed that $\beta=1$ and $y=0$.

The bremsstrahlung matrix element which was calculated from the static theory (I52) is interpreted as applying in the c.m. system, and the momenta $\mathbf{p}$ and $\mathbf{K}$ are taken to be the momenta of the initial meson and the photon in the c.m. system. However, we shall interpret $\mathbf{q}$ as referring to the final momentum of the meson in the system in which the center of mass of itself and the nucleon is at rest, if the last interaction of the meson is with the nucleon; otherwise, if the last interaction of the meson is the emission of the photon, we replace $\mathbf{q}$ by $\mathbf{q}'$, where $\mathbf{q}'$ is the momentum of the meson in the total c.m. system. The relation between $\mathbf{q}$ and $\mathbf{q}'$ is

$$\mathbf{q}'\approx\mathbf{q}-\frac{\omega_q}{M+\omega_q}\mathbf{K},\quad \omega_{q'}\approx\omega_q-\frac{\mathbf{q}\cdot\mathbf{K}}{M+\omega_q}, \tag{6}$$

where we have assumed that K/M is small. We shall also use the relations

$$\begin{gathered}K\omega_q-\mathbf{K}\cdot\mathbf{q}'\approx\Gamma^{-1}(K\omega_q-\mathbf{K}\cdot\mathbf{q}),\quad \Gamma=\gamma_p/\gamma_q,\\ \omega_p{}^*=K+\omega_q+\tfrac{1}{2}q^2/M+\tfrac{1}{2}K^2(M+\omega_q+\tfrac{1}{2}q^2/M)^{-1}=E,\\ dE/d\omega_q\approx\gamma_q{}^{-1},\\ d^3q'=d^3q\omega_{q'}/\omega_q\approx d^3q[1-\mathbf{q}\cdot\mathbf{K}\omega_q{}^{-1}(M+\omega_q)^{-1}].\end{gathered} \tag{7}$$

IV. RESULTS

In place of the matrix M we write

$$M=4\pi e(8K\omega_p\omega_q)^{-\frac{1}{2}}X.$$

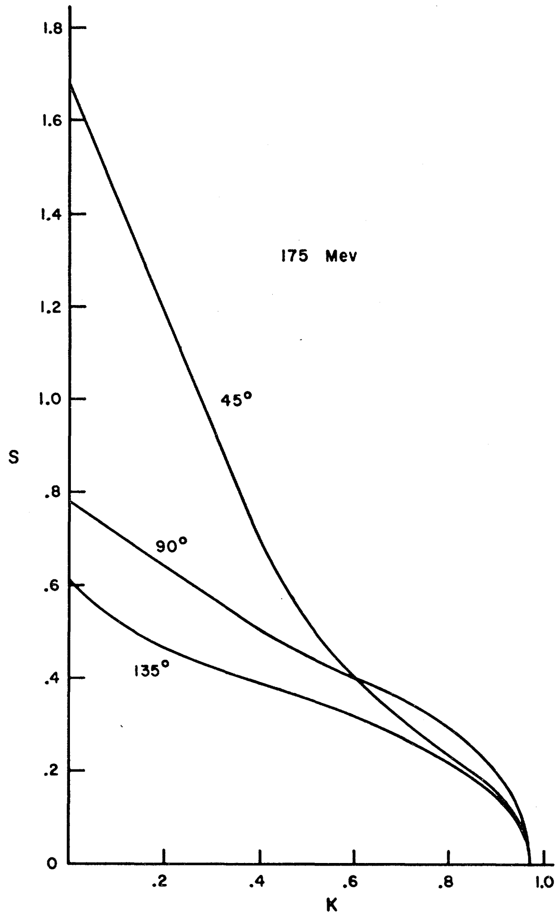

FIG. 3. Same as Fig. 2, with laboratory energy 175 Mev.

Then our final result for the cross section for radiative meson scattering is

$$d\sigma=\frac{\alpha}{4\pi^2}\frac{dK}{K}d\Omega_K d\Omega_q \gamma_p \gamma_q \frac{qK^2}{p} \times\left(1-\frac{\mathbf{q}\cdot\mathbf{K}}{\omega_q(M+\omega_q)}\right)\sum_{\text{spins}}|X|^2. \quad (8)$$

For π^+-p scattering, we use, with $X=X_1+X_2+X_3+X_4$:

$$X_1=\beta h_p e^{i\delta_p}(3\boldsymbol{\varepsilon}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\mathbf{p}) +\beta e^{i\delta_q}h_q(3\boldsymbol{\varepsilon}\cdot\mathbf{q}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}), \quad (9.1)$$

$$X_2=\Gamma\mathbf{q}\cdot\boldsymbol{\varepsilon}\left(\frac{q_i+\gamma_q K_i}{K\omega_q-\mathbf{K}\cdot\mathbf{q}}-\frac{q_i}{KM}\right)h_p e^{i\delta_p}(3p_i-\sigma_i\boldsymbol{\sigma}\cdot\mathbf{p}) -h_q e^{i\delta_q}(3q_i-\boldsymbol{\sigma}\cdot\mathbf{q}\sigma_i)\left(\frac{p_i-K_i}{K\omega_q-\mathbf{K}\cdot\mathbf{p}}-\frac{p_i}{KM}\right)\mathbf{p}\cdot\boldsymbol{\varepsilon} -\Gamma\mathbf{q}\cdot\boldsymbol{\varepsilon}[(K\omega_q-\mathbf{K}\cdot\mathbf{q})^{-1}-K^{-1}M^{-1}]a_3e^{i\alpha_p} +\mathbf{p}\cdot\boldsymbol{\varepsilon}a_3e^{i\alpha_q}[(K\omega_p-\mathbf{K}\cdot\mathbf{p})^{-1}-K^{-1}M^{-1}], \quad (9.2)$$

$$X_3=\mu_p(2MK)^{-1}\{-i\boldsymbol{\sigma}\cdot(\mathbf{K}\times\boldsymbol{\varepsilon})[3\mathbf{q}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{p}]h_p e^{i\delta_p} +[3\mathbf{q}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{p}]i\boldsymbol{\sigma}\cdot(\mathbf{K}\times\boldsymbol{\varepsilon})h_q e^{i\delta_q}, \quad (9.3)$$

$$X_4=-3_p h_q h y e^{i\delta_p+i\delta_q+i\eta}(q_i-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}\sigma_i)(\epsilon_i K_j-K_i\epsilon_j) \times(p_j-\tfrac{1}{3}\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}), -3h_p h_q z e^{i\delta_p+i\delta_q+i\zeta}(q_i-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}\sigma_i)(\epsilon_i K_j+\epsilon_j K_i) \times(p_j-\tfrac{1}{3}\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}). \quad (9.4)$$

No attempt has been made to include correctly, in X, terms which are of order M^{-2}.

We shall not write down an explicit expression for the cross section as this would take too much space. The qualitative behavior of the cross section can be inferred most easily from the numerical results.

The numerical calculations were confined to the resonance region: 130-Mev, 175-Mev, and 220-Mev laboratory energies. The experiment may be more interesting at higher energies, but the information about elastic scattering is not sufficiently detailed to permit a precise calculation. The rescattering correction also becomes much more important at higher energies.

Several unknown parameters were introduced into the matrix element (9) to represent unknown effects. For the calculations it was assumed that the electric quadrupole rescattering correction is negligible. Calculations were made with $y=3$ and $\eta=45°$, and with $y=0$ (as an illustration, to estimate the effect of the rescattering correction on the numerical results). The parameter β was given the values 1.0 and 0.8 in order to show the experimental accuracy that would be required to measure this quantity.

The spectrum of gamma rays in the c.m. system, integrated over all angles, is shown in Fig. 1. For these curves we let $\beta=1$ and $y=0$.

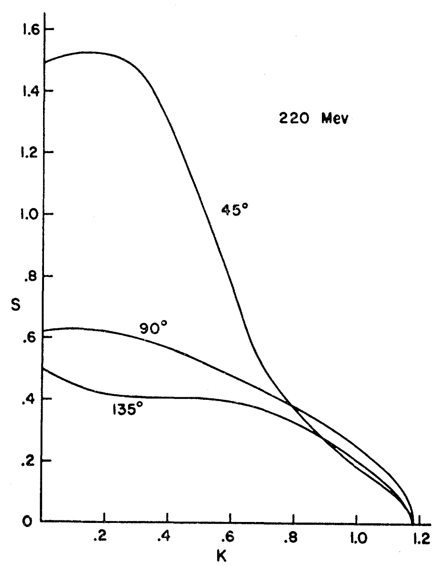

FIG. 4. Same as Fig. 2, with laboratory energy 220 Mev.

The remaining figures (Figs. 2–5) show the photon spectrum at three different angles in the c.m. systems. The quantity plotted in these curves is not the differential cross section, but a quantity S, defined so that

$$d\sigma=\alpha(\pi K)^{-1}SdKd\Omega_K.$$

The quantity S is a useful quantity to consider because it shows most conveniently the departures from the semiclassical, point-scatterer approximation. If that approximation were valid, the curves for S *vs* K would be horizontal straight lines. It is apparent that the intensity of high-energy radiation is much less than the semiclassical theory suggests. In Figs. 2–5 the unit in which the photon energy is expressed is the rest energy of the π meson, while S is given as a multiple of the square of the Compton wavelength of the π meson.

It is evident from these curves that in order to obtain significant information a very high experimental accuracy would be required. At the higher energies it is apparent that uncertainty in the rescattering correction tends to mask the more interesting effects. A measurement of the angular distribution of the outgoing meson would help to separate these effects because the rescattering correction requires the outgoing meson to be in a P-state, while the interaction current term is especially important when the outgoing meson is in an S-state. The corrections to the scattering matrix element discussed in Sec. II are likely to have an effect on the energy and angular distributions which is similar to that of the rescattering corrections, and so could not be so easily measured. It might appear that other uncertainties at the end point of the photon spectrum would appear as a result of the uncertainty in the extrapolation of the S-wave scattering matrix off the energy shell, but it is found that the contribution of S-wave scattering at this energy is only a few percent of the contribution of the interaction current.

A particularly noteworthy feature of the 220-Mev curves is the hump in the function $S(K)$ at 45°. The hump corresponds to an energy for the outgoing meson which is near the resonance energy. It will be recalled that the forward peaking is most important in the term in Eq. (9.2) which contains the scattering matrix for the final meson energy. At higher energies this hump in the $S(K)$ courve would be shifted to higher proton energies and be more pronounced.

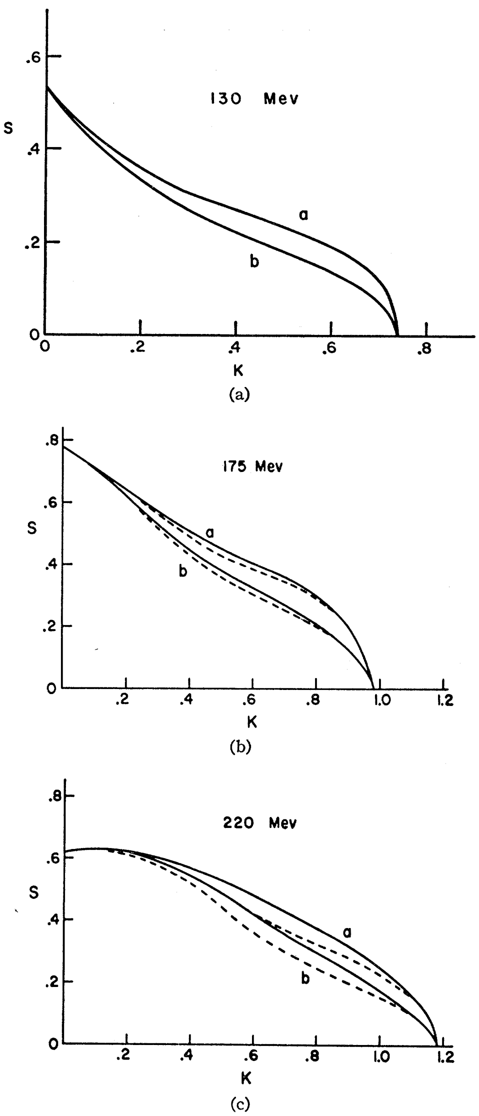

FIG. 5. Illustration of the effect of the rescattering correction and the recoil corrections to the interaction current on the spectrum of gamma rays emitted at 90° in the c.m. system. The corrections are similar at 45° and 135°. The curves marked a were calculated with $\beta=1$; for the b curves, $\beta=0.8$. The solid curves refer to $y=0$; the dashed curves to $y=3$, $\eta=45°$. The rescattering correction is negligible at 130 Mev.

ACKNOWLEDGMENTS

The author is grateful for many helpful discussions, especially with Dr. Julius Ashkin, Dr. G. Bernardini, Dr. S. DeBenedetti, Dr. A. W. Merrison, Dr. R. T. Siegel, Dr. G. C. Wick, and Dr. Lincoln Wolfenstein. Especial thanks are due Raymond Sorensen for checking many of the algebraical calculations and also performing most of the numerical ones.