β-γ Correlations in the First Forbidden Transition*

TSUNEYUKI KOTANI† AND MARC ROSS
Department of Physics, Indiana University, Bloomington, Indiana
(Received September 25, 1958)

A discussion is given of the theory of β-γ correlations in the unique and nonunique first forbidden transition. In particular, numerical results are presented for the Coulomb corrections to the β-γ directional correlation and transverse β polarization—γ correlation. The energy dependence of the contribution associated with violation of time-reversal invariance is given.

INTRODUCTION

It is the purpose of this article to show what kind of information can be obtained from various β-γ correlation and transverse polarization experiments. In particular, the Coulomb corrections are given for these processes. Some features of the β-γ directional correlation were discussed many years ago[1] while the parity nonconserving processes have been discussed by Alder, Stech, and Winther,[2] Curtis and Lewis,[3] and Morita and Morita,[4] and by the present authors.[5] The Coulomb corrections have been independently investigated by Iben.[6]

We shall discuss the first forbidden β-decay transition probability in terms of an expanison in powers of ρ, the nuclear radius. (Actually the lepton wave functions are expanded.) The orders of magnitude of successive terms in this expansion are as follows:

Allowed transition:

$$1+\alpha Z\rho+\cdots, \tag{1a}$$

Nonunique first forbidden:

$$\left(\frac{\alpha Z}{2\rho}\right)^2+\left(\frac{\alpha Z}{2\rho}\right)+1+\alpha Z\rho+\cdots, \tag{1b}$$

Unique first forbidden:

$$1+\alpha Z\rho+\cdots. \tag{1c}$$

Here ρ is in units of electron Compton wavelengths, so these expansions converge rapidly. According to the usual definitions, the second term in any of the expansions (1) will include contributions of "higher forbiddenness."[7] Thus in (1a) the $\alpha Z\rho$ term involves allowed and second forbidden contributions.[8] The second and perhaps higher terms become of interest if the first term in the expansion is unusually small due to small nuclear matrix elements (e.g., l-forbiddenness in the allowed transition), if the energy of the decay is very high, or simply if a very accurate experiment is performed. A high-energy decay causes difficulty because successive terms in the expansion generally involve one higher power of W_0, the end-point energy, in units of the electron rest energy.

The second term in the expansion for the allowed transition (1a) and nonunique transition (1b) also contains the first nonzero contributions to the β-γ directional correlation and transverse β polarizations. The magnitude of these contributions yields information about the rapidity of convergence of the expansions and about nuclear matrix elements.

In this article, we confine ourselves to the leading term in the expansion (1) which contributes to an observable. We will include, however, all Coulomb corrections and nuclear finite size effects. This approximation is seen to be somewhat better for the allowed and unique transitions than for the nonunique. Indeed, because of the structure of the expressions in the latter cases, we must also keep in mind the possibility of cancellation of nuclear matrix elements in the leading terms.

We shall assume that the (local) V and A interactions[9] and the two-component theory of the neutrino, $C_V=C_V'$ and $C_A=C_A'$,[10] characterize the β decay.

THE NONUNIQUE TRANSITION

In a previous Letter[11] we discussed the accuracy and utility of the "ξ approximation" for the nonunique first forbidden transition. In the ξ approximation only the

* Supported in part by the National Science Foundation.

† On leave from Tokyo Metropolitan University, Setagaya, Tokyo, Japan.

[1] For a list of references see H. Frauenfelder, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (Interscience Publishers, Inc., New York, 1955), p. 568.

[2] Alder, Stech, and Winther, Phys. Rev. **107**, 728 (1957).

[3] R. Curtis and R. Lewis, Phys. Rev. **107**, 543 (1957).

[4] M. Morita and R. S. Morita, Phys. Rev. **109**, 2048 (1958).

[5] T. Kotani and M. Ross, Progr. Theoret. Phys. **20**, 643 (1958).

[6] I. Iben, Jr., Phys. Rev. **111**, 1240 (1958).

[7] J. Fujita and M. Yamada, Progr. Theoret. Phys. **10**, 518 (1953); M. Morita and M. Yamada, Progr. Theoret. Phys. **13**, 114 (1955); I. Iben, Jr., Phys. Rev. **109**, 2059 (1958); M. Gell-Mann, (to be published); Boehm, Soergel, and Stech, Phys. Rev. Letters, **1**, 77 (1958); J. Bernstein and R. Lewis, Phys. Rev. **112**, 232 (1958); M. Morita, Phys. Rev. (to be published); see also, for example, Appendix A of reference 5.

[8] The first term in the expansion may also contain such higher forbidden contributions but these are small corrections to the magnitude of the term and introduce no new energy dependences.

[9] Herrmannsfeldt, Maxson, Stähelin, and Allen, Phys. Rev. **107**, 641 (1957); Goldhaber, Grodzins, and Sunyar, Phys. Rev. **109**, 1015 (1958); Burgy, Krohn, Novey, Ringo, and Telegdi, Phys. Rev. **110**, 1214 (1958); Herrmannsfeldt, Burman, Stähelin, Allen, and Braid, Phys. Rev. Letters **1**, 61 (1958).

[10] Wu, Ambler, Hayward, Hoppes, and Hudson, Phys. Rev. **105**, 1413 (1957). Also see, for example, C. S. Wu, *Proceedings of the Rehovoth Conference on Nuclear Structure* edited by N. J. Lipkin (Interscience Publishers, Inc., New York, 1957), p. 346.

[11] T. Kotani and M. Ross, Phys. Rev. Letters **1**, 140 (1958).

TABLE I. The numerical coefficient for the β-γ correlation: δ is the ratio of the matrix element for the electric quadrupole transition to that for the magnetic dipole transition, $\Delta_0=(1+\delta^2)^{-1}$, $\Delta_1=(1+2\sqrt{3}\delta-\delta^2)\Delta_0$, $\Delta_2=(1+2\sqrt{3}\delta/5+\frac{3}{5}\delta^2)\Delta_0$, $\Delta_3=(1+10\delta/\sqrt{3}-5\delta^2/21)\Delta_0$, $\Delta_4=(1+2\delta/\sqrt{3}+13\delta^2/27)\Delta_0$. See Eq. (8) for the definition of $G_{\lambda\lambda'}(n)$.

J_0	J_1	J_2	$G_{02}(2)$	$G_{11}(2)$	$G_{12}(2)$	$G_{01}(1)$	$G_{11}(1)$	Examples
0	1	0	0	$-\Delta_0/\sqrt{6}$	0	0	$-\Delta_0/\sqrt{2}$	$_{58}Ce^{144}$ (?)
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0	$-\Delta_1/2\sqrt{6}$	$\Delta_1/2\sqrt{10}$	0	$-5\Delta_2/6\sqrt{2}$	$_{49}In^{117}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\Delta_1/2\sqrt{5}$	$\frac{1}{5}\Delta_1(\frac{2}{3})^{\frac{1}{2}}$	$\Delta_1/5\sqrt{2}$	$-\frac{1}{2}\Delta_2(5/3)^{\frac{1}{2}}$	$-\Delta_2/3\sqrt{2}$	$_{79}Au^{199}$
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0	$-\Delta_1/10\sqrt{6}$	$-\frac{1}{10}\Delta_1(7/6)^{\frac{1}{2}}$	0	$\Delta_2/2\sqrt{2}$	$_{80}Hg^{203}$ (?)
1	2	0	0	$1/2\sqrt{6}$	$-1/2\sqrt{6}$	0	$-1/2\sqrt{2}$	$_{69}Tm^{170}$
2	2	0	$-(1/14)^{\frac{1}{2}}$	$-1/2\sqrt{6}$	$-1/2\sqrt{14}$	$-1/\sqrt{6}$	$-1/6\sqrt{2}$	$_{17}Cl^{38}$, $_{19}K^{42}$, $_{59}Pr^{142}$, $_{79}Au^{198}$
$\frac{7}{2}$	$\frac{7}{2}$	$\frac{5}{2}$	$-\frac{1}{2}\Delta_3(3/35)^{\frac{1}{2}}$	$-\Delta_3(\frac{2}{3})^{\frac{1}{2}}/7$	$-\Delta_3/7\sqrt{10}$	$-3\Delta_4/2\sqrt{7}$	$-\Delta_4/7\sqrt{2}$	$_{58}Ce^{141}$

leading term for any observable is kept in the expansion (1b) in descending powers of $\xi\equiv\alpha Z/2\rho$.

In this approximation the energy spectrum has the allowed shape. The shape correction factor is

$$C(W)=|V|^2+|Y|^2, \tag{2}$$

where V and Y represent certain combinations of nuclear parameters:

$$V=\xi' v_0+2\xi w_0(1+\gamma_1)^{-1},$$
$$Y=\xi' y_0-2\xi[u_0(1)+x_0(1)](1+\gamma_1)^{-1}, \tag{3}$$

where $\gamma_k=[k^2-(\alpha Z)^2]^{\frac{1}{2}}$, and the subsidiary nuclear parameters have the significance:

$$\eta=-C_V\int \mathbf{r}R(-2), \qquad \eta w_0=C_A\int \boldsymbol{\sigma}\cdot\mathbf{r}R(1),$$

$$\xi'\eta v_0=C_A\int i\gamma_5 R(-1), \qquad \eta u_0(1)=C_A\int i\boldsymbol{\sigma}\times\mathbf{r}R(1), \tag{4}$$

$$\xi'\eta y_0=-C_V\int i\boldsymbol{\alpha}R(-1), \quad \eta x_0(1)=-C_V\int \mathbf{r}R(1).$$

These parameters can all be taken as real if there is time reversal invariance. The energy independent quantity $R(\kappa)$ entering in the matrix element accounts for finite nuclear size effects[12] (κ is an electron angular momentum quantum number). The parameters $u_0,\cdots,y_0$ are defined so that they may be of order unity. One of the interesting unsolved problems of forbidden β decay is to determine the magnitude of the parameter ξ' relating the relativistic to the nonrelativistic matrix elements.[11,13]

Certain other observables in the nonunique decay depend just on these same parameters V and Y. It can be shown that for every observable of the type that normally occurs in the allowed transition [i.e., that occurs in the leading term of (1a)], the expression for the same quantity in the nonunique first forbidden transition can be obtained in the ξ approximation by using the substitution:

$$C_V M_{\mathrm{F}}\rightarrow\eta V,$$
$$C_A M_{\mathrm{GF}}\rightarrow\eta Y. \tag{5}$$

For these observables, as in the allowed transition, there are then no Coulomb corrections. In (5), C_{F}, M_{F}, C_{GT}, and M_{GT} are coupling constants and matrix elements for the Fermi and Gamow-Teller transitions, respectively. Thus, for example, the measurement of circular γ polarization—β correlation is defined by

$$P_\gamma\equiv\frac{N(R)-N(L)}{N(L)+N(R)}\equiv\omega(p/W)\cos\theta, \tag{6}$$

where $N(R)$ and $N(L)$ are the numbers of γ rays with right and left circular polarization, respectively. Here p and W are momentum and energy of the β ray, respectively. ω depends on the relative magnitude of V and Y. We have, in the ξ approximation,[14,5]

$$\omega=[2G_{01}(1)\,\mathrm{Re}(VY^*)-\sqrt{2}G_{11}(1)|Y|^2]C(W)^{-1}. \tag{7}$$

Here $C(W)$ is given by (2), and, for the transition $J_0-(\beta)\rightarrow J_1-(\gamma)\rightarrow J_2$,

$$G_{\lambda\lambda'}(n)=(-1)^{J_1-J_0}W(J_1J_1\lambda\lambda';nJ_0)(2J_1+1)^{\frac{1}{2}}$$
$$\times[\textstyle\sum_{LL'}(-1)^{L+L'}F_n(LL'J_2J_1)\delta_L^*\delta_{L'}/\sum_L|\delta_L|^2], \tag{8}$$

where

$$F_n(LL',J_2J_1)$$
$$=(-1)^{J_1-J_2-1}[(2J_1+1)(2L+1)(2L'+1)]^{\frac{1}{2}}$$
$$\times C(LL'n,1-1)W(J_1J_1LL';nJ_2).$$

The F coefficients, $F_n(LL',J_2J_1)$, are tabulated by Alder, Stech, and Winther.[2] The δ_L,s are reduced matrix elements for 2^L-pole γ-ray emission.‡ Numerical values of $G_{\lambda\lambda'}(n)$ are listed in Table I for some β decays.

Similarly, the longitudinal β^- polarization with or

[12] See Sec. 3A of reference 5.

[13] Crudely perhaps, $\xi=\xi'$. It is however, indicated in reference 11 that this relation probably does not hold for low Z.

[14] F. Boehm and A. Wapstra, Phys. Rev. **109**, 456 (1958).

‡ *Note added in proof.*—Concerning the definition of δ, a sign ambiguity is pointed out by Biedenharn and Rose [Sec. II-F of their paper, Revs. Modern Phys. **25**, 729 (1953)]. We define the reduced matrix element for the γ-ray transition from the J_1 to J_2 states as follows:

$$\delta_L\equiv(J_1\|L\|J_2),$$

and the ratio δ is

$$\delta\equiv\delta_{L+1}/\delta_L.$$

TABLE II. λ_1 as function of Z and electron momentum p. The units are $\hbar=m_e=c=1$. The nuclear radius is taken to be $\rho=1.14\times10^{-13}A^{\frac{1}{3}}$ cm. The values of A used were 43, 93, 148, 200, and 258, respectively. The same numbers apply to electron and positron decay. If a different nuclear radius is used, all the numbers in any column should be multiplied by a constant according to Eq. (14). The change would be very unimportant.

p \ Z	20	40	60	80	100
0.3	1.173	1.592	2.013	2.191	1.926
0.4	1.078	1.265	1.440	1.470	1.248
0.5	1.035	1.117	1.177	1.140	0.935
0.6	1.012	1.038	1.037	0.963	0.769
0.7	0.998	0.991	0.953	0.859	0.670
0.8	0.990	0.962	0.901	0.793	0.609
0.9	0.985	0.943	0.867	0.750	0.568
1.0	0.981	0.930	0.845	0.721	0.542
1.2	0.977	0.915	0.818	0.687	0.511
1.4	0.974	0.908	0.805	0.671	0.498
1.6	0.973	0.904	0.799	0.664	0.494
1.8	0.973	0.903	0.797	0.663	0.495
2.0	0.973	0.902	0.797	0.664	0.499
2.2	0.973	0.903	0.798	0.668	0.505
2.4	0.973	0.904	0.801	0.672	0.512
2.6	0.973	0.905	0.803	0.677	0.520
2.8	0.974	0.907	0.807	0.683	0.528
3.0	0.974	0.908	0.810	0.688	0.536
3.5	0.975	0.912	0.818	0.703	0.558
4.0	0.976	0.916	0.827	0.717	0.579
4.5	0.977	0.920	0.835	0.731	0.599
5.0	0.978	0.923	0.843	0.745	0.619
6.0	0.980	0.930	0.857	0.769	0.656
7.0	0.981	0.936	0.870	0.792	0.690
8.0	0.982	0.941	0.881	0.812	0.722
9.0	0.984	0.945	0.891	0.831	0.751

without a γ-ray coincidence, $P_L{}^{\gamma}$ or P_L, in the ξ approximation is seen by (5) to be

$$P_L{}^{(\gamma)}=-p/W. \tag{9}$$

Angular dependence between β and γ enters to an order $1/\xi$ smaller than this, as does different energy dependence.

The β-γ correlations which are very small ($\alpha Z\rho$) corrections in the allowed transition occur as small (i.e., $1/\xi$) effects in the nonunique first forbidden transi-

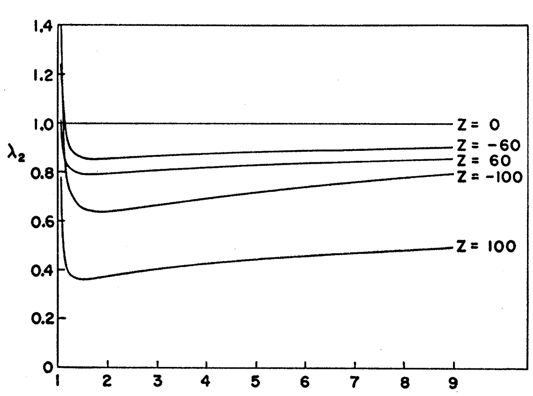

FIG. 1. λ_2 as a function of Z and electron energy W. The units are $\hbar=m_e=c=1$. Negative values of Z apply to positron decay.

tion. In the latter case, in the ξ approximation, the β-γ directional correlation has the form

$$N=1+\epsilon(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}), \tag{10}$$

where[5]

$$\epsilon=\lambda_2\left(\frac{p^2}{W}\right)\frac{R_3}{C(W)}\left[1+\alpha Z\left(\frac{W\lambda_3}{p\lambda_2}\right)\frac{I_3}{R_3}\right]. \tag{11}$$

The λ_i's here, and in (21) and (22) below, contain Coulomb corrections of order ($\alpha ZW/p$). If these are neglected, $\lambda_i=1$. In particular,

$$\lambda_2=\lambda_1^{\frac{1}{2}}A[\cos(\theta_2-\theta_1)+y\sin(\theta_2-\theta_1)/(\gamma_2+2\gamma_1)], \tag{12}$$

and

$$\lambda_3=\tfrac{4}{3}\lambda_1^{\frac{1}{2}}A[y^{-1}\sin(\theta_2-\theta_1)-\cos(\theta_2-\theta_1)/(\gamma_2+2\gamma_1)], \tag{13}$$

where

$$\lambda_1=(2+\gamma_2)F_1(Z,W)/2(1+\gamma_1)F_0(Z,W)$$
$$=\frac{(2+\gamma_2)}{2(1+\gamma_1)}(2p\rho)^{2(\gamma_2-\gamma_1-1)}\left(\frac{12\Gamma(1+2\gamma_1)}{\Gamma(1+2\gamma_2)}\right)^2\times\left|\frac{\Gamma(\gamma_2+iy)}{\Gamma(\gamma_1+iy)}\right|^2, \tag{14}$$

$$A=\tfrac{1}{4}(3+\gamma_2-\gamma_1)(2+2\gamma_1)^{\frac{1}{2}}(2+\gamma_2)^{-\frac{1}{2}},$$
$$\theta_k=\arg\Gamma(\gamma_k+iy)+\tfrac{1}{2}\pi(\gamma_k-k), \tag{15}$$

and

$$y=\alpha ZW/p. \tag{16}$$

Numerical values for λ_1 and λ_2 are presented in Tables II and III, respectively. Curves for λ_1 have been given by Davidson[15] [the actual numbers shown by him differ from ours because the size of the nucleus has changed, and he made an $(\alpha Z)^2$ approximation]. Curves for λ_2 are given in Fig. 1, and for $W\lambda_3/p\lambda_2$ in Fig. 2.

In Eq. (11), R_3 and I_3 are energy-independent quantities. If the time-reversal assumption is correct for the weak interaction, I_3 vanishes. The detailed expressions are

$$M_3=(\tfrac{2}{3})^{\frac{1}{2}}\{G_{02}(2)[z_0(-2)V]^*-2G_{11}(2)[(1-\tfrac{1}{2}u_0(-2))Y^*]\}-G_{12}(2)[z_0(-2)Y^*], \tag{17}$$

$$R_3=\mathrm{Re}M_3,\quad I_3=\tfrac{3}{4}\,\mathrm{Im}M_3, \tag{18}$$

where $G_{\lambda\lambda'}(n)$ was defined in (8). Some numerical values of $G_{\lambda\lambda'}(2)$ are listed in Table I. It is seen that to describe ϵ we need two more matrix element ratio

[15] J. Davidson, Phys. Rev. 82, 48 (1951).

TABLE III. λ_2 as a function of Z and electron momentum p. See caption of Table II for units. The negative Z values apply to positron decay.

Z \ A \ p	20	40	60	80	100	−20	−40	−60	−80	−100
	43	93	148	200	258	38	90	143	199	256
0.3	1.022	1.068	1.088	1.018	0.763	1.028	1.117	1.247	1.386	1.467
0.4	0.999	0.986	0.931	0.797	0.532	1.003	1.024	1.054	1.081	1.067
0.5	0.989	0.950	0.863	0.705	0.440	0.993	0.981	0.966	0.943	0.883
0.6	0.984	0.932	0.830	0.661	0.398	0.987	0.960	0.920	0.869	0.786
0.7	0.981	0.922	0.812	0.637	0.377	0.984	0.947	0.893	0.826	0.729
0.8	0.980	0.917	0.802	0.625	0.367	0.982	0.939	0.877	0.799	0.694
0.9	0.979	0.913	0.796	0.618	0.362	0.981	0.935	0.867	0.783	0.671
1.0	0.978	0.911	0.793	0.615	0.361	0.980	0.931	0.860	0.772	0.657
1.2	0.977	0.910	0.791	0.614	0.362	0.979	0.928	0.853	0.760	0.642
1.4	0.977	0.910	0.791	0.616	0.366	0.979	0.927	0.851	0.756	0.636
1.6	0.977	0.910	0.794	0.620	0.372	0.979	0.927	0.851	0.756	0.636
1.8	0.977	0.911	0.796	0.624	0.377	0.979	0.928	0.852	0.757	0.638
2.0	0.978	0.913	0.798	0.628	0.383	0.980	0.929	0.853	0.760	0.643
2.2	0.978	0.914	0.801	0.633	0.388	0.980	0.930	0.855	0.763	0.648
2.4	0.978	0.915	0.803	0.637	0.394	0.980	0.931	0.857	0.767	0.653
2.6	0.978	0.916	0.806	0.641	0.399	0.980	0.932	0.860	0.771	0.659
2.8	0.979	0.917	0.809	0.645	0.403	0.980	0.933	0.862	0.775	0.664
3.0	0.979	0.918	0.811	0.649	0.408	0.981	0.934	0.864	0.778	0.670
3.5	0.980	0.921	0.817	0.658	0.419	0.982	0.936	0.869	0.788	0.684
4.0	0.981	0.924	0.822	0.666	0.429	0.983	0.939	0.874	0.796	0.698
4.5	0.981	0.926	0.826	0.674	0.437	0.983	0.941	0.879	0.805	0.710
5.0	0.982	0.928	0.831	0.681	0.445	0.984	0.943	0.883	0.812	0.722
6.0	0.983	0.931	0.839	0.693	0.460	0.985	0.946	0.891	0.826	0.744
7.0	0.983	0.934	0.845	0.703	0.472	0.986	0.950	0.898	0.838	0.763
8.0	0.984	0.937	0.851	0.713	0.483	0.986	0.952	0.904	0.849	0.781
9.0	0.985	0.939	0.856	0.721	0.493	0.986	0.954	0.909	0.859	0.797

parameters, in addition to V and Y,

$$z_0(-2)=\eta^{-1}C_A\int B_{ij}R(-2), \tag{19}$$

$$u_0(-2)=\eta^{-1}C_A\int i\boldsymbol{\sigma}\times\mathbf{r}R(-2). \tag{20}$$

The transverse β polarization in the plane of β and γ (in the direction $[\mathbf{p}_e\times\mathbf{p}_\gamma]\times\mathbf{p}_e$) is[3,5]

$$P_{T\parallel}=-\tfrac{3}{2}\sin\theta\cos\theta\lambda_6\left(\frac{p}{W}\right)\frac{R_3}{C(W)}\times\left[1+\tfrac{4}{3}\alpha Z\left(\frac{W\lambda_7}{p\lambda_6}\right)\frac{I_3}{R_3}\right], \tag{21}$$

and the transverse β polarization perpendicular to the plane of β and γ (in the direction $\mathbf{p}_e\times\mathbf{p}_\gamma$) is[16]

$$P_{T\perp}=(9/8)\alpha Z\sin\theta\cos\theta\lambda_8\frac{p}{W}\frac{R_3}{C(W)}\times\left[1+\tfrac{4}{3}\alpha Z\left(\frac{W\lambda_9}{p\lambda_8}\right)\frac{I_3}{R_3}\right]. \tag{22}$$

Here, in both expressions, R_3 and I_3 are the same as in ϵ, (11), and $C(W)$ is as in (2). Also,

$$\lambda_6=\lambda_1^{\frac{1}{2}}[(\gamma_2+1+\gamma_1)/(3+\gamma_2-\gamma_1)]A\cos(\theta_2-\theta_1) \tag{23}$$

and

$$\lambda_8=\tfrac{4}{3}[(\gamma_1+\gamma_2+3)/(1+\gamma_1)(1+\gamma_1+\gamma_2)]\lambda_6. \tag{24}$$

λ_6 is plotted in Fig. 3. The Coulomb coefficients of the time reversal violating term are essentially the same in (21) and (22). That is,

$$W\lambda_7/p\lambda_6=W\lambda_9/p\lambda_8=\tan(\theta_2-\theta_1)/\alpha Z. \tag{25}$$

This quantity is shown in Fig. 4.

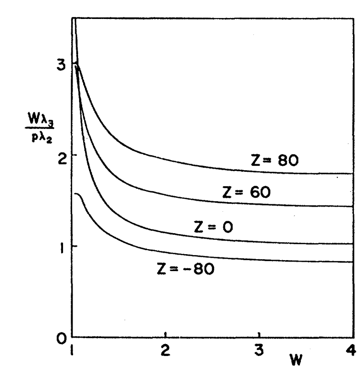

FIG. 2. $W\lambda_3/p\lambda_2$ as a function of Z and W.

[16] In the case of very low Z nuclei, where this leading term becomes fairly small because of the coefficient αZ, it may be necessary to look for the $(1/\xi)$ correction terms which have no αZ coefficient. This additional contribution to $P_{T\perp}$ among many other terms is

$$\sin\theta\cos\theta\lambda_{10}(p^2/W)[I_4/C(W)],$$

where,

$$\lambda_{10}=6\lambda_1^{\frac{1}{2}}A(1+\gamma_1)^{-1}(2\gamma_1+1)^{-1}[\gamma_1\cos(\theta_2-\theta_1)+y\sin(\theta_2-\theta_1)].$$

$I_4/C(W)$ is of order of less than $(1/\xi)^2$ (see Sec. 4 of reference 5) and vanishes, if time reversed assumption is correct.[3]

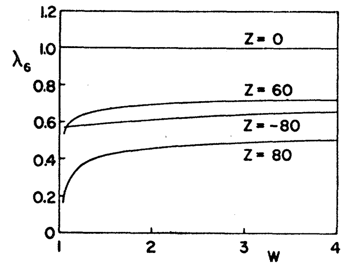

FIG. 3. λ_6 as a function of Z and W.

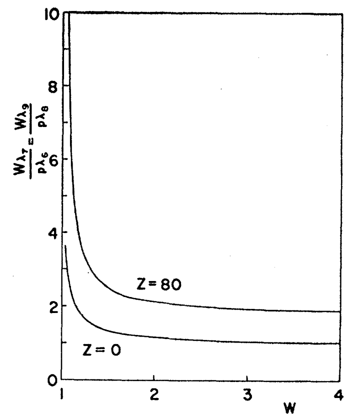

FIG. 4. $W\lambda_7/p\lambda_6 = W\lambda_9/p\lambda_8$ as a function of Z and W. For any Z in the case of positron decay, the curve lies very close to the $Z=0$ curve.

THE UNIQUE TRANSITION

The energy shape correction factor is

$$C(W)=1/12[q^2|z_0(-1)|^2+p^2\lambda_1|z_0(-2)|^2], \quad (26)$$

where q is the neutrino momentum ($q=W_0-W$), and

$$z_0(-1)=\eta^{-1}C_A\int B_{ij}R(-1), \quad (27)$$

and λ_1 is defined in Eq. (14) and tabulated in Table II. The difference between $z_0(-2)$ [Eq. (19)] and $z_0(-1)$ should be very slight.[17] The more commonly given expression for C,[18]

$$C=(q^2L_0+9L_1)[6(1+\gamma_1)]^{-1}, \quad (28)$$

differs from (26), in that certain $\alpha Z\rho$ terms [see expansion (1c)] are retained in (28). Terms of order $\alpha Z\rho$ due to higher forbidden contributions and finite nuclear size effects are not retained in (28), however, so there is little to choose between (26) and (28). This is just to say that the accuracy of (26) is the same as that of the ordinary allowed transition expression.

Letting $z_0(-1)=z_0(-2)$, the β-γ correlations in the unique first forbidden transition are, briefly,[5]

$$P_L^{(\gamma)}=-p/W, \quad (29)$$

$$\epsilon=-(\tfrac{7}{2})^{\frac{1}{2}}G_{22}(2)p^2\lambda_1/(q^2+p^2\lambda_1), \quad (30)$$

$$\omega=\frac{(5q^2+3\lambda_1p^2)G_{22}(1)-3\lambda_1p^2(5\cos^2\theta-3)G_{22}(3)}{10^{\frac{1}{2}}(q^2+\lambda_1p^2)[1+\epsilon(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2})]}. \quad (31)$$

There are no transverse β polarization—γ correlations in the unique forbidden transitions.

DISCUSSION

As far as general properties of the β decay are concerned, in principle, a test of time reversal invariance of the β interaction can be made by searching for the I_3 term in the nonunique decay. A careful energy-dependence measurement is needed to distinguish the two terms R_3 and I_3. From Fig. 2 it is seen that the difference in energy dependence of the two terms is not large, in the β-γ directional correlation. The situation is more favorable in the case of transverse polarization, Fig. 4. This experiment is, however, much more difficult than the directional correlation. In either case, even if there is no deviation from the energy dependence of the first term, namely, $\lambda_2(p^2/W)$ for ϵ or $\lambda_6(p/W)$ for either $P_{T\parallel}$ or $P_{T\perp}$, limits on the accuracy of time-reversal invariance could not be deduced without some information about the unknown nuclear parameters. This is true because I_3 involves a different nuclear parameter combination than R_3. Therefore many accurate experiments for various quantities in the same decay would have to be done to get some relations among the involved unknown nuclear parameters, as discussed below. If there is a different energy dependence than $\lambda_2(p^2/W)$ for ϵ or $\lambda_6(p/W)$ for the transverse polarization of the β ray, it would have to be shown that contributions from the term of order unity in (1b), like R_3'' in (33) below, could not be responsible. It seems that these measurements do not offer a possibility for a clear cut check of time reversal invariance.

Up to the present time, it has not been possible to determine nuclear matrix elements reliably in the nonunique decay. For the study of nuclear matrix elements it is convenient to consider the four parameters V, Y, z_0, and $u_0(-2)$. In the ξ approximation, the ratio V/Y can be determined from observables of the type that occur in the allowed transition. Thus the circular γ-ray polarization, Eq. (7), yields a quadratic expression in V/Y. Measurement of the β-ν correlation by means of resonance fluorescent scattering of the γ would be useful to fix the quantity V/Y. The theoretical expression for this process may be obtained by the substitution

[17] See Sec. 3D of reference 5. According to Sec. 3A of reference 5, and the results of Matumoto and Yamada, Progr. Theoret. Phys. **19**, 285 (1958),

$$[z_0(-2)-z_0(-1)]/z_0(-1)\approx\{[(R(-2)-R(-1)]/R(-1)\}_{r=\rho}\approx 0.04(\alpha Z)^2$$

for a uniform distribution of charge in the nucleus.

[18] M. E. Rose, in *Beta and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (Interscience Publishers, Inc., New York, 1955), p. 884.

of (5) in the allowed transition expression.[19] Other β-γ correlations discussed here involve the combination R_3 (and also I_3). Clearly, measurements that can be wholly described within the ξ approximation will not yield enough information to determine simple matrix element ratios. Such measurements can be used, however, to determine spins, and in special cases we may obtain limited information about matrix elements. In a 0-1 β decay, for example, $z_0(\kappa)$ and V vanish. Measurement of the β-γ directional correlation would yield a value for $\mathrm{Re}\{[1-\frac{1}{2}u_0(-2)]Y^*\}/|Y|^2$. In this case the circular polarization coefficient is just $\omega=-\sqrt{2}G_{11}(1)$, depending only on the mixing of different multipole γ rays. In a $\frac{1}{2}$-$\frac{1}{2}$ β decay, the quantity

$$\mathrm{Re}\{[1-\tfrac{1}{2}u_0(-2)]Y^*\}/|Y|^2$$

could be deduced by combining measurements of these two quantities since $z_0(\kappa)=0$.

The situation becomes more promising if, in addition, deviations from the ξ approximation or lack of them can be measured for the same decay. As illustration, we note that very accurate measurements of the deviation or its lack from the allowed shape energy spectrum, give values for, or upper limits on, the constants, a, b, and c,[5] defined by

$$C(W)=k[1+aW+(b/W)+cW^2]. \qquad (32)$$

The constants, a, b, and c, are functions of unknown nuclear parameters. The a and b arise from the ξ-order term of (1b) and so should be of order $1/\xi$ for heavy nuclei.[20] (In a case where a and b are comparable with unity, like RaE, the expressions in this paper are not sufficient.) By including the next term of the ξ expansion, the β-γ directional correlation coefficient has the form[5]

$$\epsilon=\lambda_2\frac{p^2}{W}\frac{R_3}{C(W)}\left[1+\alpha Z\frac{W\lambda_3}{p\lambda_2}\frac{I_3}{R_3}+R_3'+WR_3''\right], \qquad (33)$$

where R_3' and R_3'' are energy independent to order $(\alpha ZW/p)^2$ and are of order $1/\xi$. $C(W)$ must be defined by (32). Here some $1/\xi$ terms are neglected, which vanish if time reversal invariance is correct. In addition, one can, in principle at least, look for deviations from: (1) the p/W energy dependence in longitudinal polarization [see Eq. (3–20) of reference 5], (2) isotropy in the longitudinally polarized β-γ correlation [see Eq. (4–20) of reference 5], (3) p/W energy dependence in the circularly polarized γ-β correlation [see Eq. (4–18) of reference 5], or (4) $\lambda_6(p/W)$ energy dependence of $P_{T\parallel}$ or $P_{T\perp}$ [where expressions similar to (33) apply].

Let us consider the simplest case, a 0-1 β decay. Here we introduced the two nuclear matrix element parameters Y and $u_0(-2)$. When we examine deviations from the ξ approximation, we find that other parameters such as $u_0(1)$ and $u_0(-1)$ occur explicitly.[5] These parameters differ only in the finite size effect function, $R(\kappa)$. It would seem a reasonable procedure to calculate $R(\kappa)$ for a particular charge distribution and then to assume approximate relations between the various $u(\kappa)$, $y(\kappa)$, and $x(\kappa)$ that occur.[21] If this is done we will have two parameters, say Y and u. Measurement of the β-γ directional correlation, ϵ (and thus R_3), gives us one relation. Measurement, then, of one of the deviations from the ξ approximation such as a of (32) or R_3'' of (33) would suffice in principle to determine the nuclear matrix elements (the parameter η is fixed by the ft value). In this discussion we have ignored the case of no time-reversal invariance. In that case we would have one additional parameter.

In other β decays, we must consider more nuclear parameters. The problem of determining them experimentally is similar, but correspondingly more difficult.

In this article the two-component theory of the neutrino has been assumed. To remove this assumption, the following changes should be made: $C_iC_j^*\rightarrow\frac{1}{2}K_{ij}\equiv\frac{1}{2}(C_iC_j^*+C_i'C_j'^*)$ in the expressions for C, ϵ, and $P_{T\perp}$, and $C_iC_j^*\rightarrow\frac{1}{2}L_{ij}\equiv\frac{1}{2}(C_iC_j'^*+C_i'C_j^*)$ for P_γ, P_L^γ, and $P_{T\parallel}$, where i and j stand for either V or A.

All the results in this article are shown for the β^- decay. The corresponding expressions for the β^+ decay are obtained by making transformations: $(Z,P_\gamma,P_{T\parallel},P_L^\gamma,C_A,C_V)\rightarrow(-Z,-P_\gamma,-P_{T\parallel},-P_L^\gamma,+C_A^*,-C_V^*)$. If the two-component theory of the neutrino is not assumed, the transformation is $(Z,K_{jk},L_{jk})\rightarrow(-Z,\pm K_{jk}^*,\mp L_{jk}^*)$, where the upper sign applies to the case with $i=j$ and the lower sign to $i\neq j$.

ACKNOWLEDGMENT

The authors would like to thank the Research Computing Center of Indiana University for use of their facilities in the numerical computations.

[19] S. B. Treiman, Phys. Rev. **110**, 448 (1958); Frauenfelder' Jackson, and Wyld, Phys. Rev. **110**, 451 (1958); R. R. Lewis and R. B. Curtis, Phys. Rev. **110**, 910 (1958); Morita, Morita, and Yamada, Phys. Rev. **111**, 237 (1958); M. Morita and R. S. Morita Phys. Rev. **111**, 1130 (1958); C. C. Bouchiat, Phys. Rev. **112**, 877 (1958).

[20] The measurement of longitudinal polarization of the γ ray can help to check the numerical values of a and b as discussed in references 5 and 11.

[21] See Sec. 3A and Appendix C of reference 5.