evidence tends to support the supposition that I^{123} decays entirely by electron capture.

The energy of the gamma ray determined from the internal conversion spectrum and from the photoline as seen in the scintillation spectrum is 159 ± 1 kev. The $K/(L+M)$ ratio for the internal conversion line is 6.6 ± 0.1.

ACKNOWLEDGMENTS

The authors are indebted to Professor M. B. Sampson and the cyclotron group for making the bombardments, to Dr. George Bulbenko for the chemical separations, and to Professor R. G. Wilkinson for the use of his magnetic spectrometer.

PHYSICAL REVIEW VOLUME 113, NUMBER 2 JANUARY 15, 1959

Spacings of Nuclear Energy Levels

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(Received September 29, 1958)

A simple statistical model is suggested in terms of which the general features of the level spacing distribution can be understood, and which involves no special assumptions, other than that of Porter and Thomas.

INTRODUCTION

In the last few years a wealth of experimental data concerning the widths and spacings of nuclear energy levels has become available.[1] The initial theoretical response to the information these data supplied was directed toward the understanding of the statistical properties of the neutron widths[2]; and these investigations culminated in the very successful paper of Porter and Thomas.[3] These authors inferred a normal distribution for the reduced neutron width from the plausible assumption of a highly complex, rapidly varying wave function for compound nuclear states, which is not highly correlated with the wave functions of nearby states. In their paper, Porter and Thomas show their inference to be strongly supported by experimental evidence.

With the success of this simple approach in mind, Wigner suggested an analogous examination of the distribution of level spacings.[4] In particular, he pointed out that the distribution of spacings between adjacent eigenvalues of matrices whose elements were randomly chosen would show a deficiency of small spacings, contrary to the expectation if the eigenvalues themselves were uncorrelated. This so-called "level repulsion" effect has been observed experimentally in connection with the nuclear resonance levels.[5] Blumberg and Porter[6] demonstrated the level repulsion effect numerically by diagonalizing random matrices of fairly large order, all of whose elements had the same normal distribution. Rosenzweig[7] made more accurate numerical calculations of the same type and obtained a rather detailed histogram which agreed very well with the distribution of spacings originally suggested by Wigner [see Eq. (8)], except for the largest spacings.†

It is the purpose of this paper to indicate a simple statistical model in terms of which the general features of the level spacing distribution can be understood, and which involves no special assumptions, other than that of Porter and Thomas.[3]

THEORY

Nuclear resonance states are determined by the zeros of the function[8]

$$f_c(E)\equiv\int_S\left(\frac{\partial X}{\partial n}+b_cX\right)\Phi_c dS, \tag{1}$$

where E is the energy of the bombarding particle; X is a solution of the Hamiltonian equation $HX=EX$ inside the nuclear surface, S, in configuration space; n is the outward normal to S; Φ_c is a channel wave function in channel c; and b_c is a real arbitrary constant. By appropriate choice of b_c the level shift may be made

* Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

[1] J. A. Harvey and D. J. Hughes, Phys. Rev. **109**, 471 (1958), and references cited therein.

[2] J. A. Harvey and D. J. Hughes, Phys. Rev. **99**, 1032 (1955), and references cited therein.

[3] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).

[4] E. P. Wigner, Oak Ridge National Laboratory Report ORNL-2309, November 1, 1956 (unpublished), p. 67.

[5] J. A. Harvey, Phys. Rev. **98**, 1162 (1955); I. I. Gurevich and M. I. Pevsner, J. Exptl. Theoret. Phys. U.S.S.R. **31**, 162 (1956) [translation: Soviet Phys. JETP **4**, 278 (1957)]; also Nuclear Phys. **2**, 575 (1957).

[6] S. Blumberg and C. E. Porter, Phys. Rev. **110**, 786 (1958).

[7] N. Rosenzweig, Phys. Rev. Letters **1**, 24 (1958).

† *Note added in proof.*—Professor E. P. Wigner kindly pointed out to the author that this slight disagreement arose from a failure to account for the dependence of the local average level spacing on the eigenvalue (energy). When Rosenzweig's results were corrected for this effect the disagreement for large spacings was removed.

[8] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Cambridge, 1955), p. 291.

to vanish locally, and the roots of $f_c(E)$ are the real resonance energies.

According to the argument of Porter and Thomas, it is the integral

$$\int X_\lambda \Phi_c dS \propto \gamma_{\lambda c} \quad (2)$$

evaluated at a root E_λ of f_c, which is normally distributed. By a trivial extension of their argument one might infer that $f_c(E)$ is a random function distributed normally at every energy. That is to say, the integral in Eq. (1) can be decomposed into contributions from many "cells" of the nuclear surface S. Each cell has an effective area of about one neutron wavelength squared. In view of the assumption of Porter and Thomas mentioned in the first paragraph, the contribution from each cell is randomly distributed and independent of that from other cells. Using the central limit theorem, one then infers that $f_c(E)$ is normally distributed at every energy.

Our problem thus is reduced to finding the distribution of intervals between successive zeros of a random function. This latter problem, while formidable, has a considerable literature, having been studied in oceanographic[9] and communication[10] applications. It proves convenient in these applications to define a random function $f(X)$, which is normally distributed at every X, by the trigonometric sum

$$f(X)=\sum_{n=1}^{N} C_n \cos(\omega_n X-\phi_n), \quad (3)$$

where ϕ_n is a random phase uniformly distributed in the interval $(0,2\pi)$. It furthermore proves convenient to define the correlation function

$$\psi(\tau)=\langle f(X)f(X+\tau)\rangle_{\text{corr}}=\tfrac{1}{2}\sum_{n=1}^{N} C_n^2 \cos(\omega_n\tau), \quad (4)$$

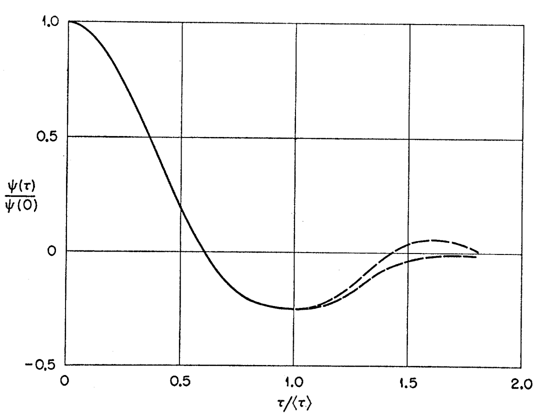

FIG. 1. The correlation function for resonance levels.

[9] M. S. Longuet-Higgins, Proc. Roy. Soc. (London) **A246**, 99 (1958).

[10] S. O. Rice, Bell System Tech. J. **23**, 282 (1944), and **24**, 46 (1945).

where the notation $\langle\ \rangle_{\text{corr}}$ denotes either an average over all X, or an average with respect to the phases ϕ_n. The finiteness of the sums is unimportant since in all applications of Eq. (3) it is possible to let $N\rightarrow\infty$ at an appropriate point in the analysis whence sums are replaced by integrals, etc.

For random functions of the type described, Rice[10] has proved the following properties for $p(\tau)$, the probability of finding an interval τ between two successive roots:

$$p(0)=0, \quad (5a)$$

$$\left(\frac{dp}{d\tau}\right)_{\tau=0}=\frac{1}{8}\left(\frac{\psi\psi^{(iv)}-(\psi'')^2}{-\psi\psi''}\right)_{\tau=0}, \quad (5b)$$

$$\langle\tau\rangle=\int_0^\infty \tau p(\tau)d\tau=\pi\left(-\frac{\psi}{\psi''}\right)^{\frac{1}{2}}_{\tau=0}. \quad (5c)$$

The first of these equations shows the root (or level) repulsion effect is a rather general property of random functions, depending essentially only on the requirement that $\psi(\tau)$ exist. For this condition requires that the sum $\sum_{n=1}^{N} C_n^2$ converge even when $N\rightarrow\infty$, and this results in a suppression of indefinitely high-frequency components, and can be thought of roughly as introducing an upper limit for ω which is approximately equal to the reciprocal of the extension in τ of the correlation function. Such a frequency cutoff inevitably suppresses periodic behavior of wavelength less than $2\pi/\omega$, and leads to a vanishingly small proportion of double roots of $f(X)$. Equation (5c) moreover states that the mean separation of roots is just π/ω, where ω is the root-mean-square angular frequency obtained from ω_n with C_n^2 as a weight. Finally, if the correlation function has a finite extension, adjacent roots of $f(X)$ separated by very large intervals must be uncorrelated in position. Consequently the interval of separation τ must have a probability of occurrence which decreases exponentially with τ for very large τ. A proof of this has been given by Kuznetsov *et al.*[11]

Finally, Longuet-Higgins[9] has given an approximation to $p(\tau)$ valid for $\tau\lesssim\tau_m$, where τ_m is the median of the distribution p, *viz.*:

$$p(\tau)\approx\frac{\langle\tau\rangle}{2\pi}\frac{d^2}{d\tau^2}\text{arc}\cos\left(-\frac{\psi(\tau)}{\psi(0)}\right), \quad (6)$$

and for which the first two exact results (5) hold. This result can be inverted[9] to give an approximation to the correlation function also valid for $\tau\lesssim\tau_m$, *viz.*,

$$\frac{\psi(\tau)}{\psi(0)}\approx\cos\left[\frac{2\pi}{\langle\tau\rangle}\int_0^\tau d\tau'\int_{\tau_m}^{\tau'} p(\tau'')d\tau''\right]. \quad (7)$$

[11] Kuznetsov, Statonovich, and Tikhonov, J. Tech. Phys. **24**, 103 (1954) (translation: N. R. Goodman, Scientific Paper No. 5, Engineering Statistics Group, New York University College of Engineering, 1956).

It can also be shown that the median of $p(\tau)$, τ_m, is approximately at the first minimum of the correlation function, $\psi(\tau)$,[9] so that Eq. (7) determines the correlation function at best up to the neighborhood of its first minimum. Finally it is of interest to note that $p(\tau)$ is independent of the variance of $f(X)$, which is entirely determined by its normalization. This point has been remarked on by Rosenzweig.[7]

CONCLUSIONS

Besides the level repulsion effect, which is observed, the only other prediction of this work which is immediately amenable to test is that the distribution of level spacings is asymptotically a negative exponential. Examination of Fig. 7 of reference 1 shows that for $\tau/\langle\tau\rangle$ exceeding about $\frac{3}{2}$ the experimental data are fairly well fitted by an exponential curve. On the other hand, reference to Fig. 9 of the same article shows that Wigner's distribution,

$$p(\tau)=\frac{\pi}{2}\frac{\tau}{\langle\tau\rangle^2}\exp\left[-\frac{\pi}{4}\left(\frac{\tau}{\langle\tau\rangle}\right)^2\right], \tag{8}$$

predicts too few large spacings, a fact also noticed by Rosenzweig.† However, Wigner's distribution is a good fit for $\tau\lesssim\langle\tau\rangle\sim\tau_m$, and in this range predicts a correlation function of the form

$$\frac{\psi(\tau)}{\psi(0)}=\cos\left[2\pi\,\mathrm{erf}\left(\frac{\sqrt{\pi}\tau}{2\langle\tau\rangle}\right)-\pi\frac{\tau}{\langle\tau\rangle}\right]\qquad \tau\lesssim\langle\tau\rangle. \tag{9}$$

Plotted in Fig. 1 is $\psi(\tau)/\psi(0)$ *vs* $\tau/\langle\tau\rangle$. The solid portion of the curve is derived from Eq. (9), the dotted portions of the curve are merely illustrations of possible behaviors which, however, become vanishingly small for $\tau/\langle\tau\rangle\gtrsim\frac{3}{2}$.

PHYSICAL REVIEW VOLUME 113, NUMBER 2 JANUARY 15, 1959

Scintillation Studies of Some Neutron Deficient Isotopes of Lutecium*

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Lu^{170}, Lu^{172}, Lu^{173}, and Lu^{174} activities were produced by bombarding Lu_2O_3 with bremsstrahlung from the University of Illinois betatrons. Gamma rays of energy 0.083, 0.190, 0.245, and 2.04 Mev were associated with the decay of Lu^{170}. A gamma-gamma coincidence experiment showed that the 2.04-Mev gamma ray was coincident with the three low-energy transitions. Gamma rays of energy 0.079, 0.113, 0.181, 0.203, 0.325, 0.370, 0.525, 0.820, 0.900, and 1.09 Mev were associated with the electron capture decay of 6.7-day Lu^{172}, the isotope studied in the most detail. Levels of energy 0.0787, 0.2602, 0.3731, 0.5769, 0.9015, 1.082, and 1.99 Mev above the ground state have been assigned to Yb^{172} by gamma-gamma coincidence measurements and energy considerations. Gamma rays of energy 0.022, 0.079, 0.113, 0.145, 0.176, 0.274, 0.335, 0.440, 0.550, and 0.640 Mev were assigned to transitions between levels of Yb^{173} while gamma rays of energy 0.077, 0.084, 0.113, 0.176, 0.230, 0.275, 0.990, and 1.245 Mev were associated with the decay of Lu^{174}. A summary of all gamma-gamma coincidence experiments involving Lu^{173} and Lu^{174} is included in this paper. The 0.084-Mev transition associated with the decay of Lu^{174} was interpreted to be the first excited level of Hf^{174}. A rough calculation of the K-conversion coefficient of this transition yielded $\alpha_K\leqslant 2.5$.

I. INTRODUCTION

WILKINSON and Hicks[1,2] were the first experimenters to make a survey of neutron-deficient lutecium isotopes. In that work the nuclides were produced by bombarding thulium with alpha particles of various energies, and by bombarding ytterbium with 10-Mev protons. After chemical separation Wilkinson and Hicks identified half-lives of 1.7±0.1 days with Lu^{170}, 8.5±0.2 days with Lu^{171}, 6.7±0.05 days with Lu^{172}, 4.0±0.1 hours with Lu^{172m}, ~1.4 years with Lu^{173}, and 165±5 days with Lu^{174}. On the basis of absorption techniques these workers listed gamma rays at ~2.5 Mev belonging to Lu^{170}, ~1.2 Mev belonging to Lu^{171}, ~1.2 Mev belonging to Lu^{172}, ~0.22 and ~0.8 Mev belonging to Lu^{173}, and ~1 Mev belonging to Lu^{174}. In addition, they reported a β^- group of end point 0.6 Mev in Lu^{174}.

More recently Mihelich, Harmatz, and Handley[3] have made a survey of neutron-deficient rare earth isotopes, including lutecium isotopes. These workers observed the conversion electrons with permanent-magnet spectrographs, obtaining information about the low-energy transitions. From K/L and L/M conversion ratios Mihelich *et al.* were able to assign the multipolarity of certain of these. This enabled them to interpret certain transitions as being between rotational

* Supported in part by the U. S. Atomic Energy Commission and the Office of Naval Research.
† National Science Foundation Predoctoral Fellow.
[1] G. Wilkinson and H. G. Hicks, University of California Radiation Laboratory Report UCRL-429, 1949 (unpublished).
[2] G. Wilkinson and H. G. Hicks, Phys. Rev. **81**, 540 (1951).
[3] Mihelich, Harmatz, and Handley, Phys. Rev. **108**, 989 (1957).