Relativistic K-Shell Photoeffect*

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Expressions for the relativistic K-shell photoeffect cross sections, correct to first order in αZ (inclusive), are established. For this purpose a second order calculation must be carried out, that is, electronic spinors correct to second order in αZ must be employed in the matrix element. The final continuum state spinor of the electron, whose exact analytic form is not known, is described by means of the Born approximation. To avoid the divergences, peculiar to the application of this method to the pure Coulomb field, the case of the screened potential is considered at the beginning. The matrix element, which is evaluated in momentum space, remains singular in the limit of no screening. Nevertheless, it is shown that the differential cross section, as issuing from a very laborious trace evaluation, is to first order finite in this limit and has the behavior one would expect. Indeed, its zero-order approximation in αZ coincides with Sauter's formula, as it should. Further, in the nonrelativistic and extreme relativistic limits the cross section determined reduces to results established by other means.

I. INTRODUCTION

THE determination of the relativistic photoeffect cross sections is an extremely involved problem, so that no exact analytic expressions have been found.[1] In the case of the differential cross section the difficulty essentially comes from not knowing the exact form of the electron's final state spinor wave function, suitable to the problem. One can give therefore only approximate formulas for the cross section. In the case of the K shell, for light elements (small Z) and high velocities of the ejected photoelectron (β approaching 1), the problem was solved long ago by Sauter.[2] Subsequently, Sommerfeld and Maue[3] succeeded in deriving the same result by a more direct method, based on the approximate expression found by them for the final continuum state spinor. The formula of Sauter represents the zero-order approximation in αZ to the exact cross section, for $\beta\approx 1$. It can therefore be applied only to the lightest elements, for which $\alpha Z \ll 1$. Now, the few numerical calculations available[4] show that for heavy elements, for which this condition is not fulfilled, the exact cross section may differ from that of Sauter by a factor larger than 2. Such a disagreement naturally raises the problem of finding the corrective terms of Sauter's formula. This implies the use in the calculations of a higher order approximation for the final state spinor, than was done before. However, the methods used in the past for the approximation of this spinor become impracticable when applied to the determination of the higher order corrections. One can use instead the Born approximation method, whose efficiency in treating such higher order corrections has been put in evidence by Dalitz, in his study of the Coulomb scattering.[5,6] The successive Born approximations not only improve the form of the spinor in what regards the parameter αZ, but also render it useful for lower and lower asymptotic velocities β. However, as is well known, the method cannot be directly applied to the Coulomb field because of the infinite range of the potential, which causes the divergence of the occurring integrals. This mathematical difficulty can be avoided by replacing the pure Coulomb field by a screened one, of potential energy $eA_0(r) = -\alpha Z e^{-\mu r}/r$, for which the successive Born approximations are convergent. The higher order approximations of the matrix element thus evaluated remain singular in the limit $\mu \to 0$, of the pure Coulomb field. Physical arguments suggest, nevertheless, that the cross section itself cannot be singular in this limit; this can happen only if, when taking the square module of the matrix element, which appears in the cross section, a cancellation of these divergences takes place in consecutive orders of αZ. This cancellation has been checked in the case of scattering, for the first-order approximation, by Dalitz. As we shall see, it occurs to first order also in the more complicated case of the photoeffect. Since the cross section thus obtained (finite in the limit $\mu\to 0$) yields in the nonrelativistic and extreme relativistic limits the results derived by other means, the validity of the method is once again fully confirmed.

The aim of the present work is to determine the relativistic cross section of the effect, correct to first order in αZ, by means of the Born approximation method.

II. SPINOR WAVE FUNCTIONS

The photoeffect differential cross section for the ejection of a K-shell electron, of spinor $\psi_1(\mathbf{r})$, and energy $m(1-\alpha^2Z^2)^{\frac{1}{2}}$, into a continuum state of spinor $\psi_2(\mathbf{r})$,

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1 For a review of the subject see the article of H. Bethe and E. Salpeter, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 35, Part I, especially Sec. 73.

2 F. Sauter, Ann. Physik **9**, 217 (1931); **11**, 454 (1931).

3 A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg und Sohn, Braunschweig, 1939), Chap. 6, Sec. 8.

4 Hulme, McDougall, Buckingham, and Fowler, Proc. Roy. Soc. (London) **A149**, 131 (1935).

5 R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

6 Higher order Born approximations have also been considered by H. Mitter and P. Urban, Acta Phys. Austriaca **7**, 311 (1953), and subsequent papers.

energy E, asymptotic momentum $\mathbf{k}$ pointing inside the solid angle $d\omega$, under the influence of an incident photon of momentum $\boldsymbol{\kappa}$, energy κ and polarization $\mathbf{s}$ ($\boldsymbol{\kappa}\cdot\mathbf{s}=0$, $\mathbf{s}^2=1$), is given by[7–9]

$$d\sigma=\frac{(2\pi)^2\alpha}{\kappa}|M|^2d\omega, \tag{1}$$

where

$$M=\int\bar{\psi}_2(\mathbf{r})\boldsymbol{s}\exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\psi_1(\mathbf{r})d^3r. \tag{2}$$

In Eq. (2), s is the four-component quantity $(\mathbf{s},0)$. Here and subsequently, given a quantity w with the four components $w_\mu=(\mathbf{w},iw_0)$, we shall denote $\boldsymbol{w}\equiv w_\mu\gamma_\mu=\mathbf{w}\cdot\boldsymbol{\gamma}+iw_0\gamma_4$.[10] Further, given the spinor φ, we will denote $\bar{\varphi}=\varphi^*\gamma_4$. The spinor $\psi_1(\mathbf{r})$ is supposed to be normalized to unity, whereas $\psi_2(\mathbf{r})$ is supposed to be normalized per energy interval and element of solid angle. In order that it should describe the final state of the ejected electron, $\psi_2(\mathbf{r})$ must represent the superposition of a relativistic plane wave and a spherical incoming wave (both distorted by the Coulomb field).[11] As we desire to find the expression of the matrix element M correct to *first* order in αZ, we shall have to use approximations correct to *second* order in αZ for the position space spinors $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$. This rather surprising circumstance is peculiar to the photoeffect.[12] It is caused (as will be explained in Sec. III) by the special analytic form of the initial bound state spinor, which can produce the lowering by a unit of the order of the integrals of M in which it appears.

The quantities which characterize the electron's initial and final states are related by several equations. It is sufficient for our purpose to consider their correct form only to first order in αZ. This is due to the fact that they will be used only after the first order approximation of the matrix element has been set up.[13] We have thus

$$E=m+\kappa, \tag{3}$$

$$k^2+m^2=E^2,\quad k^2-\kappa^2=2m\kappa,\quad k^2+\kappa^2=2E\kappa, \tag{4}$$

$$E=m/(1-\beta^2)^{\frac{1}{2}},\quad k=E\beta. \tag{5}$$

Equation (3) is the Einstein relation for the conservation of energy, the ground-state energy being in our approximation equal to m. In the last equation of (4), derived with the help of (3), terms of order $(\alpha Z)^2$ have likewise been neglected.

Actually, the quantity of experimental interest, which we will subsequently calculate, is not (1) but the cross section of the process summed up over the two K-shell electrons and the two possible spin directions of the final state:

$$d\sigma_k=\frac{(2\pi)^2\alpha}{\kappa}\sum_{\sigma_1\sigma_2}|M|^2d\omega. \tag{6}$$

Since we adopt the Born approximation method for the final state spinor, the integration of the matrix element will be carried out in momentum space. Considering the Fourier expansions

$$\psi_1(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int u_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}}d^3p,$$

$$\bar{\psi}_2(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int \bar{u}_2(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{r}}d^3p,$$

the matrix element (2) becomes

$$M=\int\bar{u}_2(\mathbf{p})\boldsymbol{s}u_1(\mathbf{p}-\boldsymbol{\kappa})d^3p. \tag{7}$$

The ground-state spinor may be written as

$$u_1(\mathbf{p})=\frac{1}{(4\pi)^{\frac{1}{2}}}\left[G(p)+iF(p)\gamma_4\boldsymbol{\gamma}\cdot\frac{\mathbf{p}}{p}\right]\chi_1, \tag{8}$$

where χ_1 is one of the constant spinors (1,0,0,0) or (0,1,0,0), according to whether we consider the state of magnetic quantum number $m=\frac{1}{2}$ or $m=-\frac{1}{2}$. For the functions $G(p)$ and $F(p)$ we shall use the following approximate forms[14]:

$$G(p)=\left(\frac{32\lambda^5}{\pi}\right)^{\frac{1}{2}}\left(1+\frac{\pi\alpha Z}{8m}p\right)\frac{1}{(p^2+\lambda^2)^2},$$
$$F(p)=\left(\frac{8\lambda^5}{\pi}\right)^{\frac{1}{2}}\frac{p}{m}\frac{1}{(p^2+\lambda^2)^2}, \tag{9}$$

where $\lambda=\alpha Zm$. It may be shown that the preceding expressions describe correctly the exact functions to first order in αZ. The form of the corrective term in the expression (9) for $G(p)$ is established under the additional assumption that $p/\lambda\gg1$,[15] whereas the expression for $F(p)$ is valid whatever p/λ.[16] We shall

[7] Reference 1, Eqs. (69.5) and (73.1).

[8] The case of hydrogen-like ions is considered; the corrections for the atomic case are discussed in reference 1, Sec. 69α.

[9] We use the natural system of units, such that $\hbar=c=1$; we have then $e^2Z=e^2Z/\hbar c\equiv\alpha Z$.

[10] The γ_μ matrices are Hermitian and defined as in reference 1, Eq. (10.12).

[11] Reference 1, p. 382.

[12] It is also present in the derivation of Sauter's formula, where a first-order calculation is needed to establish the zero-order approximation to the exact cross section. See reference 3, Chap. 6, Sec. 8B.

[13] I.e., only after Eq. (29).

[14] Their exact form may be inferred from H. Casimir, Helv. Phys. Acta 6, 287 (1933).

[15] This corrective term [u_{12} in the notation of (10)] occurs only in the matrix element M_0 of (17), where it is considered for $\mathbf{p}=\mathbf{k}-\boldsymbol{\kappa}$. Since in our case $|\mathbf{k}-\boldsymbol{\kappa}|/\lambda$ is of order $(1/\alpha Z)\gg1$, the assumption is satisfied.

[16] This approximate spinor has also been used in the calculations of Baranger, Bethe, and Feynman, Phys. Rev. 92, 482 (1953), Appendix. It actually corresponds to the Fourier transform of the position space spinor $\psi_1(\mathbf{r})$ correct to *second* order in αZ.

introduce the following notations:

$$u_1(\mathbf{p})=u_{10}(\mathbf{p})+u_{11}(\mathbf{p})+u_{12}(\mathbf{p});$$

$$u_{10}=N_1\frac{1}{(p^2+\lambda^2)^2}\chi_1,\quad u_{11}=N_1\frac{i}{2m}\frac{1}{(p^2+\lambda^2)^2}\gamma_4\boldsymbol{\gamma}\cdot\mathbf{p}\chi_1,$$

$$u_{12}=N_1\frac{\alpha Z}{2\pi^2}\phi(p)\chi_1,\quad (10)$$

where

$$N_1=\frac{1}{\pi}(8\lambda^5)^{\frac{1}{2}},\quad \lambda=\alpha Zm,\quad \phi(p)=\frac{\pi^3}{4}\frac{p}{m}\frac{1}{(p^2+\lambda^2)^2}.\quad (11)$$

The final-state spinor wave function $\bar{u}_2(\mathbf{p})$ satisfies the Dirac equation in momentum space:

$$\bar{u}_2(\mathbf{p})(ip+m)=ie\int\bar{u}_2(\mathbf{q})A(\mathbf{q}-\mathbf{p})d^3q,$$

where p is the four-component quantity $(\mathbf{p},iE)$ and A is the potential four-vector, in our case $(0,iA_0)$. The Born approximation method for the continuum state spinor $\bar{u}_2(\mathbf{p})$ consists in expanding it in powers of the external potential and keeping a suitable number of terms. By a well-known iteration procedure one finds for the second-order Born approximation the expression

$$\bar{u}_2(\mathbf{p})=N_2{}^*\bar{\chi}_2(\mathbf{k})\bigg\{\delta(\mathbf{p}-\mathbf{k})-ieA(\mathbf{k}-\mathbf{p})\frac{ip-m}{p^2-k^2-i\epsilon}$$
$$+(ie)^2\int A(\mathbf{k}-\mathbf{q})\frac{iq-m}{q^2-k^2-i\epsilon}A(\mathbf{q}-\mathbf{p})d^3q\frac{ip-m}{p^2-k^2-i\epsilon}\bigg\},\quad (12)$$

where $\chi_2(\mathbf{k})$ is the momentum space spinor for a free electron of momentum $\mathbf{k}$ and a certain spin direction, q is the four-component quantity $(\mathbf{q},iE)$ and ϵ is an infinitesimal, real quantity introduced to circumvent the poles. The sign of ϵ is essential in the determination of the nature of the solution (12). In order that $u_2(\mathbf{p})$ should represent a plane wave plus a spherical incoming wave, as is demanded by our problem, ϵ must be chosen positive in (12).[17] The coefficient N_2 originates in the change of normalization, from that of the $\mathbf{k}$ scale to that of the energy and solid angle. It may be shown that its magnitude is independent of the existence of the external field, being given as in the case of an unperturbed plane wave by

$$|N_2|^2=kE,\quad (13)$$

where k and E are related by (4).

Since the pure Coulomb potential has strictly speaking no Fourier transform, we shall consider, as discussed in the Introduction, the case of the screened potential. The Fourier transform of the latter, defined as it occurs in (12), is given by

$$eA_0(\mathbf{q})=\frac{1}{(2\pi)^3}\int eA_0(r)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r=-\frac{\alpha Z}{2\pi^2}\frac{1}{(q^2+\mu^2)}.\quad (14)$$

Hence, the spinor $\bar{u}_2(\mathbf{p})$ may be written in the form

$$\bar{u}_2(\mathbf{p})=\bar{u}_{20}(\mathbf{p})+\bar{u}_{21}(\mathbf{p})+\bar{u}_{22}(\mathbf{p});$$

$$\bar{u}_{20}=N_2{}^*\bar{\chi}_2(\mathbf{k})\delta(\mathbf{p}-\mathbf{k}),$$

$$\bar{u}_{21}=-\frac{\alpha Z}{2\pi^2}N_2{}^*\bar{\chi}_2(\mathbf{k})\frac{\gamma_4}{[(\mathbf{p}-\mathbf{k})^2+\mu^2]}\frac{ip-m}{(p^2-k^2-i\epsilon)},$$

$$\bar{u}_{22}=\left(\frac{\alpha Z}{2\pi^2}\right)^2N_2{}^*\bar{\chi}_2(\mathbf{k})\int\frac{\gamma_4}{[(\mathbf{q}-\mathbf{k})^2+\mu^2]}\frac{iq-m}{(q^2-k^2-i\epsilon)}$$
$$\times\frac{\gamma_4}{[(\mathbf{q}-\mathbf{p})^2+\mu^2]}d^3q\frac{ip-m}{(p^2-k^2-i\epsilon)}.\quad (15)$$

III. MATRIX ELEMENT

For our purpose we need the expression of the matrix element (6), correct to first order in αZ. Employing the expressions (10) and (15) for the spinor wave functions, we may split the matrix element into three terms M_j, giving the contributions of the successive Born approximations $\bar{u}_{2j}$ of (15),

$$M=M_0+M_1+M_2.\quad (16)$$

The integration in M_0 is immediate, the whole spinor $u_1(\mathbf{p})$ of (10) being required in its expression, to the order we are interested. We get

$$M_0=N_1N_2{}^*\frac{1}{[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]^2}$$
$$\bar{\chi}_2\mathbf{s}\left[1+\frac{i}{2m}(\mathbf{k}-\boldsymbol{\kappa})\gamma_4\boldsymbol{\gamma}+\frac{\alpha Z}{2\pi^2}\phi(|\mathbf{k}-\boldsymbol{\kappa}|)\right]\chi_1.\quad (17)$$

In the expressions for M_1 and M_2 only a part of the terms of $u_1(\mathbf{p})$ should be taken into consideration, to first order in αZ. To emphasize this, we introduce the notation

$$M_{ij}=\int\bar{u}_{2i}(\mathbf{p})\mathbf{s}u_{1,j}(\mathbf{p}-\boldsymbol{\kappa})d^3p.\quad (18)$$

Then, to first order, M_1 is given by

$$M_1=M_{10}+M_{11};$$

$$M_{10}=-\frac{\alpha Z}{2\pi^2}N_1N_2{}^*(\bar{\chi}_2\gamma_4I\mathbf{s}\chi_1),\quad (19)$$

$$M_{11}=-\frac{\alpha Z}{2\pi^2}N_1N_2{}^*\frac{i}{2m}(\bar{\chi}_2\gamma_4\mathbf{J}\mathbf{s}\gamma_4\boldsymbol{\gamma}\chi_1),\quad (20)$$

[17] See, for instance, reference 1, Sec. 9β.

where we have put

$$I=\int\frac{ip-m}{[(\mathbf{p}-\mathbf{k})^2+\mu^2][(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2]^2(p^2-k^2-i\epsilon)}d^3p, \quad (21)$$

$$\mathbf{J}=\int\frac{ip-m}{[(\mathbf{p}-\mathbf{k})^2+\mu^2][(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2]^2(p^2-k^2-i\epsilon)}\times(\mathbf{p}-\boldsymbol{\kappa})d^3p. \quad (22)$$

The term M_2 reduces in this approximation to M_{20}:

$$M_2=M_{20}=\left(\frac{\alpha Z}{2\pi^2}\right)^2 N_1N_2^*(\bar{\chi}_2\gamma_4 Ks\chi_1), \quad (23)$$

where

$$K=\int\int\frac{1}{[(\mathbf{q}-\mathbf{k})^2+\mu^2]}\frac{(iq-m)}{(q^2-k^2-i\epsilon)}\frac{\gamma_4}{[(\mathbf{q}-\mathbf{p})^2+\mu^2]}\times\frac{(ip-m)}{(p^2-k^2-i\epsilon)}\frac{1}{[(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2]^2}d^3pd^3q. \quad (24)$$

Thus the matrix element (16) is given in our approximation by

$$M=M_0+M_{10}+M_{11}+M_{20}. \quad (25)$$

We shall now show that all the terms of Eq. (25) are indeed of zero or first order in αZ and that M contains no others of this order of magnitude.

It is clear that all the terms of M_0 are of zero or first order. Consideration of a higher order approximation for $u_1(\mathbf{p})$ would lead to the occurrence in M_0 of terms of order $(\alpha Z)^2$, which we neglect throughout. Contrary to appearances, M_{10} is of zero order in αZ; indeed, because of the special analytic structure of the bound state spinor, the integral I of (21), considered as a function of the parameter $\lambda=\alpha Zm$, has a simple pole in $\lambda=0$. We shall show this taking advantage of the fact that one of the analytic forms of the $\delta(\mathbf{p}-\boldsymbol{\kappa})$ function is[18]

$$\delta(\mathbf{p}-\boldsymbol{\kappa})=\frac{1}{\pi^2}\lim_{\lambda\to 0}\frac{\lambda}{[(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2]^2}.$$

We then find

$$\lim_{\lambda\to 0}\lambda I=\frac{\pi^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\frac{it-m}{\kappa^2-k^2},$$

t being the four-component quantity $(\boldsymbol{\kappa},iE)$. I can therefore be written

$$I=\frac{1}{\lambda}\frac{\pi^2}{\kappa^2-k^2}\frac{it-m}{(\mathbf{k}-\boldsymbol{\kappa})^2}+I^{(0)}, \quad (26)$$

where $I^{(0)}$ contains the zero and higher order contributions in λ. Since we are interested in the expression of M_{10} only to first order in αZ, $I^{(0)}$ needs to be evaluated only to zero order in λ. M_{11} is of first order in αZ. Indeed, upon introducing similarly the function $\delta(\mathbf{p}-\boldsymbol{\kappa})$ in the integral $\mathbf{J}$ of (22), one finds that owing to the occurrence of the factor $(\mathbf{p}-\boldsymbol{\kappa})$ in the integrand, the integral is of zero order, $\mathbf{J}=\mathbf{J}^{(0)}$; we are interested in $\mathbf{J}^{(0)}$ only to this lowest order. The next term M_{12} of M_1 [see Eq. (18)] should be neglected since it contains the product of $(\alpha Z)^2$ with an integral which, owing to the presence of the factor $|\mathbf{p}-\mathbf{k}|$ in the integrand, has no pole in $\lambda=0$. M_{20} is of first order in αZ, for, applying again the δ-function procedure, it follows that the integral K has a simple pole in $\lambda=0$. To this lowest order, the only one needed, K reduces to

$$K=\frac{1}{\lambda}\frac{\pi^2}{\kappa^2-k^2}L\gamma_4(it-m), \quad (27)$$

where[19]

$$L=\int\frac{ip-m}{[(\mathbf{p}-\mathbf{k})^2+\mu^2][(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2](p^2-k^2-i\epsilon)}d^3p. \quad (28)$$

The next term M_{21} of M_2 should be neglected by similar arguments as in the case of M_{12} [this time on account of the occurrence of the factor $(\mathbf{p}-\boldsymbol{\kappa})$ in the integrand]. As regards the next correction to $\bar{u}_2(\mathbf{p})$ (the third Born approximation), it yields a contribution of order $(\alpha Z)^2$, which is therefore negligible.

Summing up the different contributions to Eq. (25) and using Eqs. (26) and (27), the matrix element M becomes

$$M=\bar{\chi}_2(P+Q)\chi_1, \quad (29)$$

where P and Q contain the zero- and first-order terms in αZ, respectively. As regards the expression of P it should be noticed that M_0 of (17) contains in the denominator the quantity $(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2$. In our case, using Eqs. (4) and (5), it may be shown that for relativistic velocities β, the ratio $\lambda^2/(\mathbf{k}-\boldsymbol{\kappa})^2$ is of order of magnitude $(\alpha Z)^2$. We may thus neglect λ^2 as being a second order quantity.[20] Taking this into account, P and Q become, with the help of the second equation of (4):

$$P=\frac{N_1N_2^*}{(\mathbf{k}-\boldsymbol{\kappa})^4}\left[s+s\gamma_4\gamma\frac{i}{2m}(\mathbf{k}-\boldsymbol{\kappa})+\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2\kappa}\gamma_4(it-m)s\right], \quad (30)$$

$$Q=\frac{\alpha Z}{2\pi^2}N_1N_2^*\left[s\phi(|\mathbf{k}-\boldsymbol{\kappa}|)-\gamma_4I^{(0)}s-\frac{i}{2m}\gamma_4\mathbf{J}^{(0)}s\gamma_4\gamma-\frac{1}{4m^2\kappa}\gamma_4L\gamma_4(it-m)s\right]. \quad (31)$$

[18] All the results we shall obtain by the use of the δ function may be rederived by using the exact formulas of the Appendix and the approximations we work in.

[19] By applying the δ-function procedure, in the expression (28) for L, μ^2 should occur in the place of λ^2. Since in our approximation this change is of no consequence, we prefer for convenience the expression (28), which, owing to Eqs. (A.23) and (A.24), is actually the exact one for L.

[20] Practically everywhere in this paper the quantity $(\mathbf{k}-\boldsymbol{\kappa})^2$ occurs in the denominators in the form $(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2$. We shall neglect λ^2 throughout.

Since s has a vanishing fourth component and $\boldsymbol{\kappa}\cdot\mathbf{s}=0$, it follows that

$$s\gamma_4=-\gamma_4 s,\quad ts=-st.$$

Thus, P may be put into the form

$$P=\frac{N_1N_2^*}{(\mathbf{k}-\boldsymbol{\kappa})^4}s\gamma_4(ia+b),\tag{32}$$

where b and the four-component quantity $a(\mathbf{a},ia_0)$ are given by

$$\mathbf{a}=\frac{1}{2m}(\mathbf{k}-\boldsymbol{\kappa})+\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2\kappa}\boldsymbol{\kappa},\quad a_0=\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2\kappa}E-1,$$
$$b=\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m\kappa}.\tag{33}$$

Equation (31) for Q may be written

$$Q=\frac{\alpha Z}{2\pi^2}N_1N_2^*\sum_{\mu=1}^{4}R_\mu,\tag{34}$$

where

$$R_1=s\phi(|\mathbf{k}-\boldsymbol{\kappa}|),\qquad R_2=-\gamma_4 I^{(0)}s,$$
$$R_3=-\frac{i}{2m}\gamma_4\mathbf{J}^{(0)}s\gamma_4\boldsymbol{\gamma},\quad R_4=-\frac{1}{4m^2\kappa}\gamma_4L\gamma_4(it-m)s.\tag{35}$$

In view of finding the explicit form of the matrix element (29), the integrations in the expressions of I, $\mathbf{J}$, L must be carried out. Putting in evidence the γ matrices in these expressions, we may write

$$I=i\gamma_jA_j-(E\gamma_4+m)A_0,\tag{36}$$

$$J_i=i\gamma_jA_{ij}-(E\gamma_4+m)A_i-\kappa_iI,\tag{37}$$

$$L=i\gamma_jB_j-(E\gamma_4+m)B_0,\tag{38}$$

where B_0, B_i, A_0, A_i, A_{ij} are the momentum space integrals (A.1), (A.21) discussed in the appendix. All these integrals are divergent in the limit $\mu\rightarrow 0$; since the first and higher order terms in μ vanish in this limit, they will be neglected from the beginning. In this approximation the results derived are correct in λ, but only the required order of magnitude should be kept in evaluating the expressions (36), (37), (38). If we denote the zero and higher order terms in λ of A_0, A_i, A_{ij} by $A_0^{(0)}$, $A_i^{(0)}$, $A_{ij}^{(0)}$ the dependence of $I^{(0)}$ and $J_i^{(0)}$ on the newly introduced quantities is similar to that given by formulas (36) and (37). (Actually, as we have previously shown, $\mathbf{J}_i=\mathbf{J}_i^{(0)}$, since $\mathbf{J}_i$ has no terms in $1/\lambda$.) We may then put

$$I^{(0)}=ic+d,\tag{39}$$

$$J_i^{(0)}=i\gamma_jA_{ij}^{(0)}-(E\gamma_4+m)A_i^{(0)}-\kappa_i(ic+d),\tag{40}$$

$$L=ie+f,\tag{41}$$

The notation introduced here is

$$c=(\mathbf{A}^{(0)},iEA_0^{(0)}),\quad e=(\mathbf{B},iEB_0),$$
$$d=-mA_0^{(0)},\quad f=-mB_0,\tag{42}$$

c and e being four-component quantities. The explicit form of the quantities of (42) is not needed for the calculations of the next section.

IV. EVALUATION OF THE TRACES

The cross section (6) is expressed in terms of $\sum|M|^2$ where the summation is to be performed over all the possible transitions from the K shell to the continuum final state of asymptotic momentum $\mathbf{k}$. Due to the form (8) adopted for the spinor $u_1(\mathbf{p})$, the matrix element M has the same aspect for all these transitions: the matrices P and Q are the same, only the spinors χ_1 and $\bar{\chi}_2$ are different from case to case. On the other hand, the spinors $\bar{\chi}_2$ and χ_1 satisfy the equalities

$$\bar{\chi}_2(\mathbf{k})(ik+m)=0,\quad (il+m)\chi_1=0,$$

where k is the momentum four-vector $(\mathbf{k},ik_0)$ of a free particle in motion $(k_0=(\mathbf{k}^2+m^2)^{\frac{1}{2}}=E)$ and l is that of a free particle at rest $(0,im)$. The first equality is satisfied by $\bar{\chi}_2$ by definition, the second one may be checked immediately taking into account the explicit form of χ_1. In these conditions the sum $\sum|M|^2$ is found, by a well-known formula, to be

$$\sum_{\sigma_1\sigma_2}|M|^2=\frac{1}{4Em}\,\mathrm{Sp}[(P+Q)(il-m)(\bar{P}+\bar{Q})(ik-m)],\tag{43}$$

where $\bar{P}$ and $\bar{Q}$ are defined by means of the equality $\bar{O}=\gamma_4O^\dagger\gamma_4$. Neglecting the second-order terms in αZ of Eq. (43) and taking into account that the two of first order are complex conjugates of each other, we find

$$\sum_{\sigma_1\sigma_2}|M|^2=\frac{1}{4Em}\,\mathrm{Sp}[P(il-m)\bar{P}(ik-m)]+\frac{2}{Em}\,\mathrm{Re}\{\tfrac{1}{4}\,\mathrm{Sp}[Q(il-m)\bar{P}(ik-m)]\}.\tag{44}$$

In our case, owing to (32), we have

$$\bar{P}=\frac{N_1N_2^*}{(\mathbf{k}-\boldsymbol{\kappa})^4}\gamma_4(ia'-b)s.\tag{45}$$

Here we have denoted by a' the four-component quantity $(\mathbf{a},-ia_0)$. Using (32), (34), and (45), Eq. (44) may be written in the form

$$\sum_{\sigma_1\sigma_2}|M|^2=\frac{|N_1N_2|^2}{Em}\frac{1}{(\mathbf{k}-\boldsymbol{\kappa})^8}\times\left[\Omega_0+\frac{\alpha Z}{\pi^2}(\mathbf{k}-\boldsymbol{\kappa})^4\,\mathrm{Re}\sum_{\mu=1}^{4}\Omega_\mu\right],\tag{46}$$

where we have introduced the following abbreviations:

$$\Omega_0=\tfrac{1}{4}\,\mathrm{Sp}[\boldsymbol{s}\gamma_4(i\boldsymbol{a}+b)(i\boldsymbol{l}-m)\gamma_4(i\boldsymbol{a}'-b)\boldsymbol{s}(i\boldsymbol{k}-m)], \quad (47)$$

$$\Omega_\mu=\tfrac{1}{4}\,\mathrm{Sp}[R_\mu(i\boldsymbol{l}-m)\gamma_4(i\boldsymbol{a}'-b)\boldsymbol{s}(i\boldsymbol{k}-m)]. \quad (48)$$

The evaluation of the traces (47), (48) is a very laborious task. It may be performed with the help of the following formulas. Let us put

$$W=\tfrac{1}{4}\,\mathrm{Sp}\Pi=W_\gamma+W_\theta, \quad (49)$$

$$X=\tfrac{1}{4}\,\mathrm{Sp}(\boldsymbol{x}\Pi)=X_\gamma+X_\theta, \quad (50)$$

$$Y=\tfrac{1}{4}\,\mathrm{Sp}(\boldsymbol{x}\boldsymbol{y}\Pi)=Y_\gamma+Y_\theta, \quad (51)$$

$$\Pi=\boldsymbol{s}(i\boldsymbol{l}-m)\gamma_4(i\boldsymbol{a}'-b)\boldsymbol{s}(i\boldsymbol{k}-m), \quad (52)$$

where x and y are arbitrary four-component quantities, with $(\mathbf{x},ix_0)$ and $(\mathbf{y},iy_0)$, respectively. We then find (with $k_0=E$)

$$W_\gamma=2m(\mathbf{a}\cdot\mathbf{s})(\mathbf{k}\cdot\mathbf{s}),$$
$$W_\theta=-m[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)], \quad (53)$$

$$X_\gamma=2im(\mathbf{a}\cdot\mathbf{s})[(\mathbf{x}\cdot\mathbf{s})(k_0+m)-x_0(\mathbf{k}\cdot\mathbf{s})], \quad (54)$$

$$X_\theta=im\{x_0[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)] -(\mathbf{k}\cdot\mathbf{x})(a_0-b)-(\mathbf{a}\cdot\mathbf{x})(k_0+m)\}, \quad (55)$$

$$Y_\gamma=2m(\mathbf{a}\cdot\mathbf{s})\{(\mathbf{y}\cdot\mathbf{s})[\mathbf{x}\cdot\mathbf{k}-x_0(k_0+m)] -(\mathbf{x}\cdot\mathbf{s})[\mathbf{y}\cdot\mathbf{k}-y_0(k_0+m)] +(\mathbf{k}\cdot\mathbf{s})(\mathbf{x}\cdot\mathbf{y}-x_0y_0)\}, \quad (56)$$

$$Y_\theta=m\{[\mathbf{a}\cdot\mathbf{x}+x_0(a_0-b)][\mathbf{k}\cdot\mathbf{y}-y_0(k_0+m)] -[\mathbf{a}\cdot\mathbf{y}+y_0(a_0-b)][\mathbf{k}\cdot\mathbf{x}-x_0(k_0+m)] -(\mathbf{x}\cdot\mathbf{y}-x_0y_0)[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)]\}. \quad (57)$$

By inserting suitably the product $\boldsymbol{s}\cdot\boldsymbol{s}=1$, the expressions in square brackets following the Sp symbols in Eqs. (47) and (48) may easily be transformed so as to contain the matrix Π as a last factor. Taking then into account Eqs. (35), (39), (40), and (41), the resulting expressions for the traces may be evaluated using the formulas for W, X, Y. If we denote by Γ the $\mathbf{s}$-dependent terms thus found and by Θ those independent of $\mathbf{s}$, we may write generally

$$\Omega_\nu=\Gamma_\nu+\Theta_\nu,\quad \nu=0,1,2,3,4. \quad (58)$$

We thus finally get

$$\Gamma_0=-4m(\mathbf{a}\cdot\mathbf{s})(\mathbf{k}\cdot\mathbf{s})(a_0-b) \quad (59)$$

$$\Theta_0=m(k_0+m)\left(\mathbf{a}+\mathbf{k}\frac{a_0-b}{k_0+m}\right)^2, \quad (60)$$

$$\Gamma_1=\phi(|\mathbf{k}-\boldsymbol{\kappa}|)W_\gamma,\quad \Theta_1=\phi(|\mathbf{k}-\boldsymbol{\kappa}|)W_\theta, \quad (61)$$

$$\Gamma_2=2m(\mathbf{a}\cdot\mathbf{s})[(\mathbf{c}\cdot\mathbf{s})(k_0+m)+(\mathbf{k}\cdot\mathbf{s})(c_0+d)], \quad (62)$$

$$\Theta_2=-m(k_0+m)\times\left[\left(\mathbf{c}+\mathbf{k}\frac{c_0+d}{k_0+m}\right)\cdot\left(\mathbf{a}+\mathbf{k}\frac{a_0-b}{k_0+m}\right)\right], \quad (63)$$

$$\Gamma_3=-(\mathbf{a}\cdot\mathbf{s})[(\mathbf{k}\cdot\mathbf{s})(\sum_{j=1}^{3}A_{jj}^{(0)}-\boldsymbol{\kappa}\mathbf{c}) -A_{ij}^{(0)}s_ik_j+(\mathbf{c}\cdot\mathbf{s})(\mathbf{k}\cdot\boldsymbol{\kappa})]+A_{ij}^{(0)}s_ia_j(\mathbf{k}\cdot\mathbf{s}) -A_{ij}^{(0)}s_is_j[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)] -(E+m)(a_0-b)(\mathbf{k}\cdot\mathbf{s})(\mathbf{A}^{(0)}\cdot\mathbf{s}), \quad (64)$$

$$\Theta_3=\tfrac{1}{2}\Bigg\{[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)](\sum_{j=1}^{3}A_{jj}^{(0)}-\boldsymbol{\kappa}\cdot\mathbf{c}) +\left[\mathbf{a}\cdot\mathbf{c}+(\mathbf{k}\cdot\mathbf{c})\frac{a_0-b}{k_0+m}\right][\mathbf{k}\cdot\boldsymbol{\kappa}+(k_0+m)^2] -\left[\mathbf{a}\cdot\boldsymbol{\kappa}+(\mathbf{k}\cdot\boldsymbol{\kappa})\frac{a_0-b}{k_0+m}\right][\mathbf{k}\cdot\mathbf{c}+(k_0+m)(c_0-d)]\Bigg\}, \quad (65)$$

$$\Gamma_4=\frac{(\mathbf{a}\cdot\mathbf{s})}{2m\kappa}\{(\mathbf{e}\cdot\mathbf{s})[\mathbf{t}\cdot\mathbf{k}-(t_0-m)(k_0+m)] -(\mathbf{k}\cdot\mathbf{s})[\mathbf{e}\cdot\mathbf{t}+(t_0-m)(e_0+f)]\}, \quad (66)$$

$$\Theta_4=\frac{1}{4m\kappa}\Bigg\{[\mathbf{e}\cdot\mathbf{t}+(t_0-m)(e_0+f)] \times[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)] +[\mathbf{e}\cdot\mathbf{k}+(e_0-f)(k_0+m)]\left[\mathbf{a}\cdot\mathbf{t}+(\mathbf{k}\cdot\mathbf{t})\frac{a_0-b}{k_0+m}\right] -\left[\mathbf{e}\cdot\mathbf{a}+(\mathbf{e}\cdot\mathbf{k})\frac{a_0-b}{k_0+m}\right][\mathbf{k}\cdot\mathbf{t}-(t_0-m)(k_0+m)]\Bigg\}. \quad (67)$$

In Eqs. (66) and (67) we have used the notation $\mathbf{t}=\boldsymbol{\kappa}$, $t_0=E$, and the fact that $\mathbf{t}\cdot\mathbf{s}=0$.

V. DIFFERENTIAL CROSS SECTION

It should be noted that owing to the fact that the quantities $\mathbf{a}$, a_0, and b of (33) are real, Ω_0 given by (58), (59), and (60) is also real. This was to be expected since Ω_0 corresponds to the zero-order approximation in αZ of the real sum $\sum|M|^2$ of (46). Ω_1 given by (58) and (61) is also real. However Ω_2, Ω_3, and Ω_4 are complex and divergent in the limit $\mu\rightarrow 0$, because they involve the quantities defined in (42). Only the real parts of these traces are required for the calculation of the sum of Eq. (46). Since the complex quantities $\mathbf{A}_0^{(0)}$, $A_0^{(0)}$, $\mathbf{B}$, B_0, and $A_{ij}^{(0)}$ occur in Ω_2, Ω_3, and Ω_4 multiplied by real factors, only their real parts are needed. These, as shown in the appendix, are finite in the limit of the pure Coulomb potential ($\mu\rightarrow 0$). The same will also be true, because of Eq. (46), for the cross-section (6), correct to first order in αZ. Thus, the cancellation of the specific divergences of the Born method is established also for the case of the photoeffect.

We now set out to find the explicit form of the quantities Ω_0 and $\mathrm{Re}\Omega_\mu$. To this end we shall use the definitions (33) and the relations which derive from them:

$$\mathbf{a}\cdot\mathbf{s}=\frac{1}{2m}(\mathbf{ks}),\quad a_0-b=\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2}-1, \tag{68}$$

$$\mathbf{a}+\mathbf{k}\frac{a_0-b}{k_0+m}=\frac{\kappa}{2m^2}\left[\mathbf{k}\frac{k^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{k^2}+\boldsymbol{\kappa}\frac{\kappa^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{\kappa^2}\right]. \tag{69}$$

Taking the scalar product of the vector (69) with $\mathbf{k}$ and $\boldsymbol{\kappa}$, we find

$$\mathbf{a}\cdot\mathbf{k}+k^2\frac{a_0-b}{k_0+m}=\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)$$
$$=\frac{1}{2m^2\kappa}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2], \tag{70}$$

$$\mathbf{a}\cdot\boldsymbol{\kappa}+(\mathbf{k}\cdot\boldsymbol{\kappa})\frac{a_0-b}{k_0+m}=\frac{\kappa}{2m^2k^2}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]. \tag{71}$$

In the case of Ω_0, employing Eqs. (68) and (69), we obtain for Γ_0 and Θ_0 of (59) and (60), the expressions

$$\Gamma_0=2(\mathbf{k}\cdot\mathbf{s})^2\left[1-\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2}\right], \tag{72}$$

$$\Theta_0=\frac{1}{4m^3\kappa}(\mathbf{k}-\boldsymbol{\kappa})^2[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]. \tag{73}$$

The terms Γ_1 and Θ_1 of Ω_1, are given by Eqs. (61), (53), (68), and (69). Expressing $\phi(|\mathbf{k}-\boldsymbol{\kappa}|)$, defined in Eq. (11) by means of $\mathcal{C}$ of Eq. (A.18),[20] we get

$$\Gamma_1=\frac{1}{4m}\frac{(\mathbf{k}\cdot\mathbf{s})^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\mathcal{C}, \tag{74}$$

$$\Theta_1=-\frac{1}{8m^2\kappa}\frac{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]}{(\mathbf{k}-\boldsymbol{\kappa})^2}\mathcal{C}. \tag{75}$$

Taking into account the definition (42) of the quantities $\mathbf{c}$, c_0, d, their real parts can be obtained with the aid of Eqs. (A.26) of the appendix. Employing also Eqs. (68) and (69), we find for the two terms of $\mathrm{Re}\Omega_2$

$$\mathrm{Re}\Gamma_2=2E\mathcal{A}(\mathbf{k}\cdot\mathbf{s})^2, \tag{76}$$

$$\mathrm{Re}\Theta_2=-\frac{E}{m\kappa}\mathcal{A}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]. \tag{77}$$

Using the same expressions for the real parts of $\mathbf{c}$, c_0, d as above and the formula (A.33) for $\mathrm{Re}\sum A_{jj}^{(0)}$, we find after some elementary transformations based upon (3) and (4)

$$\mathrm{Re}\Theta_3=\frac{E}{m\kappa}\mathcal{A}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]=-\mathrm{Re}\Theta_2. \tag{78}$$

The calculation of the real part of Γ_3, given in (76), is more tedious. From Eqs. (A.27), (A.33), and $\boldsymbol{\kappa}\cdot\mathbf{s}=0$ it follows firstly that

$$\mathrm{Re}\Gamma_3=-\mathcal{V}[\mathbf{a}\cdot\mathbf{k}+(k_0-m)(a_0-b)]$$
$$+(\mathbf{k}\cdot\mathbf{s})^2\left\{-\frac{k^2}{2m}\mathcal{A}+\frac{1}{m}\mathcal{V}+\mathcal{S}\left[\mathbf{a}\cdot\boldsymbol{\kappa}+(\mathbf{k}\cdot\boldsymbol{\kappa})\frac{a_0-b}{k_0+m}\right]\right.$$
$$+\left(\frac{1}{2m}-\frac{a_0-b}{k_0+m}\right)[k^2\mathcal{U}+(\mathbf{k}\cdot\boldsymbol{\kappa})\mathcal{S}]$$
$$\left.-(E+m)(a_0-b)\mathcal{A}\right\}. \tag{79}$$

Using the expression (A.31), (A.29) for $\mathcal{U}$ and $\mathcal{S}$, one then finds

$$k^2\mathcal{U}+(\mathbf{k}\cdot\boldsymbol{\kappa})\mathcal{S}=k^2\mathcal{A}+\tfrac{1}{4}\mathcal{C}\left[\frac{\mathbf{k}\cdot\boldsymbol{\kappa}-k^2}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}+\frac{1}{(\mathbf{k}-\boldsymbol{\kappa})^2}\right]$$
$$+\tfrac{1}{4}\pi^3\frac{k}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}.$$

We shall replace the other $\mathcal{S}$ occurring in Eq. (79) by $\mathcal{T}$, given in (A.30), since $\mathcal{S}=\mathcal{T}$; the expression of $\mathcal{V}$ is given in (A.32), and from (80) and (4) it follows

$$\frac{1}{2m}-\frac{a_0-b}{k_0+m}=\frac{1}{m}-\frac{\kappa}{2m^2k^2}(k^2-\mathbf{k}\cdot\boldsymbol{\kappa}).$$

Thus, using also Eqs. (68), (70), and (71), we find in the end after rearranging the terms conveniently,

$$\mathrm{Re}\Gamma_3=-\frac{1}{4}\frac{(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})\mathcal{C}-\pi^3k}{2m^2\kappa}$$
$$+(\mathbf{k}\cdot\mathbf{s})^2\left\{2E\left[1-\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2}\right]\mathcal{A}\right.$$
$$+\frac{\mathcal{C}}{4m}\left[\frac{1}{(\mathbf{k}-\boldsymbol{\kappa})^2}+\frac{\kappa^2}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\right]$$
$$\left.-\tfrac{1}{4}\pi^3\frac{\kappa}{2m^2k}\frac{k^2+\mathbf{k}\cdot\boldsymbol{\kappa}}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\right\}. \tag{80}$$

In order to find the expression of $\mathrm{Re}\Omega_4$, we note firstly that, given the definition (42) of the quantities $\mathbf{e}$, e_0, f, as well as that of $\mathbf{t}$ and t_0, Eq. (66) for Γ_4 may be given the form

$$\Gamma_4=\frac{(\mathbf{k}\cdot\mathbf{s})}{4m^2\kappa}[(\mathbf{B}\cdot\mathbf{s})(\mathbf{k}\cdot\boldsymbol{\kappa}-k^2)-(\mathbf{k}\cdot\mathbf{s})(\mathbf{B}\cdot\boldsymbol{\kappa}+B_0\kappa^2)]. \quad (81)$$

The calculation of $\mathrm{Re}\Gamma_4$ is done by using the formulas

$$\mathrm{Re}\mathbf{B}\cdot\mathbf{k}=\mathcal{B}k^2+\tfrac{1}{2}\mathcal{C}, \quad (82)$$

$$\mathrm{Re}\mathbf{B}\cdot\boldsymbol{\kappa}=\mathcal{B}(\mathbf{k}\cdot\boldsymbol{\kappa})+\tfrac{1}{2}\mathcal{C}-\tfrac{1}{2}\pi^3/k, \quad (83)$$

and $\mathrm{Re}\mathbf{B}\cdot\mathbf{s}=\mathcal{B}_k(\mathbf{k}\cdot\mathbf{s})$, which all derive from (A.20). We find on their account, after some manipulations, the result

$$\mathrm{Re}\Gamma_4=\frac{(\mathbf{k}\cdot\mathbf{s})^2}{4m^2\kappa}\Bigg\{-\mathcal{B}(k^2+\kappa^2)+\tfrac{1}{2}\mathcal{C}\frac{(\mathbf{k}\cdot\boldsymbol{\kappa})(\mathbf{k}-\boldsymbol{\kappa})^2-2[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}+\tfrac{1}{2}\pi^3\frac{k(\kappa^2-\mathbf{k}\cdot\boldsymbol{\kappa})}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\Bigg\}. \quad (84)$$

In the case of Θ_4, it should be noted that the real parts of the terms of Eq. (67) may be calculated using Eqs. (82), (83), (70), and (71). We thus finally find

$$\mathrm{Re}\Theta_4=\frac{1}{4m\kappa}\Bigg\{\frac{2E}{m^2}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]\mathcal{B}+\frac{\mathcal{C}}{4m^2k^2\kappa}\Big[[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](k^2+\kappa^2)+[2k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2+\kappa^2)](k^2-\mathbf{k}\cdot\boldsymbol{\kappa})\Big]-\frac{\pi^3}{4m^2k\kappa}[2k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2+\kappa^2)]\Bigg\}. \quad (85)$$

We have to consider next the summation of the individual contributions $\mathrm{Re}\Gamma_\mu$ and $\mathrm{Re}\Theta_\mu$. Addition of the results contained in Eqs. (74), (76), (80), and (84) yields, after expressing $\mathcal{B}$ in terms of $\mathcal{A}$ and conveniently rearranging the terms, the result

$$\mathrm{Re}\sum_{\mu=1}^{4}\Gamma_\mu=-\frac{1}{4}\frac{(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})\mathcal{C}-\pi^3k}{2m^2\kappa}+(\mathbf{k}\cdot\mathbf{s})^2\Bigg\{4E\mathcal{A}\Big[1-\frac{(\mathbf{k}-\boldsymbol{\kappa})^2}{4m^2}\Big]+\frac{\mathcal{C}}{4m}\Big[\frac{2}{(\mathbf{k}-\boldsymbol{\kappa})^2}+\frac{E}{m}\frac{\mathbf{k}\cdot\boldsymbol{\kappa}-\kappa^2}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\Big]-\frac{\pi^3}{8m^2k\kappa}\frac{(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2+\kappa^2)}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\Bigg\}. \quad (86)$$

Similarly, adding Eqs. (75), (77), (78), and (85), we find

$$\mathrm{Re}\sum_{\mu=1}^{4}\Theta_\mu=\frac{E}{2m^3\kappa}[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](\mathbf{k}-\boldsymbol{\kappa})^2\mathcal{A}+\frac{\mathcal{C}}{16m^3\kappa^2}\Big[-\frac{2m\kappa[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]}{(\mathbf{k}-\boldsymbol{\kappa})^2}+2k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2+\kappa^2)+\kappa^2(\mathbf{k}-\boldsymbol{\kappa})^2\Big]-\frac{\pi^3}{16m^3\kappa^2k}[2k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2+\kappa^2)]. \quad (87)$$

It is to be noticed that the $\mathcal{A}$-dependent term of Eq. (86) may be put, owing to (72), into the form $2E\mathcal{A}\Gamma_0$. Likewise, on account of (73), the $\mathcal{A}$-dependent term of Eq. (87) may be written as $2E\mathcal{A}\Theta_0$. With the results (58), (86), and (87) and the preceding remarks, Eq. (46) may be given the form

$$\sum_{\sigma_1\sigma_2}|M|^2=\frac{|N_1N_2|^2}{Em}\times\Bigg\{\frac{1}{(\mathbf{k}-\boldsymbol{\kappa})^8}\Omega_0\Big[1+2E\mathcal{A}(\mathbf{k}-\boldsymbol{\kappa})^4\frac{\alpha Z}{\pi^2}\Big]+\pi\alpha Z\Xi\Bigg\}, \quad (88)$$

In view of expressing the differential cross section (6), we now introduce the usually adopted coordinate system in which $\boldsymbol{\kappa}$ points in the positive z direction and $\mathbf{s}$ in the positive x direction. Let θ and φ be the polar angles of $\mathbf{k}$ in this coordinate system. Then, on account of (4), we have[21]

$$(\mathbf{k}-\boldsymbol{\kappa})^2=2E\kappa(1-\beta\cos\theta),\ k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2=k^2\kappa^2\sin^2\theta. \quad (89)$$

By use of the preceding formulas and the definition of $\mathcal{C}$ of Eq. (A.18), the quantities Ω_0 and Ξ occurring in Eq. (88) may be written

$$\frac{1}{(\mathbf{k}-\boldsymbol{\kappa})^8}\Omega_0=\frac{2k^2}{(2E\kappa)^4(1-\beta\cos\theta)^3}\times\Bigg\{\sin^2\theta\cos^2\varphi\Big[\frac{1}{1-\beta\cos\theta}-\frac{E\kappa}{2m^2}\Big]+\frac{E\kappa^2}{4m^3}\sin^2\theta\Bigg\}, \quad (90)$$

$$\Xi=\frac{1}{16m^3\kappa[2E\kappa(1-\beta\cos\theta)]^{\frac{5}{2}}}\times\Bigg\{4m^2\Big[\frac{k^2\sin^2\theta\cos^2\varphi}{E(1-\beta\cos\theta)}+\frac{E\kappa}{m}\cos^2\varphi\Big(\frac{k}{\kappa}\cos\theta-1\Big)\Big]-\frac{mk^2\kappa}{E}\frac{\sin^2\theta}{1-\beta\cos\theta}-4mE\kappa+4\kappa k^2\Big(1-\frac{\kappa}{k}\cos\theta\Big)\Bigg\}+\frac{1}{8m^3\kappa[2E\kappa(1-\beta\cos\theta)]^2}\times\{k(m-\kappa)+E\kappa\cos\theta-2Em\cos\theta\cos^2\varphi\}. \quad (91)$$

[21] It may be shown that the expression for $(\mathbf{k}-\boldsymbol{\kappa})^2$ given in (89), approximate because of the neglect of terms of order $(\alpha Z)^2$ in Eqs. (3) and (4), is precisely the rigorous one for $(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2$, obtained by using the exact form of the same equations. It is in fact this latter quantity which is of interest; see reference 20.

The expressions (90) and (91), as well as the coefficient $|N_1N_2|^2/E m\kappa$ may be expressed as functions of β, employing the relations (5) and (3). We also find, with the help of Eq. (A.26) for $\mathfrak{A}$,

$$1+2E\mathfrak{A}(\mathbf{k}-\boldsymbol{\kappa})^4\alpha Z/\pi^2=1-(\pi\alpha Z/\beta).$$

Thus the differential photoeffect cross section (6), correct to first order in αZ, can be given the form[22]

$$d\sigma_k=\frac{4}{m^2}\alpha^6Z^5\frac{\beta^3(1-\beta^2)^3}{[1-(1-\beta^2)^{\frac{1}{2}}]^5}\times\left\{\mathfrak{F}\left(1-\frac{\pi\alpha Z}{\beta}\right)+\pi\alpha Z\,\mathfrak{G}\right\}d\omega, \tag{92}$$

where we have abbreviated

$$\mathfrak{F}=\frac{\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^4}-\frac{1-(1-\beta^2)^{\frac{1}{2}}}{2(1-\beta^2)}\frac{\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^3}+\frac{[1-(1-\beta^2)^{\frac{1}{2}}]^2}{4(1-\beta^2)^{\frac{3}{2}}}\frac{\sin^2\theta}{(1-\beta\cos\theta)^3}, \tag{93}$$

$$\begin{aligned}\mathfrak{G}=\frac{[1-(1-\beta^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{2^{7/2}\beta^2(1-\beta\cos\theta)^{\frac{5}{2}}}&\times\Bigg[\frac{4\beta^2}{(1-\beta^2)^{\frac{1}{2}}}\frac{\sin^2\theta\cos^2\varphi}{1-\beta\cos\theta}+\frac{4\beta}{1-\beta^2}\cos\theta\cos^2\varphi\\&-4\frac{1-(1-\beta^2)^{\frac{1}{2}}}{1-\beta^2}(1+\cos^2\varphi)-\beta^2\frac{1-(1-\beta^2)^{\frac{1}{2}}}{1-\beta^2}\frac{\sin^2\theta}{1-\beta\cos\theta}\\&+4\beta^2\frac{1-(1-\beta^2)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{3}{2}}}-4\beta\frac{[1-(1-\beta^2)^{\frac{1}{2}}]^2}{(1-\beta^2)^{\frac{3}{2}}}\cos\theta\Bigg]\\&+\frac{1-(1-\beta^2)^{\frac{1}{2}}}{4\beta^2(1-\beta\cos\theta)^2}\Bigg[\frac{\beta}{1-\beta^2}-\frac{2}{1-\beta^2}\cos\theta\cos^2\varphi\\&+\frac{1-(1-\beta^2)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{3}{2}}}\cos\theta-\beta\frac{1-(1-\beta^2)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{3}{2}}}\Bigg].\end{aligned} \tag{94}$$

The zero-order approximation of the cross section (92) is precisely the formula of Sauter. It should also be noticed that contrary to $\mathfrak{F}$, the corrective term $\mathfrak{G}$ does not vanish for $\theta=0,\pi$.

VI. TOTAL CROSS SECTION

Performing the angle integrations in Eq. (92), the contribution of the $\mathfrak{F}$ term, found by Sauter, is given by

$$\mathfrak{M}\equiv\frac{(1-\beta^2)^2}{\pi}\int\mathfrak{F}d\omega=\frac{4}{3}+\frac{1-3(1-\beta^2)^{\frac{1}{2}}+2(1-\beta^2)}{\beta^2(1-\beta^2)^{\frac{1}{2}}}\times\left[1+\frac{1-\beta^2}{2\beta}\ln\frac{1-\beta}{1+\beta}\right]. \tag{95}$$

The integration of $\mathfrak{G}$ can be carried out with the aid of the following formulas

$$\int_0^\pi\frac{\sin^3\theta}{(1-\beta\cos\theta)^{7/2}}d\theta=\frac{8}{15\beta^2}\left\{\frac{2}{\beta}\frac{(1-\beta)^{\frac{1}{2}}-(1+\beta)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{1}{2}}}+\frac{(1-\beta)^{\frac{3}{2}}+(1+\beta)^{\frac{3}{2}}}{(1-\beta^2)^{\frac{3}{2}}}\right\},$$

$$\int_0^\pi\frac{\sin\theta\cos\theta}{(1-\beta\cos\theta)^{\frac{5}{2}}}d\theta=\frac{4}{3\beta}\left\{\frac{1}{\beta}\frac{(1-\beta)^{\frac{1}{2}}-(1+\beta)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{1}{2}}}+\frac{1}{2}\frac{(1-\beta)^{\frac{3}{2}}+(1+\beta)^{\frac{3}{2}}}{(1-\beta^2)^{\frac{3}{2}}}\right\},$$

$$\int_0^\pi\frac{\sin\theta}{(1-\beta\cos\theta)^{\frac{5}{2}}}d\theta=-\frac{2}{3\beta}\frac{(1-\beta)^{\frac{3}{2}}-(1+\beta)^{\frac{3}{2}}}{(1-\beta^2)^{\frac{3}{2}}},$$

$$\int_0^\pi\frac{\sin\theta}{(1-\beta\cos\theta)^2}d\theta=\frac{2}{1-\beta^2},$$

$$\int_0^\pi\frac{\sin\theta\cos\theta}{(1-\beta\cos\theta)^2}d\theta=\frac{2}{\beta(1-\beta^2)}+\frac{1}{\beta^2}\ln\frac{1-\beta}{1+\beta}.$$

With the above, the contribution of $\mathfrak{G}$ becomes

$$\begin{aligned}\mathfrak{N}\equiv\frac{(1-\beta^2)^2}{\pi}\int\mathfrak{G}d\omega=\frac{[1-(1-\beta^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{\sqrt{2}\beta^2}&\Bigg\{\frac{(1-\beta)^{\frac{1}{2}}-(1+\beta)^{\frac{1}{2}}}{\beta}\left[-\frac{4}{3}+\frac{46}{15}(1-\beta^2)^{\frac{1}{2}}-\frac{8}{15}(1-\beta^2)\right]\\&+[(1-\beta)^{\frac{3}{2}}+(1+\beta)^{\frac{3}{2}}]\left[-\frac{2}{3}\frac{1}{1-\beta^2}+\frac{23}{15}\frac{1}{(1-\beta^2)^{\frac{1}{2}}}-\frac{4}{15}\right]+[(1-\beta)^{\frac{3}{2}}-(1+\beta)^{\frac{3}{2}}]\left[-\frac{2}{3}\frac{\beta}{1-\beta^2}+\left(\frac{1}{\beta}+\frac{2\beta}{3}\right)\frac{1}{(1-\beta^2)^{\frac{1}{2}}}-\frac{1}{\beta}\right]\Bigg\}\\&+\frac{1-(1-\beta^2)^{\frac{1}{2}}}{\beta^3}[(1-\beta^2)^{\frac{1}{2}}-2(1-\beta^2)]\left[1+\frac{1}{2\beta}\ln\frac{1-\beta}{1+\beta}\right].\end{aligned} \tag{96}$$

[22] The cross section in ordinary gauss-cgs units is obtained by multiplying formula (92) by $\hbar^2/c^2$ and giving α and β their usual values.

Equation (96) may be given a more suitable form, making use of the elementary identities

$$(1-\beta)^{\frac{1}{2}}-(1+\beta)^{\frac{1}{2}}=-\{2[1-(1-\beta^2)^{\frac{1}{2}}]\}^{\frac{1}{2}},$$

$$(1-\beta)^{\frac{3}{2}}-(1+\beta)^{\frac{3}{2}}=-\{2[1-(1-\beta^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}[2+(1-\beta^2)^{\frac{1}{2}}],$$

$$(1-\beta)^{\frac{3}{2}}+(1+\beta)^{\frac{3}{2}}=2\beta\{2[1-(1-\beta^2)^{\frac{1}{2}}]\}^{-\frac{1}{2}}[2-(1-\beta^2)^{\frac{1}{2}}].$$

Thus $\mathfrak{N}$ may be expressed as a function of the powers of $(1-\beta^2)^{\frac{1}{2}}$. One finds finally[23]

$$\mathfrak{N}=\frac{1}{\beta^3}\left\{-\frac{4}{15}\frac{1}{(1-\beta^2)^{\frac{1}{2}}}+\frac{34}{15}-\frac{63}{15}(1-\beta^2)^{\frac{1}{2}}+\frac{25}{15}(1-\beta^2)+\frac{8}{15}(1-\beta^2)^{\frac{3}{2}}+(1-\beta^2)^{\frac{1}{2}}[1-3(1-\beta^2)^{\frac{1}{2}}+2(1-\beta^2)]\frac{1}{2\beta}\ln\frac{1-\beta}{1+\beta}\right\}. \quad (97)$$

With Eqs. (95) and (97), the total cross section becomes

$$\sigma_k=\tfrac{3}{2}\varphi_0\alpha^4Z^5\frac{\beta^3(1-\beta^2)}{[1-(1-\beta^2)^{\frac{1}{2}}]^5}\times\left[\mathfrak{M}\left(1-\frac{\pi\alpha Z}{\beta}\right)+\pi\alpha Z\mathfrak{N}\right], \quad (98)$$

where φ_0 is the Thomson scattering cross section.

VII. DISCUSSION

We will now consider the limiting behavior of the total cross section (98). The fully nonrelativistic limit is found by retaining only the zero-order terms of the expansion of the cross section in powers of $1/c$ (that is to say, in powers of β and αZ simultaneously). To do this, the formula (98) must be considered as written in ordinary units with $\alpha=e^2/\hbar c$, $\beta=v/c$, $\varphi_0=8\pi e^4/3m^2c^4$. We then find

$$\mathfrak{M}\simeq\tfrac{4}{3},\quad \mathfrak{N}=\mathcal{O}(\beta),\quad 1-(\pi\alpha Z/\beta)\equiv 1-\pi(e^2Z/\hbar v).$$

With the above, the nonrelativistic limit of the cross section (98) becomes

$$\sigma_k{}^{\mathrm{NR}}=4\sqrt{2}\alpha^4Z^5\varphi_0\left(\frac{mc^2}{h\nu}\right)^{7/2}\left(1-\pi\frac{e^2Z}{\hbar v}\right). \quad (99)$$

This result entirely agrees with the one obtained by making the corresponding approximations in the exact nonrelativistic formula of Fischer. To facilitate the comparison, we remark that, in the Sommerfeld version of this formula,[24] the quantity τ occurring there is actually identically defined[25] with the one we introduced in (A.9)—the arc tan function being determined in the same way.[26] Hence, expanding the formula of Fischer in powers of $e^2Z/\hbar v$ one finds indeed, to first order, the expression (99).[27]

In the extreme relativistic limit $\beta\rightarrow 1$, by keeping only the lowest power of $(1-\beta^2)^{\frac{1}{2}}$, we find

$$\sigma_k{}^{\mathrm{ER}}=\tfrac{3}{2}\alpha^4Z^5\varphi_0(1-\beta^2)^{\frac{1}{2}}\left(1-\pi\alpha Z-\frac{4}{15}\pi\alpha Z\right). \quad (100)$$

The result (100) is precisely the one obtained from the extreme relativistic formula (exact in αZ) of Hall,[28] if only first order terms in αZ are retained.[29]

We finally discuss the range of validity of Eqs. (92) and (98). The way they involve the quantity $\pi\alpha Z/\beta$, as well as the aspect of Hall's formula,[28] suggest that the error with which these equations stand for the exact cross sections is of order of magnitude $(\pi\alpha Z/\beta)^2$. Equations (92) and (98) can therefore be applied to heavier elements and to smaller velocities β than those of Sauter. These conclusions are corroborated by the comparison of the values obtained from Eq. (98) with those interpolated from the exact computations of Hulme *et al.*[4] (Table I).

TABLE I. The values of $\sigma_k/\alpha^4Z^5\varphi_0$, for Al ($Z=13$) and A ($Z=18$) as given by the Sauter formula, Eq. (98), and the exact evaluation.[a]

$h\nu/mc^2$	Sauter	Formula (98)		Exact evaluation	
		Al	A	Al	A
0.693	29.47	19.7	16.2	22.3	19.9
2.21	1.68	1.19	1.00	1.24	1.12

[a] See reference 4.

VIII. ACKNOWLEDGMENT

The author wishes to express his gratitude to Professor Serban Titeica for the benefit of invaluable dis-

[23] The terms of $\mathfrak{N}$ in $1/(1-\beta^2)$ cancel.

[24] Reference 3, Chap. 6, Sec. 5, Eq. (6) for $\mu_{\mathrm{at}}=\sigma_k$.

[25] Reference 3, Eq. (VI,5,19b).

[26] Indeed, from the definition of the generalized Laguerre function $L_n(\rho)$, which appears in Eqs. (15a) and (15b) of reference 3, Chap. 6, Sec. 4, we have $L_n(0)=1$. This requires that the determination of the imaginary power $[x/(x-1)]^n$ occurring in the integrand of $L_n(\rho)$ should be chosen so that $\lim[x/(x-1)]^n=1$, for $|x|\rightarrow\infty$; hence

$$-\pi<\arg[x/(x-1)]<+\pi.$$

The same is true, using the notations of Eq. (VI,4,16e), also for $\tau\equiv\arg[x_0/(x_0-1)]$; since the imaginary part of $[x_0/(x_0-1)]$ is negative, it follows that $\tau<0$. We thus find that the determination of τ is the one given by our formula (A.9).

[27] To this end it should be noted that employing the *nonrelativistic* energy conservation relation

$$k^2(1+|n|^2)=2\kappa m,$$

where $|n|=e^2Z/\hbar v=\lambda/k$, one finds

$$\kappa^2-k^2+\lambda^2=-k^2\left[1-\frac{\kappa}{2m}+\mathcal{O}(|n|^2)\right].$$

Since in a truly nonrelativistic calculation the quantity $\kappa/2m$ should be neglected in comparison to 1, it follows that

$$\tau=\arctan 2|n|=-\pi+\mathcal{O}(|n|).$$

Hence, in our approximation: $\tau|n|=-\pi|n|$.

[28] H. Hall, Revs. Modern Phys. **8**, 358 (1936); the formula for $\tau_k=N_0\sigma_k$ on p. 395.

[29] The evaluation, to first order in αZ, of the implicit formula of Hall has been given by M. Gavrila, Nuovo cimento **9**, 327 (1958).

cussions on quantum electrodynamical topics, as well as for his kind interest in this work.

APPENDIX. EVALUATION OF THE MOMENTUM SPACE INTEGRALS

All the integrals we shall evaluate depend critically on the screening parameter μ. Since in the final result the limit $\mu \to 0$ is to be taken, it will be sufficient to evaluate them neglecting from the beginning the first and higher order terms in μ, which vanish anyway in this limit. However, the λ dependence of the constant and divergent (for $\mu \to 0$) terms in μ will be determined exactly. Eventual λ approximations will be performed only in the end. The method of integration we follow is due to Dalitz.[5]

Out of the group of integrals

$$(B_0,B_j)=\int \frac{(1,p_j)}{[(\mathbf{p}-\mathbf{k})^2+\mu^2][(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2](p^2-k^2-i\epsilon)}d^3p$$
$$=\int \frac{(1,p_j)}{[1][2](3)}d^3p, \quad \text{(A.1)}$$

where $\mathbf{k}$ and $\boldsymbol{\kappa}$ are at the outset arbitrary, we start by evaluating B_0 and shall work exactly in μ, for the time being.[30] Using one of Feynman's identities, B_0 may be written as

$$B_0=\int_0^1 dx \int \frac{1}{[(\mathbf{p}-\mathbf{P})^2+\Lambda^2]^2(p^2-k^2-i\epsilon)}d^3p$$

where we have put

$$\mathbf{P}=\boldsymbol{\kappa}x+\mathbf{k}(1-x),$$
$$\Lambda^2=-x^2(\mathbf{k}-\boldsymbol{\kappa})^2+x[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2-\mu^2]+\mu^2. \quad \text{(A.2)}$$

With one of the formulas of Dalitz,[31] B_0 becomes

$$B_0=-\frac{\pi^2}{2ki(\mathbf{k}-\boldsymbol{\kappa})^2}B_0'; \quad B_0'=\int_0^1 \frac{dx}{y(ux+v+y)}. \quad \text{(A.3)}$$

Here we have denoted by

$$y=+\left\{-x^2+x\left[1+\frac{\lambda^2-\mu^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\right]+\frac{\mu^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\right\}^{\frac{1}{2}},$$
$$u=\frac{k^2-\kappa^2-\lambda^2+\mu^2}{2ki|\mathbf{k}-\boldsymbol{\kappa}|}, \quad v=-\frac{\mu^2}{2ki|\mathbf{k}-\boldsymbol{\kappa}|}. \quad \text{(A.4)}$$

Changing the integration variable according to $x=(x_1t^2+x_2)/(t^2+1)$ where x_1 and $x_2 (x_1>x_2)$ are the roots of equation $y^2=0$, and putting

$$t_{1,2}=\frac{(x_1-x_2)\pm[(x_1-x_2)^2-4(x_1u+v)(x_2u+v)]^{\frac{1}{2}}}{2(x_1u+v)},$$
$$t'=-\left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}, \quad t''=-\left(\frac{1-x_2}{x_1-1}\right)^{\frac{1}{2}}, \quad \text{(A.5)}$$

we find

$$B_0'=-\frac{2}{(x_1u+v)(t_1-t_2)}\int_{t'}^{t''}\left(\frac{1}{t-t_1}-\frac{1}{t-t_2}\right)dt$$
$$=-\frac{2}{[(x_1-x_2)^2-4(x_1u+v)(x_2u+v)]^{\frac{1}{2}}}\ln T. \quad \text{(A.6)}$$

We now proceed to approximate calculations in μ. Using the explicit form of the quantities appearing in (A.5), we obtain

$$T\equiv\frac{(t''-t_1)(t'-t_2)}{(t''-t_2)(t'-t_1)}=\mu\frac{k^2-\kappa^2-\lambda^2+2ki\lambda}{2ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}+\mathcal{O}(\mu^2). \quad \text{(A.7)}$$

With (A.3), (A.6), and (A.7) the integral B_0 then becomes

$$B_0=\frac{\pi^2}{ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}\ln\left(\mu\frac{k^2-\kappa^2-\lambda^2+2ki\lambda}{2ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}\right)+\mathcal{O}(\mu). \quad \text{(A.8)}$$

We must now specify which of the many values of the logarithm appearing in (A.8) has to be chosen. To this end we have to follow the variation of the arguments of $(t-t_1)$ and $(t-t_2)$ along the integration path in (A.6). One finds that, whatever the sign of $(\kappa^2-k^2+\lambda^2)$, one may write

$$\ln T=\ln|T|+i(\tau+\tfrac{1}{2}\pi)+\mathcal{O}(\mu);$$
$$-\pi\leqslant\tau\equiv\arctan\left(\frac{-2k\lambda}{\kappa^2-k^2+\lambda^2}\right)\leqslant 0. \quad \text{(A.9)}$$

It should be noticed that B_0 of (A.8) is divergent in the limit of no screening ($\mu\to 0$).

The group of integrals (A.1) occurs in Eqs. (28) and (38), where k and κ are now related by (3) and (4). In this case the result is required only to lowest order in λ. B_0 of (A.8) becomes then[32]

$$B_0=\frac{\pi^2}{ki(\mathbf{k}-\boldsymbol{\kappa})^2}\ln\left(\mu\frac{k^2-\kappa^2}{2ki(\mathbf{k}-\boldsymbol{\kappa})^2}\right). \quad \text{(A.10)}$$

The imaginary part of the logarithm of (A.10) is given by (A.9), taken for $\lambda\to 0$ and our case $\kappa^2-k^2+\lambda^2<0$; we thus find $\tau=-\pi$. For the real part of B_0 of Eq.

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[30] This integral has been calculated also in reference 5, Appendix, under an assumption ($\lambda=\mu\to 0$) not suitable for our case. Besides, the formula (A.9) derived there is misprinted, the conclusions drawn being correct.

[31] Reference 5, Eq. (A.3).

[32] According to (4), for relativistic energies the difference $k^2-\kappa^2$ is of the order m^2; $k\lambda$ and λ^2 will then be, respectively, quantities of first and second order. See also reference 20.

(A.10) we accordingly obtain the value, finite in the limit $\mu \to 0$,[33]

$$\mathfrak{B} \equiv \mathrm{Re}B_0 = -\pi^3/[2k(\mathbf{k}-\boldsymbol{\kappa})^2]. \tag{A.11}$$

The integral B_j of (A.1) may be put into the form[34]

$$2B_j = \xi k_j + \eta \kappa_j + \zeta(\mathbf{k}\times\boldsymbol{\kappa})_j. \tag{A.12}$$

It may be shown that in our case the coefficients ξ and η are given by

$$\begin{aligned}\xi &= \frac{1}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]}\{\kappa^2[(2k^2+\mu^2)B_0 - C_1 + C_3] \\ &\qquad -(\mathbf{k}\cdot\boldsymbol{\kappa})[(\kappa^2+k^2+\lambda^2)B_0 - C_2 + C_3]\}, \\ \eta &= \frac{1}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]}\{k^2[(k^2+\kappa^2+\lambda^2)B_0 - C_2 + C_3] \\ &\qquad -(\mathbf{k}\cdot\boldsymbol{\kappa})[(2k^2+\mu^2)B_0 - C_1 + C_3]\},\end{aligned} \tag{A.13}$$

(where we have put $\epsilon=0$), whereas $\zeta=0$. In (A.13) the following notations have been introduced:

$$C_1 = \int \frac{d^3p}{[2](3)},\quad C_2 = \int \frac{d^3p}{[1](3)},\quad C_3 = \int \frac{d^3p}{[1][2]}. \tag{A.14}$$

Using one of Dalitz's formulas,[35] the integrals C_1 and C_2 become

$$C_1 = -\frac{i\pi^2}{\kappa}\ln\frac{k-\kappa+i\lambda}{k+\kappa+i\lambda},\quad C_2 = -\frac{i\pi^2}{k}\ln\frac{i\mu}{2k+i\mu}. \tag{A.15}$$

The analysis of the quoted formula[35] shows that the imaginary part of the logarithm appearing in C_1 is $i\mathcal{O}(\lambda)$, while that of the one appearing in C_2 is: $i[(\pi/2)+\mathcal{O}(\mu)]$.

The integral C_3 may be reduced, by use of one of the Feynman identities, to

$$C_3 = \int_0^1 dx \int \frac{d^3p}{[(\mathbf{p}-\mathbf{P})^2+\Lambda^2]^2} = \pi^2 \int_0^1 \frac{dx}{\Lambda}, \tag{A.16}$$

the notations being the same as in (A.2). Changing the integration variable in the same way as for B_0' and using the notations of (A.4) and (A.5), C_3 becomes

$$C_3 = \frac{\pi^2}{|\mathbf{k}-\boldsymbol{\kappa}|}\int_0^1 \frac{dx}{y} = \frac{i\pi^2}{|\mathbf{k}-\boldsymbol{\kappa}|}\int_0^1 \left(\frac{1}{t-i} - \frac{1}{t+i}\right)dt.$$

We obtain finally, after neglecting the first-order terms in μ,

$$C_3 = \frac{i\pi^2}{|\mathbf{k}-\boldsymbol{\kappa}|}\ln\frac{\lambda - i|\mathbf{k}-\boldsymbol{\kappa}|}{\lambda + i|\mathbf{k}-\boldsymbol{\kappa}|} + \mathcal{O}(\mu). \tag{A.17}$$

Following the variation of the arguments of $(t-i)$ and $(t+i)$ along the integration path, we find for the imaginary part of the logarithm of (A.17) the value $i[-\pi+\mathcal{O}(\lambda)]$.

To lowest order in λ, the integrals C_1 and C_3 of (A.15) and (A.16) yield

$$C_1 = -\frac{i\pi^2}{\kappa}\ln\frac{k-\kappa}{k+\kappa},\quad C_3 = \frac{\pi^3}{|\mathbf{k}-\boldsymbol{\kappa}|} \equiv \mathfrak{C}. \tag{A.18}$$

In this case we note that C_1 is purely imaginary, whereas C_3 is real.[36]

The expression for B_j, to zero order in μ exact in what concerns λ, is obtained combining Eqs. (A.12), (A.13), (A.8), (A.15), and (A.17). In contrast to that, to lowest order in λ, B_j is expressed with the aid of (A.12), ξ and η being given by

$$\begin{aligned}\xi &= 2B_0 - \frac{1}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]} \\ &\quad\times\left\{\kappa^2(C_1-C_3) + (\mathbf{k}\cdot\boldsymbol{\kappa})C_3 - \frac{i\pi^2}{k}(\mathbf{k}\cdot\boldsymbol{\kappa})\ln\frac{\kappa^2-k^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\right\}, \\ \eta &= \frac{1}{[k^2\kappa^2-(\mathbf{k}\boldsymbol{\kappa})^2]} \\ &\quad\times\left\{(\mathbf{k}\cdot\boldsymbol{\kappa})(C_1-C_3) + k^2C_3 - i\pi^2 k\ln\frac{\kappa^2-k^2}{(\mathbf{k}-\boldsymbol{\kappa})^2}\right\}.\end{aligned} \tag{A.19}$$

In these formulas, the expressions (A.10) and (A.18) should be used for B_0, C_1, and C_3, the imaginary part of $\ln[(\kappa^2-k^2)/(\mathbf{k}-\boldsymbol{\kappa})^2]$ being $-i\pi$. One thus sees that B_j is divergent in the limit $\mu\to 0$. However, $\mathrm{Re}B_j$ is finite in this limit; indeed from (A.12) and (A.19) we find

$$\begin{aligned}\mathrm{Re}\mathbf{B} &= \mathfrak{B}_k\mathbf{k} + \mathfrak{B}_\kappa\boldsymbol{\kappa}; \\ \mathfrak{B}_k &= \mathfrak{B} + \frac{1}{2}\frac{(\kappa^2-\mathbf{k}\cdot\boldsymbol{\kappa})\mathfrak{C} + \pi^3\mathbf{k}\cdot\boldsymbol{\kappa}/k}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}, \\ \mathfrak{B}_\kappa &= \frac{1}{2}\frac{(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})\mathfrak{C} - \pi^3 k}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2},\end{aligned} \tag{A.20}$$

where we have introduced $\mathfrak{C}$ defined in (A.18).

We now proceed to the evaluation of the integrals

$$(A_0, A_j, A_{ij}) = \int \frac{(1, p_j, p_ip_j)}{[(\mathbf{p}-\mathbf{k})^2+\mu^2][(\mathbf{p}-\boldsymbol{\kappa})^2+\lambda^2]^2(p^2-k^2-i\epsilon)}d^3p, \tag{A.21}$$

where $\mathbf{k}$ and $\boldsymbol{\kappa}$ are again at the outset arbitrary. This

[33] The conclusions drawn from (A.10) and (A.9) for $\mathrm{Re}B_0$ are in agreement with those obtained in reference 5, following Eq. (A.9).
[34] Reference 6, Appendix.
[35] Reference 5, Eq. (A.2).
[36] The results (A.18) and others that can be derived from (A.15) reduce, in particular cases, to those of reference 6, Eq. (A.14).

task reduces to that of the evaluation of the integrals in (A.1), since

$$A_0=-\frac{1}{2\lambda}\frac{\partial B_0}{\partial\lambda},\quad A_j=\kappa_jA_0+\frac{1}{2}\frac{\partial B_0}{\partial\kappa_j}=-\frac{1}{2\lambda}\frac{\partial B_j}{\partial\lambda},$$
$$A_{ij}=\kappa_jA_i+\frac{1}{2}\frac{\partial B_i}{\partial\kappa_j}. \tag{A.22}$$

In evaluating A_0, one must use, on account of the differentiation involved, the expression (A.8) for B_0, exact in λ except for terms of order $\mathcal{O}(\mu)$. (Differentiation with respect to λ does not change the orders of magnitude in μ). As seen from (A.22), A_j may be calculated in two different ways, the first alternative being the shorter one.[37] We thus find for A_0 and A_j

$$A_0=-\frac{1}{\lambda}\frac{\pi^2}{ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}\times\left[\frac{ki-\lambda}{k^2-\kappa^2-\lambda^2+2ki\lambda}-\frac{\lambda}{(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2}\right]+\frac{\pi^2}{ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]^2}\ln\left(\mu\frac{k^2-\kappa^2-\lambda^2+2ki\lambda}{2ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}\right)+\mathcal{O}(\mu), \tag{A.23}$$

$$A_j=-\frac{1}{\lambda}\kappa_j\frac{\pi^2}{(k^2-\kappa^2-\lambda^2+2ki\lambda)[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}+k_j\frac{\pi^2}{ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]^2}\left[1+\ln\left(\mu\frac{k^2-\kappa^2-\lambda^2+2ki\lambda}{2ki[(\mathbf{k}-\boldsymbol{\kappa})^2+\lambda^2]}\right)\right]+\mathcal{O}(\mu), \tag{A.24}$$

where the logarithm has the same imaginary part as in (A.9). The integrals A_0 and A_j occur in Eqs. (36) and (37), which should be evaluated to zero order in λ. We obtain for their real parts, in this approximation,

$$\mathrm{Re}A_0=-\frac{1}{\lambda}\frac{\pi^2}{(\mathbf{k}-\boldsymbol{\kappa})^2(k^2-\kappa^2)}+\mathrm{Re}A_0^{(0)},$$
$$\mathrm{Re}A_j=-\frac{1}{\lambda}\kappa_j\frac{\pi^2}{(\mathbf{k}-\boldsymbol{\kappa})^2(k^2-\kappa^2)}+\mathrm{Re}A_j^{(0)}, \tag{A.25}$$

$$\mathrm{Re}A_0^{(0)}\equiv\mathcal{A}=-\frac{\pi^3}{2k(\mathbf{k}-\boldsymbol{\kappa})^4},\quad \mathrm{Re}A_j^{(0)}=\mathcal{A}k_j, \tag{A.26}$$

where now k and κ are related by (4).

The expression of the integral A_{ij}, appearing in (37), is needed itself only to zero order in μ and λ. Hence, for evaluating its real part, in this approximation, use may be made of Eq. (A.22) combined with (A.20) and (A.25). We thus find

$$\mathrm{Re}A_{ij}=\mathcal{R}\kappa_i\kappa_j+\mathcal{S}k_i\kappa_j+\mathcal{T}\kappa_ik_j+\mathcal{U}k_ik_j+\mathcal{V}\delta_{ij}, \tag{A.27}$$

where we have denoted

$$\mathcal{R}=-\frac{1}{\lambda}\frac{\pi^2}{(\mathbf{k}-\boldsymbol{\kappa})^2(k^2-\kappa^2)}-\tfrac{1}{4}\mathcal{C}\left[\frac{2k^2(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}+\frac{k^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](\mathbf{k}-\boldsymbol{\kappa})^2}\right]+\tfrac{1}{2}\pi^3\frac{k^3}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}, \tag{A.28}$$

$$\mathcal{S}=\tfrac{1}{4}\mathcal{C}\left[\frac{2(\mathbf{k}\cdot\boldsymbol{\kappa})(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}-\frac{\kappa^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](\mathbf{k}-\boldsymbol{\kappa})^2}\right]-\tfrac{1}{2}\pi^3\frac{k(\mathbf{k}\cdot\boldsymbol{\kappa})}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}, \tag{A.29}$$

$$\mathcal{T}=\tfrac{1}{4}\mathcal{C}\left[\frac{2k^2(\mathbf{k}\cdot\boldsymbol{\kappa})-(\mathbf{k}\cdot\boldsymbol{\kappa})^2-k^2\kappa^2}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}+\frac{k^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](\mathbf{k}-\boldsymbol{\kappa})^2}\right]-\tfrac{1}{2}\pi^3\frac{k(\mathbf{k}\cdot\boldsymbol{\kappa})}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}, \tag{A.30}$$

$$\mathcal{U}=\mathcal{A}+\tfrac{1}{4}\mathcal{C}\left[\frac{2\kappa^2(\mathbf{k}\cdot\boldsymbol{\kappa})-(\mathbf{k}\cdot\boldsymbol{\kappa})^2-k^2\kappa^2}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}+\frac{\kappa^2-\mathbf{k}\cdot\boldsymbol{\kappa}}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2](\mathbf{k}-\boldsymbol{\kappa})^2}\right]+\frac{\pi^3}{4k}\frac{k^2\kappa^2+(\mathbf{k}\cdot\boldsymbol{\kappa})^2}{[k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2]^2}, \tag{A.31}$$

$$\mathcal{V}=\frac{1}{4}\left[\frac{(k^2-\mathbf{k}\cdot\boldsymbol{\kappa})\mathcal{C}-\pi^3k}{k^2\kappa^2-(\mathbf{k}\cdot\boldsymbol{\kappa})^2}\right]. \tag{A.32}$$

It should be noted that $\mathcal{S}=\mathcal{T}$ as it should, since $A_{ij}=A_{ji}$; also, out of the preceding quantities, only $\mathcal{R}$ has a term in $1/\lambda$, which does thus not appear in the expression of $\mathrm{Re}A_{ij}^{(0)}$. From the above formulas one may derive the result

$$\mathrm{Re}\sum_{j=1}^{3}A_{jj}=-\frac{1}{\lambda}\frac{\pi^2}{(\mathbf{k}-\boldsymbol{\kappa})^2(k^2-\kappa^2)}\kappa^2+k^2\mathcal{A};$$
$$\mathrm{Re}\sum_{j=1}^{3}A_{jj}^{(0)}=k^2\mathcal{A}. \tag{A.33}$$

As one sees, while the integrals (A.21) are divergent for $\mu\to 0$, their real parts (A.25) and (A.27) are finite in this limit.

[37] In the second alternative, the expression for B_j exact in λ [to terms of order $\mathcal{O}(\mu)$] must be used. One ascertains that both alternatives lead to the same result.