# Theory of Ultrasonic Absorption in Metals: the Collision-Drag Effect

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A basic assumption of the semiclassical treatments of ultrasonic absorption in metals is that of "collision drag." This assumption states that, in the presence of an impressed ultrasonic wave, the velocity distribution toward which the conduction electrons relax is a Fermi distribution centered, not at the origin of velocity space, but at a point, $\mathbf{v}_i$, equal to the local, impressed lattice-displacement velocity. In the present paper, the explanation of this collision-drag effect in terms of basic electron-lattice theory is investigated for the case of collisions with thermal vibrations. The effect is found to originate from those higher-order terms in the electron-lattice interaction whose matrix elements are bilinear in the amplitudes of impressed and thermal lattice displacements. In the conventional perturbation-theory treatment, these matrix elements give rise to transitions in which both a thermal phonon and an impressed phonon are simultaneously either absorbed or emitted. However, in such a treatment, no collision-drag effects appear. In order to obtain them, it is necessary to alter the standard perturbation treatment so as to provide for space-time localization of collisions to within an interval small compared to the wavelength and period of the impressed sound wave. It is then found that the impressed ultrasonic wave produces a modification in the energy-conservation law of electron-lattice collisions, $\epsilon_{\mathbf{k}'}=\epsilon_{\mathbf{k}}\pm\hbar\omega_\lambda$, in which the effective electron energy, $\epsilon_{\mathbf{k}}$, is the Bloch energy, augmented by a term proportional to the impressed displacement velocity [Eq. (2.30)]. When this modification is introduced into the collision term of the Boltzmann transport equation, the equilibrium distribution (defined as that for which the collision term vanishes) is found to be a Fermi distribution centered about a point in $\mathbf{k}$ space equal to $m\mathbf{v}_i/\hbar$; for the free-electron model, this result is in accord with the collision-drag assumption as stated above. An additional result of the treatment is that the crystal-momentum conservation law of electron-lattice collisions is altered by the inclusion of terms linearly proportional to the impressed strain; this modification, however, turns out to have no effect in ultrasonic absorption. The final section of the paper is devoted to an investigation of the effects on energy transfer arising from the bilinear matrix elements; these effects are shown to be describable, also, in terms of the collision-drag picture.

## INTRODUCTION

IN theoretical treatments of ultrasonic absorption in metals the interaction of the impressed ultrasonic wave with the conduction electrons of the metallic sample is generally assumed to take place in two ways. One of these is via the standard first-order electron-lattice interaction linear in the lattice-displacement amplitude; it is in fact the same interaction as that responsible for the scattering of electrons by the thermal vibrations of the lattice.

A second basic mechanism of interaction—designated in this paper as the *collision-drag effect*—comes into play when the electrons undergo collisions either with impurity centers or with thermal lattice vibrations. In published theoretical treatments this interaction has generally been assumed[1] to take the following form. Namely in a lattice which is vibrating under the action of an impressed sound wave, the velocity distribution towards which the electrons relax, by virtue of collisions, is a *displaced* Fermi distribution centered, not at the origin of velocity space—as would be the case in a lattice at rest—but rather at a point, $\mathbf{v}_i$, equal to the instantaneous local, lattice-displacement velocity. This assumption corresponds to a physical picture in which the electron is viewed as colliding with an effective "collision" center; the latter, oscillating in response to the impressed wave, "drags" the electrons with it.

In the present paper, the validity of the collision-drag picture will be investigated from the standpoint of basic electron-lattice interaction theory. The treatment will be focused primarily on the case of collisions with thermal vibrations. In the case of electron-impurity encounters, the circumstance that the collision centers themselves move with the local lattice displacement velocity provides an intuitively obvious explanation of collision drag. For electron-phonon collisions, however, the situation is not comparably obvious. In particular, if thermal and impressed vibrations are assumed not to be directly coupled to each other—i.e., if anharmonic terms in the lattice displacement energy are ignored—it would seem that the collisions of electrons with thermal phonons are not influenced at all by the impressed sound wave. This conclusion is valid, however, only in the approximation in which electron-lattice interactions are treated to the first order in the lattice displacements. When account is taken of higher-order interactions, specifically those *bilinear* in the amplitudes of the two types of displacements—thermal and externally impressed—collision-drag effects are obtained.

In Sec. 1, the matrix elements of the bilinear electron-lattice interaction are computed explicitly; they assume a relatively simple form for the case of interest, in which the wave vector of the impressed vibration, $\mathbf{q}_i$, is small compared to that of a typical thermal vibration, $\mathbf{q}_\lambda$. The introduction of these matrix elements into the conventional first-order perturbation calculation—that of transitions between plane-wave Bloch states—yields nonvanishing probabilities for transitions of the type $\mathbf{k}\rightarrow\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i$, in which an impressed and a thermal phonon are simultaneously either absorbed or emitted. This calculation is carried out in Appendix II, together with the resultant energy transfer from the impressed ultrasonic wave to the combined system of conduction electrons and thermal phonons.

As pointed out at the beginning of Sec. 2, however, such a treatment does not, and, in fact, cannot be ex-

[1] A. B. Pippard, Phil. Mag. 41, 1104 (1955).

pected to yield collision-drag effects. The reason is as follows. Collision drag by its very nature—namely, its dependence on the instantaneous local displacement velocity—implies a physical description in which collision events are localized to within a space-time interval small compared to the ultrasonic wavelength and period. On the other hand, a calculation of transitions of the type $\mathbf{k}\rightarrow\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i$ implies a physical description in which the crystal momenta and energies of the electrons are defined to within intervals small compared to $q_i$ and $\hbar\omega_i$, respectively ($\omega_i$= ultrasonic frequency). By virtue of the uncertainty principle, these two descriptions are not simultaneously realizable. In order to explain collision drag, one requires an approach which features localizability, and which therefore makes no predictions about the probabilities of individual impressed-phonon processes. An approach of this type is developed in Sec. 2. It consists of a modification of the conventional first-order perturbation treatment, in which the requisite collision localization is achieved by the use of Bloch wave packets (in place of states of sharply defined crystal momentum). The results of the treatment are contained in explicit expressions [Eqs. (2.35), (2.36), and (2.37)] for the probabilities of transitions, $\mathbf{k}\rightarrow\mathbf{k}\pm\mathbf{q}_\lambda$, in which a thermal phonon is either emitted or absorbed. The influence of the bilinear matrix elements is found to manifest itself in an alteration of the energy and crystal-momentum conservation laws of electron-lattice scattering. Of particular relevance for collision drag is the energy conservation law, $\epsilon_{\mathbf{k}'}=\epsilon_{\mathbf{k}}\pm\hbar\omega_\lambda$, in which each "effective" energy, $\epsilon_{\mathbf{k}}$, is equal to the corresponding Bloch energy, $E_{\mathbf{k}}$, minus a term $m\mathbf{v}_i\cdot\mathbf{v}_{\mathbf{k}}$ (where $\mathbf{v}_i$ is the above-defined local impressed displacement velocity, $m$ the electron mass, and $\mathbf{v}_{\mathbf{k}}\equiv(1/\hbar)\,\mathrm{grad}_{\mathbf{k}}E_{\mathbf{k}}$ the expectation velocity in a Bloch state, $\mathbf{k}$). The connection between this result and the collision-drag effect is established in Sec. 3 via the introduction of the transition probability expressions of Sec. 2 into the collision term, $(\partial f/\partial t)_{\mathrm{coll}}$, of the Boltzmann transport equation. Defining the "relaxed" distribution as that for which $(\partial f/\partial t)_{\mathrm{coll}}=0$, one finds that it is equal to a Fermi distribution, $f_0(\epsilon_{\mathbf{k}})$, differing from the conventional one in that the energy argument is $\epsilon_{\mathbf{k}}$ rather than $E_{\mathbf{k}}$; this replacement corresponds to the fact that $\epsilon_{\mathbf{k}}$ (rather than $E_{\mathbf{k}}$) is a so-called "summational invariant" of electron-lattice collisions. Taking account of the fact that, to the first order in the amplitude of the impressed ultrasonic wave, $\epsilon_{\mathbf{k}}=E_{\mathbf{k}}-m\mathbf{v}_i/\hbar$, one sees that $f_0(\epsilon_{\mathbf{k}})$ is a Fermi distribution whose centroid in $\mathbf{k}$ space is located at the point $m\mathbf{v}_i/\hbar$. This result constitutes a generalized statement of the collision-drag effect, valid for arbitrary $\mathbf{k}$ dependence of $E_{\mathbf{k}}$; for the special case of free electrons, the corresponding centroid in velocity space is $\mathbf{v}_i$, in accord with Pippard's original statement[1] of the effect as given above.

The final section of the paper is devoted to an investigation of the effects on energy transfer arising from the bilinear interaction; it is shown that these effects are also describable in terms of the collision-drag picture.

## 1. BILINEAR ELECTRON-PHONON INTERACTION

Let it be assumed that, in addition to the impressed sound wave, a single lattice vibration mode is thermally excited. The displacement, $\mathbf{R}_{\mathbf{g}}$, of the $\mathbf{g}$th lattice site is then a sum of the "impressed" displacement

$$\mathbf{R}_{\mathbf{g},i}=\mathbf{u}_i\exp[i(\mathbf{q}_i\cdot\mathbf{g}-\omega_i t)], \tag{1.1}$$

and the "thermal" displacement

$$R_{\mathbf{g},\lambda}=\mathbf{u}_\lambda\exp[i(\mathbf{q}_\lambda\cdot\mathbf{g}-\omega_\lambda t)]. \tag{1.2}$$

In these expressions, the $\mathbf{u}$'s are complex vector amplitudes, the $\mathbf{q}$'s wave vectors, and the $\omega$'s the associated frequencies (multiplied by $2\pi$). The subscripts $i$ and $\lambda$ refer to the impressed wave and to the $\lambda$th (traveling) mode of thermal vibration. Finally, it is to be understood that (1) and (2) are each to be supplemented by their complex conjugates, in order that $\mathbf{R}_{\mathbf{g},i}$ and $\mathbf{R}_{\mathbf{g},\lambda}$ be real[2]; this prescription will be understood to apply to all equations containing complex lattice-displacement amplitudes.

In the presence of the displacements (1.1) and (1.2), the one-electron potential takes the form

$$\begin{aligned}V(\mathbf{r})=V_0(\mathbf{r})&+\sum_{\mathbf{g}}\exp[i(\mathbf{q}_\lambda\cdot\mathbf{g}-\omega_\lambda t)]\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}}V(\mathbf{r})\\&+\sum_{\mathbf{g}}\exp[i(\mathbf{q}_i\cdot\mathbf{g}-\omega_i t)]\mathbf{u}_i\cdot\mathrm{grad}_{\mathbf{g}}V(\mathbf{r})\\&+\sum_{\mathbf{g},\mathbf{g}'}\exp[i(\mathbf{q}_\lambda\cdot\mathbf{g}-\omega_\lambda t)]\exp[i(\mathbf{q}_i\cdot\mathbf{g}'-\omega_i t)]\\&\times(\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}})(\mathbf{u}_i\cdot\mathrm{grad}_{\mathbf{g}'})V(\mathbf{r})\\&+(\cdots).\end{aligned} \tag{1.3}$$

Here, $V(\mathbf{r})$ is to be regarded as a function of the electron coordinate, $\mathbf{r}$, and of the displacements of the individual lattice sites, $\mathbf{R}_{\mathbf{g}}$; $V_0(\mathbf{r})$ denotes the electron potential of an undisturbed lattice ($\mathbf{R}_{\mathbf{g}}=0$); and $\mathrm{grad}_{\mathbf{g}}$ means the gradient in $\mathbf{R}_{\mathbf{g}}$ space. The second and third terms give the conventional electron-lattice interaction in terms of the amplitudes of thermal and impressed waves, respectively; the new, "bilinear" potential, which is the subject matter of this section, is represented by the fourth term; it will henceforth be denoted by the symbol $V_{i\lambda}(\mathbf{r})$. Finally, the terms in (1.3) denoted by $(\cdots)$ are either of third or higher order in the $\mathbf{u}$'s, or of second order in $\mathbf{u}_\lambda$; in the latter case, the inclusion of such terms would only result in corrections to the standard electron-lattice relaxation theory, uninfluenced by the presence of the impressed wave.[3]

[2] For the time being, the vector amplitudes, $\mathbf{u}_\lambda$, $\mathbf{u}_\lambda^*$, $\mathbf{u}_i$, $\mathbf{u}_i^*$, will be treated as classical amplitudes. In the eventual quantization process, the positive exponentials of (1.1) and (1.2) get associated with phonon absorption, their complex conjugates with phonon emission.

[3] Under this category one should also include terms proportional to the products, $\mathbf{u}_\lambda\mathbf{u}_{\lambda'}$, of the displacement amplitudes of different lattice-vibration modes; these terms are also uninfluenced by the impressed wave, and hence make no contribution to collision-drag effects.

Turning to $V_{i\lambda}(\mathbf{r})$, one may write this term as

$$V_{i\lambda}(\mathbf{r})=\sum_{\mathbf{gh}}\exp\{i[(\mathbf{q}_i+\mathbf{q}_\lambda)\cdot\mathbf{g}-(\omega_i+\omega_\lambda)t]\}\times\exp(i\mathbf{q}_i\cdot\mathbf{h})(\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g})(\mathbf{u}_i\cdot\mathrm{grad}_{\mathbf{g}+\mathbf{h}})V(\mathbf{r}),$$

where

$$\mathbf{h}\equiv\mathbf{g}'-\mathbf{g} \tag{1.4}$$

is a relative lattice-site vector. Let it now be assumed that $(\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g})(\mathbf{u}_i\cdot\mathrm{grad}_{\mathbf{g}+\mathbf{h}})V(\mathbf{r})$ is "short-range" in that it vanishes for sufficiently large $|\mathbf{h}|\equiv h_{\max}$. Then, if

$$q_i h_{\max}\ll 1, \tag{1.5}$$

as will here be assumed (the assumption being considered appropriate in view of the extreme largeness of the ultrasonic wavelength, $2\pi/q_i$, in comparison to lattice spacings), one may replace the factor $\exp(i\mathbf{q}_i\cdot\mathbf{h})$ by unity in (1.4). Then, upon applying the operator relation

$$\sum_{\mathbf{h}}\mathrm{grad}_\mathbf{h}+\mathrm{grad}_\mathbf{r}=0, \tag{1.6}$$

which is a statement of the invariance of $V(\mathbf{r})$ with respect to a rigid displacement of all particles (lattice atoms and electrons), one has

$$V_{i\lambda}(\mathbf{r})=-\sum_{\mathbf{g}}\exp\{i[(\mathbf{q}_i+\mathbf{q}_\lambda)\cdot\mathbf{g}-(\omega_i+\omega_\lambda)t]\}\times(\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g})(\mathbf{u}_i\cdot\mathrm{grad}_\mathbf{r})V(\mathbf{r})$$
$$=\sum_{\mathbf{g}}\exp\{i[(\mathbf{q}_i+\mathbf{q}_\lambda)\cdot\mathbf{g}-(\omega_i+\omega_\lambda)t]\}V_{i\lambda}{}^{\mathbf{g}}(\mathbf{r}), \tag{1.7}$$

where

$$V_{i\lambda}{}^{\mathbf{g}}(\mathbf{r})=-(\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g})(\mathbf{u}_i\cdot\mathrm{grad}_\mathbf{r})V(\mathbf{r}). \tag{1.8}$$

Having obtained the "bilinear" potential in the form given by (1.7) and (1.8), one may now proceed to compute its matrix element, $(n\mathbf{k}'|V_{i\lambda}{}^{(1)}|n\mathbf{k})$, between two Bloch states, $\psi_{n\mathbf{k}}$ and $\psi_{n\mathbf{k}'}$, i.e., states of differing crystal momentum, but belonging to the same band.[4] Using the operator relation

$$\mathrm{grad}_\mathbf{r}(\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(r))=\frac{im}{\hbar}[\mathbf{v},\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})], \tag{1.9}$$

where the square-bracketed expression is the commutator of the velocity operator, $\mathbf{v}=(\hbar/im)\,\mathrm{grad}_\mathbf{r}$, with the operator $\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})$, one has

$$(n\mathbf{k}'|V_{i\lambda}{}^{\mathbf{g}}|n\mathbf{k})=-\sum_{n'}\frac{im}{\hbar}\{(n|\mathbf{u}_i\cdot\mathbf{v}_{\mathbf{k}'}|n')(n'\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})|n\mathbf{k})-(n\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})|n'\mathbf{k})(n'|\mathbf{u}_i\cdot\mathbf{v}_\mathbf{k}|n)\}, \tag{1.10}$$

where use has been made of the circumstance that $\mathbf{v}$ is diagonal in $\mathbf{k}$.

[4] As will be seen later, this matrix element represents only part of the total *effective* bilinear interaction; hence, the use of the parenthesized superscript (1).

A useful simplification of (1.10) may be achieved by noting that

$$(n'\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})|n\mathbf{k})\equiv\int\psi_{n'\mathbf{k}'}{}^*(\mathbf{r})[\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r})]\psi_{n\mathbf{k}}(\mathbf{r})d\mathbf{r}$$
$$=\int\psi_{n'\mathbf{k}'}{}^*(\mathbf{r}+\mathbf{q})[\mathbf{u}_\lambda\cdot\mathrm{grad}_\mathbf{g}V(\mathbf{r}+\mathbf{q})]\psi_{n\mathbf{k}}(\mathbf{r}+\mathbf{q})d\mathbf{r}$$
$$=\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{q}]\int\psi_{n'\mathbf{k}'}{}^*(\mathbf{r})\mathbf{u}_\lambda\,\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})\psi_{n\mathbf{k}}(\mathbf{r})d\mathbf{r}$$
$$\equiv\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{q}](n'\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|n\mathbf{k}), \tag{1.11}$$

where $R_{\mathbf{g}=0}$ denotes the displacement of the "zeroth" site, located at an arbitrary origin; the next to the last equality of (1.11) obtains by virtue of the translational symmetry properties of potential and wave functions. Introducing (1.11) and (1.10) into (1.8), one then has

$$(n\mathbf{k}'|V_{i\lambda}{}^{(1)}|n\mathbf{k})=-\frac{imN}{\hbar}\delta_{\mathbf{k}+\mathbf{q}_i+\mathbf{q}_\lambda,\mathbf{k}'}\exp[-i(\omega_i+\omega_\lambda)t]\times\sum_{n'}\{(n|\mathbf{u}_i\cdot\mathbf{v}_{\mathbf{k}'}|n')(n'\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|n\mathbf{k})-(n\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|n'\mathbf{k})(n'|\mathbf{u}_i\cdot\mathbf{v}_\mathbf{k}|n)\}, \tag{1.12}$$

where $N$ is the number of unit cells in the sample, and where $\delta_{\mathbf{k},\mathbf{k}'}$ indicates the Kronecker delta symbol: unity for $\mathbf{k}'=\mathbf{k}$, zero otherwise.

At this point, it is necessary to recognize that, as indicated by the parenthesized superscript (1), the matrix elements $(n\mathbf{k}'|V_{i\lambda}{}^{(1)}|n\mathbf{k})$ do not by themselves provide a complete description of bilinear transitions, i.e., those transitions in which both an impressed and a thermal phonon are absorbed (or emitted). Transition amplitudes, $(n\mathbf{k}'|V_{i\lambda}{}^{(2)}|n\mathbf{k})$, of comparable strength arise from two-stage processes, each stage consisting of an interband (and hence virtual) transition, the matrix elements for which are provided by the linear terms of (1.3). Denoting the second and third terms of (1.3) by the symbols $V_\lambda(\mathbf{r})$ and $V_i(\mathbf{r})$, respectively, one has, according to standard second-order perturbation theory,

$$(n\mathbf{k}'|V_{i\lambda}{}^{(2)}|n\mathbf{k})=\sum_{n'\neq n}\frac{(n\mathbf{k}'|V_\lambda|n',\mathbf{k}+\mathbf{q}_i)(n',\mathbf{k}+\mathbf{q}_i|V_i|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}-\hbar\omega_i}+\sum_{n'\neq n}\frac{(nk'|V_i|n',\mathbf{k}'-\mathbf{q}_i)(n',k'-\mathbf{q}_i|V_\lambda|n\mathbf{k})}{E_{n\mathbf{k}'}-E_{n',\mathbf{k}'-\mathbf{q}_i}+\hbar\omega_i}. \tag{1.13}$$

In writing down (1.13), use has been made of the crystal-momentum selection rule for the matrix elements $(n'\mathbf{k}''|V_i|n\mathbf{k})$, according to which $\mathbf{k}''=\mathbf{k}+\mathbf{q}_i$. It should furthermore be understood that $E_{n\mathbf{k}'}=E_{n\mathbf{k}}+\hbar\omega_\lambda$

$+\hbar\omega_i$; this condition is imposed in view of the fact that (1.13) [together, of course, with (1.12)], is to be used to compute *real* transitions.

It is now proposed to utilize the extreme smallness of $q_i$ (and $\omega_i$) to approximate (1.13) by the relation

$$(n\mathbf{k}'|V_{i\lambda}^{(2)}|n\mathbf{k}) = \sum_{n'\neq n}\frac{(n\mathbf{k}'|V_\lambda|n',\mathbf{k}+\mathbf{q}_i)(n'\mathbf{k}|V_i|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n'\mathbf{k}}} + \sum_{n'\neq n}\frac{(n\mathbf{k}'|V_i|n'\mathbf{k}')(n',\mathbf{k}'+\mathbf{q}_i|V_\lambda|n\mathbf{k})}{E_{n\mathbf{k}'}-E_{n'\mathbf{k}'}}, \quad (1.14)$$

where $(n'\mathbf{k}|V_i|n\mathbf{k})$ is to be understood as the limit of

$$(n',\mathbf{k}+\mathbf{q}_i|V_i|n\mathbf{k})\equiv e^{-i\omega_i t} \times(n',\mathbf{k}+\mathbf{q}_i|\textstyle\sum_g\exp(i\mathbf{q}_i\cdot\mathbf{g})\mathbf{u}_i\cdot\mathrm{grad}_g V(\mathbf{r})|n\mathbf{k}), \quad (1.15)$$

as $q_i$ goes to zero ($\omega_i$ being kept fixed in the time exponential). By applying the translational symmetry properties of the potential and the Bloch wave functions in a manner exhibited by the various steps in (1.11), one readily establishes that the right-hand side of (1.15) is equal to

$$N(n',\mathbf{k}+\mathbf{q}_i|u_i\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n\mathbf{k})e^{-i\omega_i t},$$

which, in the limit of $\mathbf{q}_i=0$, becomes

$$N(n',\mathbf{k}|\mathbf{u}_i\cdot\mathrm{grad}_g V(\mathbf{r})|n\mathbf{k})e^{-i\omega_i t} = (n',\mathbf{k}|\textstyle\sum_g\mathbf{u}_i\cdot\mathrm{grad}_g V(\mathbf{r})|n\mathbf{k})e^{-i\omega_i t}.$$

The utilization of (1.6) then yields

$$(n'\mathbf{k}|V_i|n\mathbf{k}) = -(n'\mathbf{k}|\mathbf{u}_i\cdot\mathrm{grad}_r V(\mathbf{r})|n\mathbf{k})e^{-i\omega_i t} = -(n'\mathbf{k}|\mathbf{u}_i\cdot\mathrm{grad}_r V_0(\mathbf{r})|n\mathbf{k})e^{-i\omega_i t}, \quad (1.16)$$

the last equality holding by virtue of the fact that all derivatives are evaluated for $\mathbf{R}_g=0$, at which $V(r)=V_0(\mathbf{r})$.

It should now be noted that, since the one-electron Hamiltonian for the undisturbed crystal is

$$H_0=(1/2m)p^2+V_0(\mathbf{r}),$$

one has

$$\mathrm{grad}_r V_0(r)=(i/\hbar)(\mathbf{p}H_0-H_0\mathbf{p}) = (im/\hbar)(\mathbf{v}H_0-H_0\mathbf{v}), \quad (1.17)$$

so that

$$(n'\mathbf{k}|V_i|n\mathbf{k}) = -(im/\hbar)(n'|\mathbf{u}_i\cdot\mathbf{v}_\mathbf{k}|n)(E_{n\mathbf{k}}-E_{n'\mathbf{k}})e^{-i\omega_i t}. \quad (1.18)$$

Substituting (1.18) and the equivalent relation for $(n\mathbf{k}'|V_i|n'\mathbf{k}')$ into (1.14), one has

$$(n\mathbf{k}'|V_{i\lambda}^{(2)}|n\mathbf{k}) = -\frac{im}{\hbar}\sum_{n'\neq n}\{(n,\mathbf{k}'|V_\lambda|n',\mathbf{k}+\mathbf{q}_i)(n'|\mathbf{u}_i\cdot\mathbf{v}_{\mathbf{k}'}|n) - (n|\mathbf{u}_i\cdot\mathbf{v}_\mathbf{k}|n')(n',\mathbf{k}'-\mathbf{q}_i|V_\lambda|n\mathbf{k})\}e^{-i\omega_i t}. \quad (1.19)$$

Now

$$(n',\mathbf{k}'-\mathbf{q}_i|V_\lambda|n\mathbf{k}) = (n',\mathbf{k}'-\mathbf{q}_i|\textstyle\sum_g\exp(i\mathbf{q}_\lambda\cdot\mathbf{g})\mathbf{u}_\lambda\cdot\mathrm{grad}_g V(r)|n\mathbf{k}) \times\exp(-i\omega_\lambda t)$$
$$=N\delta_{\mathbf{k}+\mathbf{q}\lambda+\mathbf{q}_i,\mathbf{k}'}\exp(-i\omega_\lambda t) \times(n',\mathbf{k}'-q_i|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n\mathbf{k})$$
$$\cong N\delta_{\mathbf{k}+\mathbf{q}\lambda+\mathbf{q}_i,\mathbf{k}'}\exp(-i\omega_\lambda t) \times(n',\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n\mathbf{k}). \quad (1.20a)$$

Correspondingly,

$$(n\mathbf{k}'|V_\lambda|n',\mathbf{k}+\mathbf{q}_i) \cong N\delta_{\mathbf{k}+\mathbf{q}\lambda+\mathbf{q}_i,\mathbf{k}'}\exp(-i\omega_\lambda t) \times(n\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n'\mathbf{k}). \quad (1.20b)$$

Substituting (1.20a) and (1.20b) into (1.19), one has

$$(n\mathbf{k}'|V_{i\lambda}^{(2)}|n\mathbf{k}) = (im/\hbar)N\delta_{\mathbf{k}+\mathbf{q}\lambda+\mathbf{q}_i,\mathbf{k}'}\exp[-i(\omega_i+\omega_\lambda)t] \times\sum_{n'\neq n}\{(n|\mathbf{u}_i\cdot\mathbf{v}_{\mathbf{k}'}|n')(n'\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n\mathbf{k}) - (n\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|n'\mathbf{k})(n'|\mathbf{u}_i\cdot\mathbf{v}_\mathbf{k}|n)\}. \quad (1.21)$$

The total bilinear matrix element is obtained by adding (1.21) and (1.12). It is to be observed that (1.21) is the negative of all the $n'\neq n$ terms of (1.12); the final result is then given by the $n'=n$ terms of (1.12). Thus

$$(\mathbf{k}'|V_{i\lambda}|\mathbf{k}) = (i/\hbar)N\delta_{\mathbf{k}+\mathbf{q}\lambda+\mathbf{q}_i,\mathbf{k}'}\exp[-i(\omega_i+\omega_\lambda)t]m\mathbf{u}_i\cdot(\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'}) \times(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|\mathbf{k}), \quad (1.22)$$

where the band index, $n$, has been dropped, since all quantities now refer to a single band.

A significant feature of (1.22) is that the matrix element, $(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})|\mathbf{k})$, contained therein is the same as that which appears in the conventional electron-lattice interaction theory; it is, in fact, the transition amplitude for electron scattering by the $\lambda$th lattice-vibration mode. It is this feature which, as will be seen later, permits one to express the effect of the bilinear interaction in terms of such a basically simple concept as collision drag.[5]

The generalization of (1.22) to the case in which each of the two types of phonons—thermal or impressed—is either absorbed or emitted, is readily achieved, and will be stated without proof. Namely, all the quantities in (1.22) remain unchanged except for the Kronecker delta and the time exponential, which take the forms

[5] In particular, (1.22) assures the absence of new electron-lattice interaction parameters which, in general, would be expected to occur with the introduction of higher-order matrix elements.

$\delta_{\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i,\,\mathbf{k}'}$ and $\exp[-(\pm\omega_\lambda\pm\omega_i)t]$, the notation $\pm$ referring to phonon absorption and emission, respectively.

Equation (1.22) (generalized as indicated above), together with the in-band matrix elements, $(\mathbf{k}'|V_\lambda|\mathbf{k})$ and $(\mathbf{k}'|V_i|\mathbf{k})$, of the first-order terms of (1.3), constitute the basic ingredients for the treatment of problems involving the simultaneous interaction of conduction electrons with thermal and impressed lattice vibrations. In the following sections, it will be shown how, under the appropriate conditions, the transitions associated with these matrix elements give rise to collision drag.

## 2. TRANSITION PROBABILITIES

In the event that the mean free path of electrons is large compared to the wavelength of the impressed sound wave, the effects arising from the bilinear interactions may be treated straightforwardly. Namely, one inserts (1.22) into the conventional formulas of first-order perturbation theory, and thereby obtains transition probabilities from which one may compute, e.g., transfer of energy from the ultrasonic wave to the conduction electrons and thermal vibrations of the metal. This particular calculation is, in fact, carried out in Appendix II of the present paper. However, as has already been discussed in the Introduction, such an approach does not yield specific collision-drag effects—at least, none in the sense of Pippard's assumption,[1] which expresses them in terms of the *local* lattice-displacement velocity, $\mathbf{v}_i$; such effects could be expected only if the domain of collision is localizable to within a wavelength, and if its temporal duration is capable of specification to within a period of the impressed sound wave. These requirements, however, are not compatible with a description of electronic states in terms of infinite plane waves; hence, the conventional theory, in which one computes transitions between states of sharply defined wave vectors $\mathbf{k}$ and $\mathbf{k}'$, will not yield collision-drag effects. In particular, if the required space-time localization is to be achieved via a wave-packet approach, as is the case in the present paper, it will be necessary that the wave-vector and frequency spreads ($\Delta k$ and $\Delta\omega\sim v_0\Delta k$) of the wave packets be large compared to $q_i$ and $\omega_i$, respectively. ($v_0$ is the velocity at the Fermi surface.)

In the event that the electron mean free path, $l_e$, is small compared to the wavelength of the impressed sound wave, it is all the more necessary to tolerate wave-vector and frequency spreads in excess of $q_i$ and $\omega_i$. In fact, they must be large compared to $1/l_e$ and $v_0/l_e$, respectively; otherwise, quantal, multiple-scattering effects would have to be considered explicitly.

On the other hand, there does exist an upper limit to the magnitude of $\Delta k$, which is determined by the requirement that the resultant uncertainties in the wave-vector and energy conservation laws of electron-lattice scattering,

$$\mathbf{k}'=\mathbf{k}\pm\mathbf{q}_\lambda, \tag{2.1}$$

$$E_{\mathbf{k}'}=E_{\mathbf{k}}\pm\hbar\omega_\lambda, \tag{2.2}$$

be sufficiently small to permit their application to transport problems. A discussion of this requirement is presented in Appendix III; it is shown therein that it corresponds to the condition

$$\Delta k\ll\hbar v_0/\kappa T, \tag{2.3}$$

where $\kappa$ is Boltzmann's constant. Since $\Delta k$ also has to be large compared to $q_i$, it is necessary that

$$q_i\ll\hbar v_0/\kappa T. \tag{2.4}$$

In what follows, (2.4) will be assumed to hold. It is shown in Appendix III that this condition does not impose any significant restrictions on the application of the theory to experimental conditions.

A typical electron wave packet is built up as a superposition of Bloch waves, thus:

$$\psi(\mathbf{r},t)=\textstyle\sum_{\mathbf{k}} A(\mathbf{k})\psi_{\mathbf{k}}(\mathbf{r})\exp(-iE_{\mathbf{k}}t/\hbar). \tag{2.5}$$

It is convenient to choose $A(\mathbf{k})$ to be the space-Fourier transform of a function, $e^{i\mathbf{k}_0\cdot\mathbf{r}}G(\mathbf{r}-\mathbf{r}_0)$, which, in essence, represents the space-probability amplitude of the wave packet. Thus

$$A(\mathbf{k})=\frac{N^{+\frac{1}{2}}}{V}\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}G(\mathbf{r}-\mathbf{r}_0)d\mathbf{r}, \tag{2.6}$$

where $V$ is the volume and $N$ the number of unit cells of the sample, $\mathbf{k}_0$ the "wave-vector centroid" of the packet, and $G(\mathbf{r}-\mathbf{r}_0)$ a function which, on the one hand, varies little over a thermal-phonon wavelength, and, on the other, differs from zero only for $|\mathbf{r}-\mathbf{r}_0|\ll\lambda_i$. Equation (2.6) may be written in the form

$$A(\mathbf{k})=e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_0}B(\mathbf{k}), \tag{2.7}$$

where

$$B(\mathbf{k})\equiv(N^{\frac{1}{2}}/V)\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}G(\mathbf{r})d\mathbf{r} \tag{2.8}$$

is the space-Fourier transform of $e^{i\mathbf{k}_0\cdot\mathbf{r}}G(r)$. In view of the properties of $G(\mathbf{r})$, it is apparent that, whereas $B(\mathbf{k})$—and hence $A(\mathbf{k})$—differs from zero only for $|\mathbf{k}-\mathbf{k}_0|\ll q_\lambda$, it also varies sufficiently slowly that in the Taylor expansion

$$B(\mathbf{k}+\mathbf{q}_i)=B(\mathbf{k})+\mathbf{q}_i\cdot\mathrm{grad}_{\mathbf{k}}B(\mathbf{k})+\cdots, \tag{2.9}$$

each term is an order of magnitude smaller than its predecessor.[6]

It is of interest to evaluate the Fourier sum of (2.5); this evaluation may be carried out with sufficient accuracy as follows. Writing

$$\psi_{\mathbf{k}}=N^{-\frac{1}{2}}e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r})=N^{-\frac{1}{2}}e^{i(\mathbf{k}_0+\mathbf{k}_r)\cdot\mathbf{r}}U_{\mathbf{k}_0+\mathbf{k}_r}(\mathbf{r}),$$

where $U_{\mathbf{k}}(\mathbf{r})$ is the periodic part of the wave function (normalized per unit cell) and $\mathbf{k}_r=\mathbf{k}-\mathbf{k}_0$ is the relative

[6] In fact, the degree of smallness is $\sim q_i/\Delta k$, where $\Delta k$, the spread in $\mathbf{k}$ of the wave packet, is related to the spread in $\mathbf{r}$, $\Delta r$, of $G(\mathbf{r})$ by the standard uncertainty formula $\Delta k\Delta r\sim 1$.

wave vector (referred to the wave-vector centroid, $\mathbf{k}_0$), one observes that $U_{\mathbf{k}}(\mathbf{r})$ is a sufficiently slowly varying function of $\mathbf{k}$ that, over the relatively small range of $\mathbf{k}_r$, $U_{\mathbf{k}_0+\mathbf{k}_r}(\mathbf{r})$ may be approximated by $U_{\mathbf{k}_0}(\mathbf{r})$ for all $\mathbf{r}$. On the other hand, $\mathbf{k}$-dependent terms in the exponentials behave altogether differently (i.e., for sufficiently large $\mathbf{r}$ and $t$, the factors $e^{i\mathbf{k}\cdot\mathbf{r}}$ and $\exp(iE_{\mathbf{k}}t/\hbar)$ vary arbitrarily rapidly with $\mathbf{k}_r$). One thus has for (2.5)

$$\psi(\mathbf{r},t)=U_{\mathbf{k}_0}\exp\{i[\mathbf{k}_0\cdot\mathbf{r}-E_{\mathbf{k}_0}t/\hbar]\}(1/V)\times\sum_{\mathbf{k}}A(\mathbf{k})\exp[i\mathbf{k}_r\cdot(\mathbf{r}-\mathbf{v}_{\mathbf{k}_0}t)]+\cdots,$$

which, in view of (2.6) and Fourier's theorem, may be written as

$$\psi(\mathbf{r},t)=U_{\mathbf{k}_0}\exp\{i[\mathbf{k}_0\mathbf{r}-E_{\mathbf{k}_0}t/\hbar]\}G(\mathbf{r}-\mathbf{r}_0-\mathbf{v}_{\mathbf{k}_0}t). \quad (2.10)$$

According to (2.10), the wave packet, in essence, consists of the Bloch function $\psi_{\mathbf{k}_0}(\mathbf{r})\exp(-iE_{\mathbf{k}_0}t/\hbar)$, modulated by the relatively slowly varying amplitude function, $G(\mathbf{r}-\mathbf{r}_0-\mathbf{v}_{\mathbf{k}_0}t)$; this amplitude function moves uniformly with velocity $\mathbf{v}_{\mathbf{k}_0}$, and hence possesses the required quasi-classical behavior.

With these preliminaries out of the way, one may now consider the scattering undergone by the wave packet under the simultaneous action of (1.22) and the ordinary electron-lattice interaction[7]

$$(\mathbf{k}|V_\lambda|\mathbf{k})=N\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}_\lambda}\exp(-i\omega_\lambda t)\times(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|\mathbf{k}). \quad (2.11)$$

This scattering will be computed according to first-order perturbation theory. Writing

$$\psi_{\mathbf{sc}}=\sum_{\mathbf{k}'}A_{\mathbf{sc}}(\mathbf{k}')\psi_{\mathbf{k}'}(\mathbf{r})\exp(-iE_{\mathbf{k}'}t/\hbar),$$

one has

$$\begin{aligned}i\hbar\frac{\partial A_{\mathbf{sc}}(\mathbf{k}')}{\partial t}&=\exp[i(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)t/\hbar]\\&\times N(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|\mathbf{k})A(\mathbf{k})\\&+\sum_{(\pm)}\exp[i(E_{\mathbf{k}'}-E_{\mathbf{k}\mp\mathbf{q}_i}-\hbar\omega_\lambda\mp\hbar\omega_i)t/\hbar]\\&\times\left(\frac{im\mathbf{u}_i}{\hbar}\right)\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\\&\times N(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|\mathbf{k})A_{\mathbf{k}\mp\mathbf{q}_i}, \quad (2.12)\end{aligned}$$

where $\mathbf{k}$ has been fixed at the value $\mathbf{k}=\mathbf{k}'-\mathbf{q}_\lambda$ (i.e., that value which satisfies the wave-vector selection rule for ordinary electron-lattice transitions), and where the sum, $\sum_{(\pm)}$, goes over the two possibilities of absorption and emission of an impressed phonon, in accordance with the text subsequent to Eq. (1.2). (It is thereby to be understood that the vector amplitude factor, $\mathbf{u}_i$, is to be replaced by its complex conjugate, $\mathbf{u}_i^*$, in the "minus" emission term.)

Integrating with respect to time from $-T/2$ to $+T/2$, and replacing $A(\mathbf{k})$ by $B(\mathbf{k})$, according to (2.7), one has

$$\begin{aligned}&e^{-i\mathbf{k}_0\cdot\mathbf{r}_0}A_{\mathbf{sc}}(\mathbf{k}')\\&=\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)N(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|\mathbf{k})B(\mathbf{k})\\&\times e^{-i\mathbf{k}\cdot\mathbf{r}_0}+\frac{i}{\hbar}\sum_{(\pm)}\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}\mp\mathbf{q}_i}-\hbar\omega_\lambda\mp\hbar\omega_i)m\mathbf{u}_i\\&\cdot(\mathbf{v}_{\mathbf{k}}-v_{\mathbf{k}'})N(\mathbf{k}'|\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})|\mathbf{k})\\&+B_{\mathbf{k}-\mathbf{q}_i}e^{-i(\mathbf{k}\mp\mathbf{q}_i)\cdot\mathbf{r}_0}, \quad (2.13)\end{aligned}$$

where

$$\Delta(x)\equiv\frac{e^{ixT/2\hbar}-e^{-ixT/2\hbar}}{ix}=\frac{2\sin(xT/2\hbar)}{x}. \quad (2.14)$$

It is now to be observed that, apart from the exponentials, $e^{-i(\mathbf{k}\mp\mathbf{q}_i)\cdot\mathbf{r}_0}$, and the $\Delta$'s, the $\mathbf{q}_i$ dependence of the right-hand side of (2.13) is of the slowly varying type represented by (2.9). As far as the $\Delta$'s are concerned, their $\mathbf{q}_i$ (and $\omega_i$) dependence may also be considered slow provided that one limits the "interaction" interval, $T$, in (2.14) sufficiently. In particular, for $\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}\mp\mathbf{q}_i}-\hbar\omega_\lambda\mp\hbar\omega_i)$ to be representable as an expansion about the point $\mathbf{q}_i=0$, one requires

$$\mathbf{q}_i\cdot\mathrm{grad}_{\mathbf{k}}E_{\mathbf{k}}T/\hbar\pm\omega_iT=(\mathbf{q}_i\cdot v_{\mathbf{k}}\pm q_ic_s)T\ll1,$$

which, since $|\mathbf{v}_{\mathbf{k}}|$ is generally $\gg c_s$ (the velocity of sound), reduces to

$$|\mathbf{v}_{\mathbf{k}}|T\ll\lambda_i. \quad (2.15)$$

The significance of (2.15) is that interaction time intervals must be taken small enough so that the collision can be localized to a region of linear dimensions small compared to the impressed wavelength. As pointed out at the beginning of this section, such localization is a basic feature of the present treatment. Admittedly, the arbitrary time-chopping procedure embodied in (2.15) is not wholly satisfactory. It is used here in lieu of some kind of "thought-construct," which would serve to limit the duration of collisions in a physically more natural way; such a construct could conceivably be achieved by the use of localized wave packets for thermal phonon.[8]

Using (2.15), then, one arrives at the situation where,

[7] In what follows, the in-band matrix elements, $(\mathbf{k}'|V_i|\mathbf{k})$, of the impressed deformation field will be ignored. This is done primarily in the interests of simplicity. There is, in fact, no intrinsic relationship between the $(\mathbf{k}'|V_i|\mathbf{k})$ and the matrix elements of the bilinear interaction; in principle, the $(\mathbf{k}'|V_i|\mathbf{k})$ may vanish. (Indeed, in the case of a transverse wave, the conventional matrix element, applicable to a spherical energy surface, does vanish.) A discussion of the effects to be expected from the eventual inclusion of the $(\mathbf{k}'|V_i|\mathbf{k})$ will be given at the end of the section.

[8] A treatment based on the construction of *classical* lattice-vibration wave packets, in which the $\mathbf{u}_\lambda$ are to be considered as specifiable vectors (rather than as operators), will be given in a later report.

apart from the exponentials $e^{-i(\mathbf{k}\mp\mathbf{q}_i)\cdot\mathbf{r}_0}$, all the quantities on the right-hand side of (2.13) are slowly varying functions of $\mathbf{q}_i$ in the sense of (2.9). One then has, up to terms linear in $\mathbf{q}_i$ and $\omega_i$,

$$
\begin{aligned}
&e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_0}A_{\mathrm{sc}}(k')\\
&=N(k'|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}v(\mathbf{r})\,|\mathbf{k})\Big\{\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)B(\mathbf{k})\\
&\quad\times\Big[1+\frac{im}{\hbar}(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\sum_{(\pm)}\mathbf{u}_i e^{\pm iq_i\cdot\mathbf{r}_0}\Big]\\
&\quad+\frac{i}{\hbar}\sum_{(\pm)}m(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\mathbf{u}_i e^{\pm iq_i\cdot\mathbf{r}_0}\\
&\quad\times(\mp q_i\cdot\mathrm{grad}_{\mathbf{k}})[\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)B(\mathbf{k})]\\
&\quad+\frac{i}{\hbar}\sum_{(\pm)}m(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\mathbf{u}_i e^{\pm iq\cdot\mathbf{r}_0}B(\mathbf{k})\\
&\qquad\times\Big(\mp\hbar\omega_i\frac{\partial}{\partial E_{\mathbf{k}'}}\Big)\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\Big\}\\
&=N(\mathbf{k}'|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})\,|\mathbf{k})\Big\{\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)B(\mathbf{k})\\
&\quad\times\Big[1+\frac{im}{\hbar}\mathbf{u}_i(\mathbf{r}_0)\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\Big]\\
&\quad-\frac{m}{\hbar}(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\mathbf{e}_i(\mathbf{r}_0)\cdot\mathrm{grad}_{\mathbf{k}}[\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)B(\mathbf{k})]\\
&\quad+m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})B_{\mathbf{k}}\frac{\partial}{\partial E_{\mathbf{k}'}}\\
&\qquad\times[\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)]\Big\},\qquad(2.16)
\end{aligned}
$$

where

$$\mathbf{u}_i(\mathbf{r}_0)\equiv\mathbf{u}_i e^{i\mathbf{q}_i\cdot\mathbf{r}_0}+\text{c.c.},\qquad(2.17)$$

$$\mathbf{v}_i(\mathbf{r}_0)\equiv i\omega_i\mathbf{u}_i e^{i\mathbf{q}_i\cdot\mathbf{r}_0}+\text{c.c.},\qquad(2.18)$$

$$\mathbf{e}_i(\mathbf{r}_0)\equiv i\mathbf{u}_i\mathbf{q}_i e^{i\mathbf{q}_i\cdot\mathbf{r}_0}+\text{c.c.},\qquad(2.19)$$

are the displacement, displacement velocity, and strain dyadic[9] associated with the impressed wave; these quantities are all evaluated at $\mathbf{r}=\mathbf{r}_0$ and at $t=0$, i.e., the point in space and time at which the collision is presumed to occur. In general, of course, the time is some arbitrary $t_0$, so that the factor $e^{i\mathbf{q}_i\cdot\mathbf{r}_0}$ should be replaced by $e^{i[\mathbf{q}_i\cdot\mathbf{r}_0+\omega_i t_0]}$.

It will now be noted that, up to and including terms linear in the amplitude of the impressed sound wave, (2.16) is equivalent to

$$
\begin{aligned}
&e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_0}A_{\mathrm{sc}}(\mathbf{k}')\\
&=(\mathbf{k}'|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})\,|\mathbf{k})\exp[(im/\hbar)\mathbf{u}(\mathbf{r}_0)\cdot(\mathbf{v}_{\mathbf{k}}-v_{\mathbf{k}'})]\\
&\quad\times\Delta(E_{\mathbf{k}'}-E_{\mathbf{k}-\kappa_i}-\hbar\omega_i+m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'}))\\
&\qquad\times B(\mathbf{k}-\boldsymbol{\kappa}_i),\qquad(2.20)
\end{aligned}
$$

where

$$\boldsymbol{\kappa}_i\equiv+(m/\hbar)(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\mathbf{e}_i(\mathbf{r}_0).\qquad(2.21)$$

Upon taking the absolute square of (2.20), one obtains

$$
\begin{aligned}
|A_{\mathrm{sc}}(\mathbf{k}')|^2&=(N^2/\hbar^2)|(\mathbf{k}'|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})\,|\mathbf{k})|^2\\
&\quad\times\Omega[E_{\mathbf{k}'}-E_{\mathbf{k}-\kappa_i}-\hbar\omega_\lambda+m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})]\\
&\quad\times|B(\mathbf{k}-\boldsymbol{\kappa}_i)|^2,\qquad(2.22)
\end{aligned}
$$

where

$$\Omega(x)\equiv\frac{4\sin^2(xT/2\hbar)}{x^2/\hbar^2}=\hbar^2|\Delta(x)|^2\qquad(2.23)$$

is the energy-resonance function introduced in Sommerfeld and Bethe's article.[10] It has the property that

$$\lim_{T\to\infty}\Omega(x)=2\pi\hbar T\delta(x),\qquad(2.24)$$

where $\delta(x)$ is the Dirac delta function.

It is convenient to change the notation as follows: instead of $\mathbf{k}$ being taken equal to $\mathbf{k}'-\mathbf{q}_\lambda$, it is now to be defined as

$$\mathbf{k}=\mathbf{k}'-\mathbf{q}_\lambda-\boldsymbol{\kappa}_i.\qquad(2.25)$$

In terms of this new definition of $\mathbf{k}$, (2.22) takes the form[11]

$$
\begin{aligned}
|A_{\mathrm{sc}}(\mathbf{k}')|^2&=(N^2/\hbar^2)|(k'|\,u_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})\,|\mathbf{k})|^2\\
&\quad\times\Omega[E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda+m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})]\\
&\quad\times|B(\mathbf{k})|^2.\qquad(2.26)
\end{aligned}
$$

Equation (2.26) states that $|A_{\mathrm{sc}}(\mathbf{k}')|^2$, the probability that, at time $t=T/2$, the electron is to be found scattered into a state $\mathbf{k}'$, is the product of two factors. One of these, $|B(k)|^2$, is the probability that, initially, at time $t=-T/2$, the electron is in a state $\mathbf{k}$ (independent of other features of the wave packet, e.g., phase or amplitude relationships between the different $A_{\mathbf{k}}$). It is then appropriate to consider the remaining factor to represent the probability that an electron makes a transition from $\mathbf{k}$ to $\mathbf{k}'$ (while absorbing a phonon of mode $\lambda$). If this interpretation is accepted, one then has for the transition probability (per unit time) the expression

$$
\begin{aligned}
&P^{(\lambda)}(\mathbf{k}\to\mathbf{k}')\\
&=(N^2/\hbar^2T)|(\mathbf{k}'|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{g=0}V(\mathbf{r})\,|\mathbf{k})|^2\\
&\quad\times\Omega[E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda+m\mathbf{v}_i(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})]\\
&\qquad\times\delta_{\mathbf{k},\,\mathbf{k}'-\mathbf{q}_\lambda-\kappa_i(\mathbf{k},\mathbf{k}')},\qquad(2.27)
\end{aligned}
$$

[9] Actually, $\mathbf{e}_i(\mathbf{r}_0)$ is not quite the strain dyadic, since the latter is by definition a symmetrical tensor, in contrast to the right-hand side of (2.19), which is clearly unsymmetrical, and which, in fact, includes the rotational part of the displacement.

[10] A. Sommerfeld and H. Bethe, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. 24, Part 2, p. 514, Eq. (34.34).

[11] The small change in the matrix element (and in $\mathbf{v}_{\mathbf{k}}$) arising from the redefinition of $\mathbf{k}$, is ignored in (2.26); this change involves only the *magnitude* of the transition probability, but not the selection rules for wave-vector and energy conservation which, as will be seen, are the important effects of the bilinear interaction.

where the notation $\kappa_i(\mathbf{k},\mathbf{k}')$ is used in recognition of the fact that $\kappa_i$ depends on $\mathbf{k}$ and $\mathbf{k}'$ [via the factor, $\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'}$, as shown in (2.21)].

It will first be noted that, in the absence of the impressed wave, (2.27) reduces to the standard transition-probability expression for electron-lattice transitions.[12] The effects of the impressed wave are then seen to be twofold. Firstly, the wave-vector selection rule is altered, in that the difference between initial and final wave vectors, previously equal to the phonon wave vector $\mathbf{q}_\lambda$, is augmented by the term $\kappa_i(\mathbf{k},\mathbf{k}')$. Secondarily, the energy conservation rule no longer holds; instead, in the transition from $\mathbf{k}$ to $\mathbf{k}'$, the combined energy of the system of electron and thermal phonons is increased[13] by an amount $m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}'}-\mathbf{v}_\mathbf{k})$.

An alternative way of stating these results is as follows. Firstly, it is expedient to introduce a "local" wave vector

$$\mathbf{k}_l\equiv\mathbf{k}+(m/\hbar)\mathbf{v}_\mathbf{k}\cdot\mathbf{e}_i(\mathbf{r}_0). \tag{2.28}$$

Equation (2.25) then becomes

$$\mathbf{k}_l'=\mathbf{k}_l+\mathbf{q}_\lambda. \tag{2.29}$$

Similarly, one introduces a "local" energy

$$\epsilon_\mathbf{k}\equiv E_\mathbf{k}-m\mathbf{v}_i\cdot\mathbf{v}_\mathbf{k}. \tag{2.30}$$

Inserting (2.28) and (2.30) into the Kronecker delta and the energy-resonance factor of (2.27), one then has

$$P^{(\lambda)}(\mathbf{k}\to\mathbf{k}')=(N^2/\hbar^2T)\,|\,(\mathbf{k}'\,|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})\,|\,\mathbf{k})\,|^2 \times\Omega(\epsilon_{\mathbf{k}'}-\epsilon_\mathbf{k}-\hbar\omega_\lambda)\delta_{\mathbf{k}_l,\,\mathbf{k}_l'-\mathbf{q}_\lambda}. \tag{2.31}$$

It will later be seen that the occurrence of $\epsilon_\mathbf{k}$ and $\epsilon_{\mathbf{k}'}$, in place of $E_\mathbf{k}$ and $E_{\mathbf{k}'}$, in the argument of $\Omega(x)$, is of crucial significance for the explanation of the collision-drag effect.

At this point, one may readily generalize to the case in which thermal phonons are emitted, as well as absorbed. One has, in place of (2.31),

$$P_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}')=(N^2/\hbar^2T)\,|\,(\mathbf{k}'\,|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})\,|\,\mathbf{k})\,|^2 \times\Omega(\epsilon_{\mathbf{k}'}-\epsilon_\mathbf{k}\mp\hbar\omega_\lambda)\delta_{\mathbf{k}_l',\,\mathbf{k}_l\pm\mathbf{q}_\lambda}, \tag{2.32}$$

where the plus-minus sign refers to the two cases of absorption and emission of thermal phonons, respectively.

For application to transport problems, two further steps are needed. One of these, the introduction of (2.24) into (2.32), is permissible for the transport problems of interest, in which it is assumed (see Appendix III) that the energy variation of the characteristic transport quantities (e.g., distribution functions) is small over the effective width ($\sim\hbar/T$) of the resonance function $\Omega(x)$. The insertion of (2.24) into (2.32) yields

$$P_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}')=(2\pi N^2/\hbar)\,|\,(\mathbf{k}'\,|\,\mathbf{u}_\lambda\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})\,|\,k)\,|^2 \times\delta(\epsilon_{\mathbf{k}'}-\epsilon_\mathbf{k}\mp\hbar\omega_\lambda)\delta_{\mathbf{k}_l',\,\mathbf{k}_l\pm\mathbf{q}_\lambda}. \tag{2.33}$$

Equation (2.33) is the standard expression[14] for the transition rate; it possesses the two features of being independent of the collision-time interval, $T$, and of providing infinitely sharp energy selection.

The second step is to render explicit the dependence of (2.32) on the lattice-vibration state—specifically, on the initial number of phonons, $N_\lambda$, contained in the $\lambda$th mode. This dependence is contained implicitly in the complex displacement amplitude, $\mathbf{u}_\lambda$. Up to the present, $\mathbf{u}_\lambda$ has been treated as a classical field. On the basis of a simple correspondence-principle argument, one would take $|\,\mathbf{u}_\lambda\,|^2$ to be proportional to $N_\lambda$. As is well known, however, the correct recipe is

$$|\,\mathbf{u}_\lambda\,|^2\to\left\{\begin{matrix}N_\lambda\\N_\lambda+1\end{matrix}\right\}|\,\mathbf{u}_\lambda^{(0)}\,|^2, \tag{2.34}$$

where the upper and lower factors are to be used for the cases of phonon absorption and emission, respectively, and where $|\,\mathbf{u}_\lambda^{(0)}\,|^2$ is independent of $N_\lambda$. (It is, in fact, the absolute square of the matrix element of the quantized $\mathbf{u}_\lambda$ between states of $N_\lambda=0$ and $N_\lambda=1$.) It then follows that (2.33) may be written as

$$P_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}')=\left\{\begin{matrix}N_\lambda\\N_\lambda+1\end{matrix}\right\}W_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}'), \tag{2.35}$$

with a vibration-independent transition probability given by

$$W_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}')=w_{\mathbf{k}\mathbf{k}'}^{(\lambda)} \times\delta(\epsilon_{\mathbf{k}'}-\epsilon_\mathbf{k}\mp\hbar\omega_\lambda)\delta_{\mathbf{k}_l',\,\mathbf{k}_l\pm\mathbf{q}_\lambda}, \tag{2.36}$$

where

$$w_{\mathbf{k},\,\mathbf{k}'}^{(\lambda)}\equiv(2\pi N^2/\hbar)\,|\,(\mathbf{k}'\,|\,u_\lambda^{(0)}\cdot\mathrm{grad}_{\mathbf{g}=0}V(\mathbf{r})\,|\,\mathbf{k})\,|^2. \tag{2.37}$$

It remains to discuss the question, raised in reference (6), of the neglect of the first-order matrix elements, $(\mathbf{k}'\,|\,V_i\,|\,\mathbf{k})$, of the impressed deformation potential in the basic equation (2.12) of this section. These matrix elements give rise to transitions of the type $\mathbf{k}\to\mathbf{k}\pm\mathbf{q}_i$, which, in view of the smallness of $q_i$ with respect to the wave-vector spread, $\Delta k$, result in an essentially continuous alteration of a typical wave packet in $\mathbf{k}$ space—in sharp contrast to the transitions associated with the matrix elements $(\mathbf{k}'\,|\,V_\lambda\,|\,\mathbf{k})$ and $(\mathbf{k}'\,|\,V_{i\lambda}\,|\,\mathbf{k})$, which pro-

[12] See A. H. Wilson, *Theory of Metals* (Cambridge University Press, Cambridge, 1953), second edition, p. 258, Eq. (9.32.3). In comparing this equation with (2.27) of the present paper, one should note that (a) Wilson's $\Omega(x)$ is $\frac{1}{2}$ that of (2.27), and that (b) Wilson defines the transition rate as a time derivative, rather than as a time average. The two definitions are in effect equivalent since they are both used only in situations in which (2.24) holds, so that the total transition probability is linear in $T$.

[13] It is, of course, to be understood that, strictly speaking, all final energies are allowed; the phrase "selection rule" refers to that choice of $E_{\mathbf{k}'}$ for which the argument of $\Omega(x)$ vanishes.

[14] See R. Peierls, *Quantum Theory of Solids* (The Clarendon Press, Oxford, 1955), p. 40, Eq. (2.58); also pp. 123–124, Eqs. (6.37) and (6.39).

duce changes in $\mathbf{k}$ large compared to $\Delta k$ and are hence to be interpreted as scattering events. The quasi-continuous transitions, $\mathbf{k}\rightarrow\mathbf{k}\pm\mathbf{q}_i$, correspond, in fact, to the classical displacement of the electron in $\mathbf{k}$ space, in response to the action of the impressed deformation field, the latter being considered as a given function $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$ of position and wave vector.[15] This correspondence must necessarily obtain for sufficiently long impressed wavelength and, in fact, is the basis of classical treatments, such as that of Pippard.[1]

With respect to the subject matter of this paper, the question of interest is: what is the effect of this quasi-classical motion on the collisions of electrons with thermal phonons? Without going into quantitative details, one may answer this question as follows. In the limit of sufficiently large impressed wavelength and vibration period—i.e., large compared to the space-time domain of a typical collision—the deformation potential is, to a good approximation, a constant (equal to $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$ evaluated at space-time locus of the collision). In this approximation, it is clear that the sole effect of the impressed deformation field is to shift the energy of the electrons by an amount $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$; the immediate consequence of such a shift is that the argument of the delta function in (2.33) becomes[16] $(\epsilon_{\mathbf{k}'}+\mathcal{U}_{\mathbf{k}'}-\mathcal{U}_\mathbf{k}\mp\hbar\omega_\lambda)$. This modification, being essentially unrelated to collision drag, will be ignored in subsequent sections.

The above qualitative argument has been confirmed by detailed, though at present preliminary, calculations, which it is hoped to incorporate in a subsequent report.[17]

### 3. TRANSPORT THEORY

The basic expressions (2.35) and (2.36) for the transition probability of electron-phonon collision—expressions which contain the first-order effects of the impressed wave—will now be introduced into the collision term of the Boltzmann transport equation.[18] The general form of this term is

$$\begin{aligned}(\partial f/\partial t)_{\text{coll}}&=\sum_{\lambda,\mathbf{k}'}\{[N_\lambda W_{(+)}{}^{(\lambda)}(\mathbf{k}'\rightarrow\mathbf{k})+(N_\lambda+1)W_{(-)}{}^{(\lambda)}(\mathbf{k}'\rightarrow\mathbf{k})]\\&\quad\times f(\mathbf{k}')[1-f(k)]\\&\quad-[N_\lambda W_{(+)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')+(N_\lambda+1)W_{(-)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')]\\&\quad\times f(k)[1-f(\mathbf{k}')]\}.\end{aligned}\tag{3.1}$$

It will be noted from (2.36) and the Hermitian property of the matrix element contained therein, that $W_{(\pm)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')$ satisfies the basic reversibility relation

$$W_{(\pm)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')=W_{(\mp)}{}^{(\lambda)}(\mathbf{k}'\rightarrow\mathbf{k}),\tag{3.2}$$

so that (3.1) may be written as

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}&=\sum_{\lambda,\mathbf{k}'}\{[N_\lambda f(\mathbf{k}')(1-f(\mathbf{k}))-(N_\lambda+1)f(\mathbf{k})(1-f(\mathbf{k}'))]\\&\quad\times W_{(-)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')\\&\quad+[(N_\lambda+1)f(\mathbf{k}')(1-f(\mathbf{k}))-N_\lambda f(\mathbf{k})(1-f(k'))]\\&\quad\times W_{(+)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')\}.\end{aligned}\tag{3.3}$$

Corresponding to (3.1), one may write down an equation for $(\partial N_\lambda/\partial t)_{\text{coll}}$, which reads

$$\begin{aligned}\left(\frac{\partial N_\lambda}{\partial t}\right)_{\text{coll}}&=\sum_{\mathbf{k},\mathbf{k}'}f(\mathbf{k})[1-f(\mathbf{k}')]\\&\quad\times[(N_\lambda+1)W_{(-)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')\\&\quad-N_\lambda W_{(+)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')].\end{aligned}\tag{3.4}$$

With the use of (3.2), this equation takes the form

$$\begin{aligned}(\partial N_\lambda/\partial t)_{\text{coll}}&=\sum_{\mathbf{k},\mathbf{k}'}\{f(\mathbf{k})[1-f(\mathbf{k}')](N_\lambda+1)-f(\mathbf{k}')\\&\quad\times[1-f(\mathbf{k})]N_\lambda\}W_{(-)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}').\end{aligned}\tag{3.5}$$

The substitution of (2.36) into (3.1), (3.3), (3.4), and (3.5), in place of the conventional expressions for $W_{(\pm)}{}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')$ (in which $\mathbf{k}_l$ and $\epsilon_\mathbf{k}$ reduce to $\mathbf{k}$ and $E_\mathbf{k}$, respectively), automatically incorporates the effects of the bilinear interaction into transport theory.[19]

In what follows, the phonon population, $N_\lambda$, will be assumed, as a result of other unspecified interactions, to be in thermal equilibrium with a temperature reservoir,

[15] In the standard text-book case in which $(\mathbf{k}'|V_i|\mathbf{k})$ depends only on $(\mathbf{k}-\mathbf{k}')=\pm\mathbf{q}_i$, $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$ is independent of $\mathbf{k}$, and may then be regarded as an ordinary potential energy; generally, however, the $\mathbf{k}$, $\mathbf{k}'$ dependence of the matrix element will be more complex, requiring a dependence of $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$ on $\mathbf{k}$. Physically, such a situation means that the additional potential energy experienced by the electron, in response to a given lattice deformation, depends on its location in $\mathbf{k}$ space, as well as on the local strain.

[16] It may here be remarked that in the standard treatments in which $\mathcal{U}_\mathbf{k}$ is taken independent of $\mathbf{k}$, the effect disappears.

[17] It is perhaps of interest to quote an expression for $\mathcal{U}_\mathbf{k}(\mathbf{r},t)$ which was obtained in the course of these preliminary calculations. It reads

$$\mathcal{U}\mathbf{k}(\mathbf{r},t)=(\mathbf{k}+\tfrac{1}{2}\mathbf{q}_i|V_i|\mathbf{k}-\tfrac{1}{2}\mathbf{q}_i)e^{i\mathbf{q}\cdot\mathbf{r}-\omega_i t}+\text{c.c.}$$

[18] The validity of this procedure will here be assumed, in common with other treatments based on the Boltzmann transport equation. In the case at hand there is present the added feature of localization (the distribution function, $f(\mathbf{k})$, having also to be considered a function of position, specifiable to within a distance small compared to the ultrasonic wavelength). A discussion of the significance of this localization, in relation to the requirements of energy definition in transport theory, is given in Appendix III. The more general question of the validity of introducing any spatial variation into the Boltzmann equation for a degenerate Fermi gas lies outside the scope of the present paper.

[19] It should here be remarked that, since (2.36) has been established only to first order in the impressed amplitude, the above expressions for $(\partial f/\partial t)_{\text{coll}}$ and $(\partial N_\lambda/\partial t)_{\text{coll}}$ are to be employed only for calculations involving first-order quantities, such as current density. By way of contrast, as will be seen in Sec. 4, the computation of energy transfer (a quantity quadratic in the impressed amplitude) requires a special supplementary investigation of second-order effects.

so that the Planck distribution

$$N_\lambda=\frac{1}{\exp(\hbar\omega_\lambda/\kappa T)-1} \tag{3.6}$$

holds. With the insertion of (3.6) into (3.1) or (3.3), Eqs. (3.4) or (3.5) for $(\partial N_\lambda/\partial t)_{\text{coll}}$ become superfluous and will not be considered further.[20]

The point has now been reached where the validity of the collision-drag hypothesis may be examined. The problem is to determine the distribution for which $(\partial f/\partial t)_{\text{coll}}$ is zero; this is, by definition, the distribution towards which the system relaxes.

The solution of the problem may readily be inferred from the solution of the corresponding problem for the conventional collision integral, in which $\mathbf{k}$ and $\epsilon_\mathbf{k}$ are replaced by $\mathbf{k}$ and $E_\mathbf{k}$, respectively. In that case, it is simply the Fermi distribution

$$f_0(E_\mathbf{k})\equiv\frac{1}{1+\exp[(E_\mathbf{k}-\zeta)/\kappa T]}. \tag{3.7}$$

The occurrence of $E_\mathbf{k}$ in (3.7) is connected with the circumstance that in the standard case, it is this quantity which enters into the energy conservation law of electron-phonon collisions. In the case at hand, however, it is not $E_\mathbf{k}$, but rather $\epsilon_\mathbf{k}=E_\mathbf{k}-m\mathbf{v}_i\cdot\mathbf{v}_\mathbf{k}$ which is involved in energy conservation. It then follows that the "relaxed" distribution function is

$$f_0(\epsilon_\mathbf{k})=\frac{1}{1+\exp[(\epsilon_\mathbf{k}-\zeta)/\kappa T]}. \tag{3.8}$$

Equation (3.8) constitutes the generalized version of the collision-drag hypothesis—generalized in that it applies to an arbitrary wave-vector dependence of $E_\mathbf{k}$. For the special case of free electrons, one has

$$E_\mathbf{k}=\hbar^2k^2/2m=\tfrac{1}{2}m\mathbf{v}_\mathbf{k}^2$$

and

$$\epsilon_\mathbf{k}=\tfrac{1}{2}m(\mathbf{v}_\mathbf{k}^2-2\mathbf{v}_\mathbf{k}\cdot\mathbf{v}_i),$$

which, to the first order in the impressed amplitude, may be written as

$$\epsilon_\mathbf{k}=\tfrac{1}{2}m(\mathbf{v}_\mathbf{k}-\mathbf{v}_i)^2. \tag{3.9}$$

When (3.9) is inserted into (3.8), the resulting distribution is seen to be a Fermi distribution uniformly displaced in velocity space by an amount $\mathbf{v}_i$, in accordance with Pippard's[1] assumption.

In the more general case of arbitrary $\mathbf{k}$-dependence of $E_\mathbf{k}$, one has, to the first order in the impressed amplitude,

$$\epsilon_\mathbf{k}=E_{\mathbf{k}-m\mathbf{v}_i/\hbar}. \tag{3.10}$$

Substituting (3.10) into (3.8), one obtains a Fermi distribution, uniformly displaced in $\mathbf{k}$ space by an amount $m\mathbf{v}_i/\hbar$.

It is of interest to compute the average velocity

$$\langle\mathbf{v}\rangle_{\text{Av}}{}^{(1)}=(1/nV)\textstyle\sum_\mathbf{k}\mathbf{v}_\mathbf{k}f_0(\epsilon_\mathbf{k}) \tag{3.11}$$

associated with (3.8). [Here, $n$ is the electron density, and $V$ the volume of the sample; the meaning of the superscript (1) on the left-hand side will become apparent shortly.] Noting that to the required (first-order) accuracy

$$f_0(\epsilon_\mathbf{k})=f_0(E_\mathbf{k})-(m\mathbf{v}_i/\hbar)\cdot\text{grad}_\mathbf{k}f_0(E_\mathbf{k}),$$

and performing an integration by parts, one has successively

$$\begin{aligned}\langle\mathbf{v}\rangle_{\text{Av}}{}^{(1)}&=-\textstyle\sum_\mathbf{k}(m\mathbf{v}_i/nV\hbar)\cdot[\text{grad}_\mathbf{k}f_0(E_\mathbf{k})]\mathbf{v}_\mathbf{k}\\&=-(m/nV\hbar)\textstyle\sum_\mathbf{k}\mathbf{v}_\mathbf{k}\,\text{div}_\mathbf{k}[f_0(E_\mathbf{k})\mathbf{v}_i]\\&=\mathbf{v}_i\cdot\langle m/\mathfrak{m}_k{}^*\rangle_{\text{Av}},\end{aligned} \tag{3.12}$$

where

$$m/\mathfrak{m}_\mathbf{k}{}^*=(m/\hbar)\,\text{grad}_\mathbf{k}\mathbf{v}_\mathbf{k}=(m/\hbar^2)(\text{grad}_\mathbf{k})^2E_\mathbf{k} \tag{3.13}$$

is the reciprocal mass tensor, expressed in units of the free-electron mass, and where the average is taken with the states weighted according to the unperturbed distribution, $f_0(E_\mathbf{k})$.

Equation (3.12) would appear to signify that the mean electron velocity associated with the locally relaxed distribution, $f_0(\epsilon_\mathbf{k})$, is in general different from the local displacement velocity, $\mathbf{v}_i$. This interpretation is, however, incorrect, since it assumes tacitly that the expectation velocity $\mathbf{v}_\mathbf{k}$ for a state of crystal momentum $\mathbf{k}$ is unaffected by the impressed wave. Such an assumption would be valid if an electron were in a given "$n\mathbf{k}$" state belonging to a *single band*. Actually, however, virtual transitions to other bands will take place due to the presence of nonvanishing interband matrix elements, $(n',\mathbf{k}\pm\mathbf{q}_i|V_i|n\mathbf{k})$, of the first-order electron-lattice interaction. Associated with these virtual transitions is an additional "induced" component of velocity, $\mathbf{v}_\mathbf{k}{}^{(2)}$, linearly proportional to the amplitude of the impressed ultrasonic wave. This component, or rather its average over the unperturbed distribution function, $f_0(E_\mathbf{k})$, is computed in Appendix I. The result is [see Eq. (I31)]

$$\langle\mathbf{v}\rangle_{\text{Av}}{}^{(2)}\equiv\mathbf{v}_i-\mathbf{v}_i\cdot\langle m/\mathfrak{m}_\mathbf{k}{}^*\rangle_{\text{Av}}, \tag{3.14}$$

which when added to (3.12) gives $\mathbf{v}_i$. The average velocity of an electron gas in local equilibrium with its lattice "environment" is thus equal to the local displacement velocity.

It should be remarked here that, from the standpoint of ultrasonic absorption, the $\langle\mathbf{v}\rangle_{\text{Av}}{}^{(1)}$ component is of principal significance, since it is associated with variations in the distribution function. These variations are a necessary prerequisite for the occurrence of relaxation, which, in turn, constitutes the ultimate mechanism for the dissipative conversion of ultrasonic energy into

[20] Some interesting features arise when $N_\lambda$ is not assumed *a priori* to be given by (3.6). Owing primarily to considerations of space, the treatment of this case will be deferred to a later report.

heat.[21] The $\langle \mathbf{v} \rangle_{Av}^{(2)}$ component, on the other hand, exhibits an instantaneous[22] response to the impressed wave, unaffected by collisions, and is, hence, relaxationless. It therefore cannot play any direct role in ultrasonic absorption.

### 4. ENERGY TRANSFER IN COLLISIONS

The problem of ultrasonic absorption in metals is essentially that of computing the energy transfer from the ultrasonic wave to the combined system of electrons and thermal phonons. A component of this transfer, $(\partial E/\partial t)_{\text{coll}}$, results from electron-phonon collisions, as modified by the bilinear interaction; this component will be computed in the present section. In the course of the work, it will be shown that (2.36), though not *a priori* valid to second order in the impressed amplitude, is nevertheless applicable to this computation.

It is most convenient to start by writing down expressions for energy transfer in terms of the general Boltzmann collision integral. The energy density of the system of electrons and thermal phonons is

$$E = V^{-1} \sum_{\mathbf{k}} E_{\mathbf{k}} f_{\mathbf{k}} + V^{-1} \sum_{\lambda} \hbar\omega_{\lambda} N_{\lambda}, \tag{4.1}$$

so that

$$\left(\frac{\partial E}{\partial t}\right)_{\text{coll}} = V^{-1} \sum_{\mathbf{k}} E_{\mathbf{k}} \left(\frac{\partial f_{\mathbf{k}}}{\partial t}\right)_{\text{coll}} + V^{-1} \sum_{\lambda} \hbar\omega_{\lambda} \left(\frac{\partial N_{\lambda}}{\partial t}\right)_{\text{coll}}. \tag{4.2}$$

Inserting (3.1) and (3.4) into (4.1), and interchanging $\mathbf{k}$ and $\mathbf{k}'$ in the double summation of the positive terms coming from (3.1), one has

$$\begin{aligned}\left(\frac{\partial E}{\partial t}\right)_{\text{coll}} = V^{-1} \sum_{\lambda,\mathbf{k},\mathbf{k}'} \{ & (E_{\mathbf{k}'} - E_{\mathbf{k}} - h\omega_{\lambda}) \\ & \times N_{\lambda} W_{(+)}^{(\lambda)}(\mathbf{k} \to \mathbf{k}') f(\mathbf{k})[1 - f(\mathbf{k}')] \\ & + (E_{\mathbf{k}'} - E_{\mathbf{k}} + \hbar\omega_{\lambda})(N_{\lambda} + 1) \\ & \times W_{(-)}^{(\lambda)}(\mathbf{k} \to \mathbf{k}') f(\mathbf{k})[1 - f(\mathbf{k}')] \}. \end{aligned} \tag{4.3}$$

An additional simplification is achieved by interchange of $\mathbf{k}$ and $\mathbf{k}'$ in the double summation of one of the terms, say the second, followed by the use of (3.2); one thereby obtains

$$\begin{aligned}\left(\frac{\partial E}{\partial t}\right)_{\text{coll}} = V^{-1} \sum_{\lambda,\mathbf{k},\mathbf{k}'} & (E_{\mathbf{k}'} - E_{\mathbf{k}} - \hbar\omega_{\lambda}) W_{(+)}^{(\lambda)}(\mathbf{k} \to \mathbf{k}') \\ & \times \{ N_{\lambda} f(\mathbf{k})[1 - f(k')] \\ & - (N_{\lambda} + 1) f(\mathbf{k}')[1 - f(\mathbf{k})] \}. \end{aligned} \tag{4.4}$$

In proceeding further, it will first be assumed that (2.36) is valid to all orders in the impressed amplitude. It then follows that, by virtue of the energy delta function occurring in (2.36),

$$\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}} = \hbar\omega_{\lambda},$$

whereupon, with the use of (2.30), one has

$$E_{\mathbf{k}'} - E_{\mathbf{k}} - h\omega_{\lambda} = m\mathbf{v}_i \cdot (\mathbf{v}_{\mathbf{k}'} - \mathbf{v}_{\mathbf{k}}). \tag{4.5}$$

Substituting (4.5) into (4.4), one has

$$\begin{aligned}\left(\frac{\partial E}{\partial t}\right)_{\text{coll}} = V^{-1} \sum_{\lambda,\mathbf{k},\mathbf{k}'} & m\mathbf{v}_i \cdot (\mathbf{v}_{\mathbf{k}'} - \mathbf{v}_{\mathbf{k}}) W_{(+)}^{(\lambda)}(\mathbf{k} \to \mathbf{k}') \\ & \times \{ N_{\lambda} f(\mathbf{k})[1 - f(\mathbf{k}')] \\ & - (N_{\lambda} + 1) f(\mathbf{k}')[1 - f(\mathbf{k})] \}. \end{aligned} \tag{4.6}$$

An alternative, physically more suggestive form of (4.6) is arrived at as follows. One notes that the collision-induced time variation of the electron current density

$$\mathbf{j} = V^{-1} \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}} f_{\mathbf{k}}$$

is representable by an expression similar to (4.4), in which $E_{\mathbf{k}}$ is replaced by $\mathbf{v}_{\mathbf{k}}$ (and $\hbar\omega_{\lambda}$ by zero, since the phonon flux is not to be included in $\mathbf{j}$). Thus

$$\begin{aligned}\left(\frac{\partial \mathbf{j}_{\mathbf{k}}}{\partial t}\right)_{\text{coll}} = V^{-1} \sum_{\lambda,\mathbf{k},\mathbf{k}'} & (\mathbf{v}_{\mathbf{k}'} - \mathbf{v}_{\mathbf{k}}) W_{(+)}^{(\lambda)}(\mathbf{k} \to \mathbf{k}') \\ & \times \{ N_{\lambda} f(\mathbf{k})[1 - f(\mathbf{k}')] \\ & - (N_{\lambda} + 1) f(\mathbf{k}')[1 - f(\mathbf{k})] \}. \end{aligned} \tag{4.7}$$

Comparison of (4.6) and (4.7) then yields the simple relation

$$(\partial E/\partial t)_{\text{coll}} = m\mathbf{v}_i \cdot (\partial \mathbf{j}/\partial t)_{\text{coll}}. \tag{4.8}$$

From the standpoint of the impressed wave, (4.8) represents an energy loss of the form $\mathbf{F} \cdot \mathbf{v}_i$, where $\mathbf{F}$, the effective force per unit volume exerted on the lattice particles, is given by

$$\mathbf{F} = -m(\partial \mathbf{j}/\partial t)_{\text{coll}}. \tag{4.9}$$

This expression will be recognized as the so-called *Stuart-Tolman* reaction force, evoked whenever the electron current changes by virtue of interaction with the host lattice. It should be noted, however, that, strictly speaking, the Stuart-Tolman effect has to do only with the ponderomotive force exerted over the whole body; (4.9) goes further in describing how this force is distributed over individual volume elements.

The question of the applicability of (2.36) to the computation of energy transfer will now be considered. As pointed out in reference 20, the question arises because of the circumstance that the previous sections have established (2.36) only to first order in the impressed amplitude, whereas the energy transfer, as

[21] See A. Akhieser, J. Phys. U.S.S.R. **1**, 289 (1939), wherein dissipation is computed in terms of the rate of increase of entropy due to collisions.

[22] The actual time of response is of the order of $\hbar/(E_{n\mathbf{k}} - E_{n'\mathbf{k}})$, i.e., the reciprocal of an interband frequency; such time intervals are in effect zero.

represented by (4.4), is essentially a quadratic effect. In other words, if one expands $W_{(\pm)}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')$ as a power series in the impressed amplitude, viz.,

$$W_{(\pm)}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')=\sum_n W_{(\pm)}^{(\lambda,n)}(\mathbf{k}\rightarrow\mathbf{k}'), \quad (4.10)$$

and carries out the comparable expansion for the right-hand side of (2.36), it is legitimate to identify coefficients only for $n=0$, 1, to wit[23]

$$W_{(\pm)}^{(\lambda,0)}=w_{\mathbf{k}\mathbf{k}'}^{(\lambda)}\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda)\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda}, \quad (4.11)$$

$$W_{(\pm)}^{(\lambda,1)}=w_{\mathbf{k}\mathbf{k}'}^{(\lambda)}[m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\delta'(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda)\times\delta_{\mathbf{k},\mathbf{k}'\pm\mathbf{q}_\lambda}+\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda)\times m(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})\cdot\mathbf{e}_i\cdot\mathrm{grad}_{\mathbf{k}}\delta_{\mathbf{k},\mathbf{k}'\pm\mathbf{q}_\lambda}]. \quad (4.12)$$

These equations are not sufficient for the computation of $(\partial E/\partial t)_{\mathrm{coll}}$; the quadratic term, $W_{(\pm)}^{(\lambda,2)}(\mathbf{k}\rightarrow\mathbf{k}')$, is also needed.

However, it turns out, as will now be shown, that $W_{(\pm)}^{(\lambda,2)}(\mathbf{k}\rightarrow\mathbf{k}')$ enters into (4.4) in such a way as to permit establishing a correspondence between its contribution and the energy-transfer calculation of Appendix II. From this correspondence it becomes possible to show that the use of (2.36) gives correct results, at least for the *space-time average* of $(\partial E/\partial t)_{\mathrm{coll}}$.

One proceeds by substituting (4.10) into (4.4) and studying the nature of the individual contributions, $(\partial E/\partial t)_{\mathrm{coll}}^{(n)}$. Firstly, it is clear that only the $n\pm0$, 1, 2 terms play a role (except in the case of large impressed amplitude, which is not of interest here). Next, upon considering $(\partial E/\partial t)_{\mathrm{coll}}^{(0)}$ one notes that, by virtue of (4.11), the integrand contains the factor $(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)$ which is zero, so that

$$(\partial E/\partial t)_{\mathrm{coll}}^{(0)}=0. \quad (4.13)$$

For the same reason, the second term of $W_{(+)}^{(\lambda,1)}(\mathbf{k}\rightarrow\mathbf{k}')$, as given by the right-hand side of (4.12), makes no contribution to $(\partial E/\partial t)_{\mathrm{coll}}^{(1)}$. The first term of (4.12), when substituted into (4.4), gives the factor

$$(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\delta'(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)=-\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda). \quad (4.14)$$

Inserting (4.14) into the expression for $(\partial E/\partial t)_{\mathrm{coll}}^{(1)}$, and comparing with (4.11), one obtains

$$\left(\frac{\partial E}{\partial t}\right)_{\mathrm{coll}}^{(1)}=-V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})W_{(+)}^{(\lambda,0)}(\mathbf{k}\rightarrow\mathbf{k}')\times\{N_\lambda f(\mathbf{k})[1-f(\mathbf{k})]-(N_\lambda+1)f(\mathbf{k}')[1-f(\mathbf{k})]\}. \quad (4.15)$$

It is at this point to be noted that (4.15) vanishes when $f(k)$ is equal to the Fermi distribution, $f_0(E_{\mathbf{k}})$—that distribution being essentially *definable* by the requirement that

$$N_\lambda f_0(E_{\mathbf{k}})[1-f_0(E_{\mathbf{k}'})]=(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_{\mathbf{k}})] \quad (4.16)$$

for all $\mathbf{k}$, $\mathbf{k}'$, and $\lambda$ for which $W_{(+)}^{(\lambda,0)}(\mathbf{k}\rightarrow\mathbf{k}')\neq0$ (i.e., for $E_{\mathbf{k}'}=E_{\mathbf{k}}+\hbar\omega_\lambda$).

Finally, in considering the integral for $(\partial E/\partial t)_{\mathrm{coll}}^{(2)}$, one notes that, since $W_{(+)}^{(\lambda,2)}(\mathbf{k}\rightarrow\mathbf{k}')$ is already quadratic in the impressed amplitude, it is appropriate to replace $f(\mathbf{k})$ by its zeroth order approximation, $f_0(E_{\mathbf{k}})$. Thus

$$\left(\frac{\partial E}{\partial t}\right)_{\mathrm{coll}}^{(2)}=V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)W_{(+)}^{(\lambda,2)}(\mathbf{k}\rightarrow\mathbf{k}')\times\{N_\lambda f_0(E_{\mathbf{k}})[1-f_0(E_{\mathbf{k}'})]-(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_{\mathbf{k}})]\}. \quad (4.17)$$

The significant feature of (4.17) is that it is independent of the actual electron distribution, since it contains this distribution only in its zeroth order form. In fact, by virtue of the vanishing of $(\partial E/\partial t)_{\mathrm{coll}}^{(0)}$ and of $(\partial E/\partial t)_{\mathrm{coll}}^{(1)}$ for $f(k)=f_0(E_{\mathbf{k}})$ (as pointed out above), Eq. (4.17) represents the *total* $(\partial E/\partial t)_{\mathrm{coll}}$ under the condition that $f(\mathbf{k})=f_0(E_{\mathbf{k}})$, i.e., that the electron distribution in $\mathbf{k}$ space be spatially uniform and constant in time.

It may now be pointed out that the space-time average of such a transfer (to a spatially and temporarily uniform distribution of electrons) may independently be computed from the first-order bilinear interaction, (1.22), by conventional perturbation theory, as is done in Appendix II. The limitation on the use of this theory in the calculation of $W_{(+)}^{(\lambda)}(\mathbf{k}\rightarrow\mathbf{k}')$, given in Sec. 2, arises, as explained at the beginning of that section, from the requirements of space-time localization. For the special case represented by the space-time average of (4.17), however, such localization is *not required*; hence, the results of Appendix II are applicable. In particular, the space-time average of (4.17) is correctly given by the right-hand side of (II9), i.e.,

$$\langle(\partial E/\partial t)_{\mathrm{coll}}^{(2)}\rangle_{\mathrm{Av}}=V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}_\lambda}[m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})]^2\times(1/\kappa T)N_\lambda f_0(E_{\mathbf{k}})[1-f_0(E_{\mathbf{k}'})]. \quad (4.18)$$

With this result, the question of the applicability of (2.36) to the computation of $\langle(\partial E/\partial t)_{\mathrm{coll}}^{(2)}\rangle_{\mathrm{Av}}$ may now be resolved by straightforward comparison of the right-hand side of (4.18) with what would be obtained by *assuming* the validity of (2.36) to the *second order* in the impressed amplitude, thereby obtaining an explicit expression for $W_{(\pm)}^{(\lambda,2)}(\mathbf{k}\rightarrow\mathbf{k}')$ analogous to (4.11) and (4.12), and inserting it into (4.17). However, one may alternatively obtain the expression for $\langle(\partial E/\partial t)_{\mathrm{coll}}^{(2)}\rangle_{\mathrm{Av}}$

[23] Since the $\mathbf{k}$, $\mathbf{k}'$ sums, into which (4.11) and (4.12) enter, are in practice always replaced by integrals, the Kronecker delta, $\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda}$, is to be regarded as a three-dimensional delta function, and hence comparably differentiable.

which results from (2.36), simply by setting $f(\mathbf{k})$ equal to $f_0(E_\mathbf{k})$ in (4.6) [since, as noted above, $(\partial E/\partial t)_{\text{coll}}^{(0)} = (\partial E/\partial t)_{\text{coll}}^{(1)} = 0$ for $f(\mathbf{k}) = f_0(E_\mathbf{k})$]. Noting, then, that (a) the curly bracket of (4.6) vanishes for $f(\mathbf{k}) = f_0(\epsilon_\mathbf{k})$, (b) only terms thereof linear in the amplitude of the impressed wave are required, and (c) to the first order,

$$f_0(E_\mathbf{k}) = f_0(\epsilon_\mathbf{k}) + m\mathbf{v}_i\cdot\mathbf{v}_\mathbf{k}\partial f_0/\partial E_\mathbf{k} = f_0(E_\mathbf{k}) - m\mathbf{v}_i\cdot\mathbf{v}_\mathbf{k} f_0(E_\mathbf{k})[1-f_0(E_\mathbf{k})]/\kappa T,$$

one readily obtains, upon substitution of the last equality into (4.6) and use of (4.16),

$$\left(\frac{\partial E}{\partial t}\right)_{\text{coll}}^{(2)} = V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}[m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}'}-\mathbf{v}_\mathbf{k})]^2 W_{(+)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}') \times\frac{N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]}{\kappa T}. \quad (4.19)$$

Then, taking account of the fact that, since the factor $[m\mathbf{v}_i\cdot(\mathbf{v}_{\mathbf{k}'}-\mathbf{v}_\mathbf{k})]^2$ is already quadratic in the impressed amplitude, $W_{(\pm)}^{(\lambda)}(\mathbf{k}\to\mathbf{k}')$ may be replaced by its zeroth order approximation, $W_{(+)}^{(\lambda,0)}(\mathbf{k}\to\mathbf{k}')$, as given by the right-hand side of (4.11), one immediately observes that the space-time average of the resulting expression for $(\partial E/\partial t)_{\text{coll}}^{(2)}$ is identical with the right-hand side of (4.18). The applicability of (2.36) for the computation of average energy transfer, and hence of ultrasonic absorption, is thus established.

The above discussion is by no means intended to exclude the possibility that (2.36) may actually apply to the computation of instantaneous, local energy transfer. It appears in fact possible that the treatment of Secs. 1 and 2 may straightforwardly—albeit tediously—be extended to include second-order terms; the result of such an extension would quite likely constitute a second-order verification of (2.36). However, in view of the circumstance that, in practice, only the average energy transfer is required, such an extension will not be attempted here.

## APPENDIX I

As pointed out in Sec. 3 [in the paragraph preceding Eq. (3.17)], the existence of nonvanishing interband matrix elements, $(n', \mathbf{k}\pm\mathbf{q}_i|V_i|n\mathbf{k})$, of the first-order impressed electron-lattice interaction, gives rise to virtual interband transitions of the type $n\mathbf{k}\to n', \mathbf{k}\pm\mathbf{q}_i$. These transitions constitute, in effect, a modification of the basic Bloch states of the system, the modified states, $\psi_{n\mathbf{k}}^{(i)}(\mathbf{r},t)$, being given by the standard expression

$$\psi_{n\mathbf{k}}^{(i)}(\mathbf{r},t) = \left\{\psi_{n\mathbf{k}}(\mathbf{r}) + \sum_{n'\neq n}\frac{\psi_{n',\mathbf{k}+\mathbf{q}_i}(n',\mathbf{k}+\mathbf{q}_i|V_i|n\mathbf{k})}{E_{n\mathbf{k}}+\hbar\omega_i-E_{n',\mathbf{k}+\mathbf{q}_i}} + \sum_{n'\neq n}\frac{\psi_{n',\mathbf{k}-\mathbf{q}_i}(n',\mathbf{k}-\mathbf{q}_i|V_i^*|n\mathbf{k})}{E_{n\mathbf{k}}-\hbar\omega_i-E_{n',\mathbf{k}-\mathbf{q}_i}}\right\} \times\exp(-iE_{n\mathbf{k}}t/\hbar), \quad (I1)$$

where, in accordance with the text Eq. (1.3),

$$V_i(\mathbf{r},t) = \sum_\mathbf{g}\exp\{i[\mathbf{q}_i\cdot\mathbf{g}-\omega_i t]\}\mathbf{u}_i\cdot\text{grad}_\mathbf{g}V(\mathbf{r}). \quad (I2)$$

Associated with this modification is an additional "induced" component of electron current density; the computation of this induced component is the subject matter of the present appendix.

Before any calculations are entered upon, it should be stated that the actual current density at each point of the crystal is not to be computed. Rather, one is interested in a spatial average over dimensions large compared to lattice distances but nevertheless small compared to the impressed wavelength. Such an average would constitute a "macrocurrent" which, at the same time, would be sufficiently local for application to semiclassical treatments of ultrasonic absorption.

A convenient way of computing the macrocurrent density is to calculate the long-wavelength space-Fourier components of the microcurrent density. For example, the Fourier component of current density associated with a given "modified" Bloch state is

$$\mathbf{j}_{n\mathbf{k}}(\mathbf{q}) = \int e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{\hbar}{2im}[\psi_{n\mathbf{k}}^{(i)*}\,\text{grad}_\mathbf{r}\psi_{n\mathbf{k}}^{(i)} - \psi_{n\mathbf{k}}^{(i)}\,\text{grad}_\mathbf{r}\psi_{n\mathbf{k}}^{(i)*}]dV, \quad (I3)$$

where the volume integration goes over the whole crystal. It will immediately be noticed from (I1) that the only nonvanishing long-wavelength components are those for which $\mathbf{q}=\pm\mathbf{q}_i$. The macrocurrent density is then

$$\mathbf{j}_{n\mathbf{k}}(\mathbf{r},t) = \frac{1}{V}[j_{n\mathbf{k}}(\mathbf{q}_i)e^{i\mathbf{q}_i\cdot\mathbf{r}} + \mathbf{j}_{n\mathbf{k}}(-\mathbf{q}_i)e^{-i\mathbf{q}_i\cdot\mathbf{r}}], \quad (I4)$$

[$\mathbf{j}_{n\mathbf{k}}(\pm\mathbf{q}_i)$ being a function of time via the time dependence of $\psi_{n\mathbf{k}}^{(s)}(\mathbf{r},t)$].

An alternative form of (I3) is

$$\mathbf{j}_{n\mathbf{k}}(\mathbf{q}) = \int\psi_{n\mathbf{k}}^{(i)*}\mathbf{v}(\mathbf{q})\psi_{n\mathbf{k}}^{(i)}dV, \quad (I5)$$

where $\mathbf{v}(\mathbf{q})$ is defined by the operator equation

$$\mathbf{v}(\mathbf{q}) = \frac{\hbar}{2im}[e^{-i\mathbf{q}\cdot\mathbf{r}}\,\text{grad}_\mathbf{r} + \text{grad}_\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}]. \quad (I6)$$

[It will be noted that $\mathbf{v}(\mathbf{q})$ is not Hermitian, but is adjoint to $\mathbf{v}(-\mathbf{q})$.]

The insertion of (I1) into (I5) yields, without further ado,

$$\mathbf{j}_{n\mathbf{k}}(\mathbf{q}_i) = \sum_{n'\neq n}\frac{(n\mathbf{k}|\mathbf{v}(\mathbf{q}_i)|n',\mathbf{k}+\mathbf{q}_i)(n',\mathbf{k}+\mathbf{q}_i|V_i|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}+\hbar\omega_i} + \sum_{n'\neq n}\frac{(n',\mathbf{k}-q_i|V_i^*|n\mathbf{k})^*(n',\mathbf{k}-\mathbf{q}_i|\mathbf{v}(\mathbf{q}_i)|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}-\mathbf{q}_i}-\hbar\omega_i}, \quad (I7)$$

which, with the usual laws of matrix algebra (in particular that $V_i^*$ and $V_i$ are mutually adjoint), becomes

$$\mathbf{j}_{n\mathbf{k}}(\mathbf{q}_i) = \sum_{n'\neq n} \frac{(n\mathbf{k}|\,\mathbf{v}(\mathbf{q}_i)\,|n',\mathbf{k}+\mathbf{q}_i)(n',\mathbf{k}+\mathbf{q}_i|\,V_i\,|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}+\hbar\omega_i} + \sum_{n'\neq n} \frac{(n\mathbf{k}|\,V_i\,|n',\mathbf{k}-\mathbf{q}_i)(n',\mathbf{k}-\mathbf{q}_i|\,\mathbf{v}(\mathbf{q}_i)\,|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}-\mathbf{q}_i}-\hbar\omega_i}. \quad \text{(I8)}$$

The evaluation of (I8), while not intrinsically difficult, is quite tedious, and the results are not simple. Fortunately, however, what is required is not $\mathbf{j}_{n\mathbf{k}}(\mathbf{q}_i)$, itself, but rather

$$\mathbf{J}(\mathbf{q}_i)=\textstyle\sum_{\mathbf{k}}\mathbf{j}_{n\mathbf{k}}(\mathbf{q}_i)f_0(\mathbf{k}), \quad \text{(I9)}$$

where $f_0(\mathbf{k})$ is the unperturbed Fermi distribution. [The replacement of $f_0(\mathbf{k})$ by the actual distribution function would yield corrections to (I9) which are of second (or higher) order in the impressed amplitude, and hence negligible.] Now, since the states $\mathbf{k}$ and $-\mathbf{k}$ have the same energy, so that $f_0(\mathbf{k})$ and $f_0(-\mathbf{k})$ are equal, (I9) may be written as

$$\mathbf{J}(\mathbf{q}_i)=\textstyle\sum_{\mathbf{k}}\mathbf{J}_{n\mathbf{k}}(\mathbf{q}_i)f_0(\mathbf{k}), \quad \text{(I10)}$$

where

$$\mathbf{J}_{n\mathbf{k}}(\mathbf{q}_i)\equiv\tfrac{1}{2}[j_{n\mathbf{k}}(\mathbf{q}_i)+j_{n,-\mathbf{k}}(\mathbf{q}_i)]. \quad \text{(I11)}$$

The evaluation of $\mathbf{J}_{n\mathbf{k}}(\mathbf{q}_i)$ is relatively simple and will now be carried out.

The procedure is to add to (I8) the corresponding expression for $\mathbf{j}_{n,-\mathbf{k}}(\mathbf{q}_i)$. This expression, which is gotten simply by replacing $\mathbf{k}$ by $-\mathbf{k}$ in all terms of the right-hand side of (I8), will not be written down explicitly; the important step is to transform it into a form which is suitable for superposition with (I8). One is able to achieve this transformation by use of the relationships

$$E_{n\mathbf{k}}=E_{n,-\mathbf{k}}, \quad \text{(I12)}$$

$$\psi_{n\mathbf{k}}=\psi_{n,-\mathbf{k}}^*, \quad \text{(I13)}$$

which hold by virtue of the so-called "time-reversal" symmetry of the system (in the absence of a magnetic field). From (I13) and the adjoint properties of the operators $V_i$ and $\mathbf{v}(\mathbf{q})$, it is easily shown that

$$(n,-\mathbf{k}|\,\mathbf{v}(\mathbf{q}_i)\,|n',-\mathbf{k}+\mathbf{q}_i) = -(n,\mathbf{k}|\,\mathbf{v}^*(\mathbf{q}_i)\,|n',\mathbf{k}-\mathbf{q}_i) = -(n,\mathbf{k}|\,\mathbf{v}(-\mathbf{q})\,|n',\mathbf{k}-\mathbf{q}_i) = -(n',\mathbf{k}-\mathbf{q}_i|\,\mathbf{v}(\mathbf{q}_i)\,|n\mathbf{k}), \quad \text{(I14)}$$

$$(n,-\mathbf{k}|\,V_i\,|n',-\mathbf{k}-\mathbf{q}_i)=(n,\mathbf{k}|\,V_i^*\,|n',\mathbf{k}+\mathbf{q}_i)^* = (n',\mathbf{k}+\mathbf{q}_i|\,V_i\,|n\mathbf{k}), \quad \text{(I15)}$$

similarly,

$$(n',-\mathbf{k}-\mathbf{q}_i|\,\mathbf{v}(\mathbf{q}_i)\,|n,-\mathbf{k}) = -(n,\mathbf{k}|\,\mathbf{v}(\mathbf{q}_i)\,|n',\mathbf{k}+\mathbf{q}_i), \quad \text{(I16)}$$

$$(n',-\mathbf{k}+\mathbf{q}_i|\,V_i\,|n,-k)=(n,\mathbf{k}|\,V_i\,|n',\mathbf{k}-\mathbf{q}_i). \quad \text{(I17)}$$

Making use of these relations together with (I12), one may transform the expression for $\mathbf{j}_{n,-\mathbf{k}}(\mathbf{q}_i)$ to read

$$\mathbf{j}_{n,-\mathbf{k}}(\mathbf{q}_i) = -\sum_{n'\neq n} \frac{(n\mathbf{k}|\,\mathbf{v}(\mathbf{q}_i)\,|n',\mathbf{k}+\mathbf{q}_i)(n',\mathbf{k}+\mathbf{q}_i|\,V_i\,|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}-\hbar\omega_i} - \sum_{n'\neq n} \frac{(n\mathbf{k}|\,V_i\,|n',k-\mathbf{q}_i)(n',\mathbf{k}-\mathbf{q}_i|\,\mathbf{v}(\mathbf{q}_i)\,|n\mathbf{k})}{E_{n\mathbf{k}}-E_{n',\mathbf{k}-\mathbf{q}_i}+\hbar\omega_i}, \quad \text{(I18)}$$

in which, apart from the signs in front of the $\hbar\omega_i$'s in the denominators, each term is seen to be equal and opposite to a corresponding term of (I8). Hence, upon inserting (I18) and (I8) into (I11), one finds that $\mathbf{j}_{n,\mathbf{k}}(\mathbf{q}_i)$ and $\mathbf{j}_{n,-\mathbf{k}}(\mathbf{q}_i)$ almost cancel each other, the residual expression being proportional to $\hbar\omega_i \ll E_{n\mathbf{k}}-E_{n'\mathbf{k}}$.

In carrying out the evaluation of (I11), one may legitimately use approximations of the type

$$\frac{1}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}-\hbar\omega_i} = \frac{1}{E_{n\mathbf{k}}-E_{n',\mathbf{k}+\mathbf{q}_i}}+\frac{\hbar\omega_i}{(E_{n\mathbf{k}}-E_{n',\mathbf{k}})^2}, \quad \text{(I19)}$$

and let $\mathbf{q}_i$ approach zero in the matrix elements. One then has

$$\mathbf{J}_{n\mathbf{k}}(\mathbf{q}_i)=\hbar\omega_i\sum_{n'\neq n}\left\{-\frac{(n\mathbf{k}|\,\mathbf{v}\,|n'\mathbf{k})(n',\mathbf{k}|\,V_i\,|n\mathbf{k})}{(E_{n\mathbf{k}}-E_{n'\mathbf{k}})^2} + \frac{(n\mathbf{k}|\,V_i\,|n'\mathbf{k})(n'\mathbf{k}|\,\mathbf{v}\,|n\mathbf{k})}{(E_{n\mathbf{k}}-E_{n'\mathbf{k}})^2}\right\}, \quad \text{(I20)}$$

where

$$\mathbf{v}\equiv\lim_{\mathbf{q}_i\to 0}\mathbf{v}(\mathbf{q}_i)$$

is seen to be the ordinary velocity operator

$$\mathbf{v}\equiv(\hbar/im)\,\mathrm{grad}_{\mathbf{r}}, \quad \text{(I21)}$$

and where

$$(n'\mathbf{k}|\,V_i\,|n\mathbf{k})\equiv\lim_{\mathbf{q}_i\to 0}(n',\mathbf{k}+\mathbf{q}_i|\,V_i\,|n\mathbf{k}) = -(i/\hbar)(n'|\,\mathbf{u}_i\cdot p_{\mathbf{k}}\,|n) \times(E_{n\mathbf{k}}-E_{n'\mathbf{k}})e^{-i\omega_i t} \quad \text{(I22)}$$

in accordance with the text, Eq. (1.18). Introducing (I22) and the relation

$$(n'\mathbf{k}|\,\mathbf{v}\,|n\mathbf{k})=(1/m)(n'|\,\mathbf{P}_{\mathbf{k}}\,|n) \quad \text{(I23)}$$

into (I20), one has

$$\mathbf{J}_{n\mathbf{k}}(\mathbf{q}_i)=\frac{i\omega_i e^{-i\omega_i t}}{m}\sum_{n'\neq n}\left\{\frac{(n|\,\mathbf{p}_{\mathbf{k}}\,|n')(n'|\,\mathbf{u}_i\cdot\mathbf{p}_{\mathbf{k}}\,|n)}{E_{n\mathbf{k}}-E_{n'\mathbf{k}}} + \frac{(n|\,\mathbf{u}_i\mathbf{p}_{\mathbf{k}}\,|n')(n'|\,\mathbf{p}_{\mathbf{k}}\,|n)}{E_{n\mathbf{k}}-E_{n'\mathbf{k}}}\right\}. \quad \text{(I24)}$$

At this point, one may expediently use Eqs. (2.81.2), (2.81.3), and (2.81.8) of reference 12 (pp. 46 and 47); in the notation of this paper, these equations are equivalent to

$$\frac{1}{m}\sum_{n'\neq n}\frac{(n|p_{kx}|n')(n'|p_{ky}|n)+(n|p_{ky}|n')(n'|p_{kx}|n)}{E_{nk}-E_{n'k}}$$
$$=\frac{m}{h^2}\frac{\partial^2 E_{kn}}{\partial k_x\partial k_y}-\delta_{xy}=\frac{m}{\hbar}\frac{\partial v_{ky}}{\partial k_x}-\delta_{xy}$$
$$=\frac{m}{\hbar}\frac{\partial v_{kx}}{\partial k_y}-\delta_{xy}. \quad \text{(I25)}$$

Inserting (I25) into (I24), one has

$$\mathbf{J}_{nk}(\mathbf{q}_i)=i\omega_i\left[\frac{m}{\hbar}\operatorname{grad}_k(\mathbf{u}_i\cdot\mathbf{v}_k)-\mathbf{u}_i\right]e^{-i\omega_i t}$$
$$=\frac{\partial}{\partial t}\left\{\left[\frac{m}{\hbar}\operatorname{grad}_k(\mathbf{u}_i\cdot\mathbf{v}_k)-\mathbf{u}_i\right]e^{-i\omega_i t}\right\}. \quad \text{(I26)}$$

Turning, now, to the quantity $\mathbf{J}_{nk}(-\mathbf{q}_i)$, one readily establishes from (I6) that

$$\mathbf{J}_{nk}(-\mathbf{q}_i)=\mathbf{J}_{nk}{}^*(q_i)$$
$$=\frac{\partial}{\partial t}\left\{\left[\frac{m}{\hbar}\operatorname{grad}_k(\mathbf{u}_i{}^*\cdot\mathbf{v}_k)-\mathbf{u}_i{}^*\right]e^{+i\omega_i t}\right\}. \quad \text{(I27)}$$

Then, substituting (I26) and (I27) into the relation

$$\mathbf{J}_{nk}(\mathbf{r},t)=\frac{1}{V}[\mathbf{J}_{nk}(\mathbf{q}_i)e^{i\mathbf{q}_i\cdot\mathbf{r}}+J_{nk}(-\mathbf{q}_i)e^{-i\mathbf{q}_i\cdot\mathbf{r}}] \quad \text{(I28)}$$

(which is the analog of I4), one has

$$J_{nk}=-\frac{1}{V}\left[\frac{m}{\hbar}\operatorname{grad}_k(\mathbf{v}_i\cdot\mathbf{v}_k)-\mathbf{v}_i\right]$$
$$=\frac{1}{V}\left[\mathbf{v}_i-\frac{m}{\hbar}\mathbf{v}_i\cdot\operatorname{grad}_k\mathbf{v}_k\right], \quad \text{(I29)}$$

where

$$\mathbf{v}_i\equiv\frac{\partial}{\partial t}[\mathbf{u}_i e^{i(\mathbf{q}\cdot\mathbf{r}-\omega_i t)}+u_i{}^* e^{-i(q\cdot\mathbf{r}-\omega_i t)}] \quad \text{(I30)}$$

is the impressed lattice-displacement velocity [given equivalently by the text Eq. (2.18)]. The corresponding local expectation value of velocity is gotten by dividing (I29) by the electron density (which is $V^{-1}$ per occupied Bloch state), and is

$$\mathbf{v}_{nk}{}^{(\text{in})}=\mathbf{v}_i-(m/\hbar)\mathbf{v}_i\cdot\operatorname{grad}_k\mathbf{v}_k, \quad \text{(I31)}$$

which, when substituted into the equation

$$\langle\mathbf{v}\rangle_{\text{in}}\equiv\sum_{nk}\mathbf{v}_{nk}{}^{(\text{in})}f_0(\mathbf{k}), \quad \text{(I32)}$$

gives the induced component of average velocity, utilized in the text Eq. (3.25).

## APPENDIX II

In this Appendix, the transitions associated with the bilinear interaction will be computed by conventional perturbation theory. From (1.22) [and the generalizations obtained by changing the signs of either of the pairs $(\mathbf{q}_i,\omega_i)$ or $(\mathbf{q}_\lambda,\omega_\lambda)$], the transitions in question are of the type

$$\mathbf{k}\rightarrow\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i, \quad \text{(II1)}$$

in which the $\pm$ sign refers to the absorption or emission of either kind of phonon—thermal or impressed. In the light of the discussion at the beginning of Sec. 2, such transitions are to be regarded as taking place anywhere in the volume of the sample—in particular in a region whose linear dimensions are large compared to an impressed wavelength; their time duration (infinite, in principle) is to be regarded as large compared, e.g., to $1/\omega_i$. It then follows that the transition probabilities for (II1) will contain the impressed amplitude as space-time averages, rather than as local, instantaneous quantities (such as characterize the collision-drag effect). In particular, interference between the bilinear and ordinary electron-lattice interactions can no longer occur, since, starting from the same initial state, $\mathbf{k}$, the two interactions send electrons to different final states—$\mathbf{k}\pm\mathbf{q}$ and $\mathbf{k}\pm\mathbf{q}\pm\mathbf{q}_i$; the two types of transitions thus constitute independent processes, and are to be computed separately.

With these preliminaries out of the way, one may now write down the first-order perturbation expressions for the probabilities of transitions (II1). Using the notation of Sec. 2 [see (2.27)], one has from (1.22)

$$P_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')$$
$$=\frac{N^2}{\hbar^4 T}|(\mathbf{k}'|\mathbf{u}_\lambda\cdot\operatorname{grad}_{q=0}V(\mathbf{r})|\mathbf{k})|^2|m\mathbf{u}_i\cdot(\mathbf{v}_k-\mathbf{v}_{k'})|^2$$
$$\times\Omega(E_{k'}-E_k\mp\hbar\omega_\lambda\mp\hbar\omega_i)\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i},$$

or, on going from $\Omega(x)$ to the energy delta function via (2.24),

$$P_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')$$
$$=\frac{2\pi N^2}{\hbar^3}|(\mathbf{k}'|\mathbf{u}_\lambda\cdot\operatorname{grad}_{q=0}V(\mathbf{r})|\mathbf{k})|^2|m\mathbf{u}_i\cdot(\mathbf{v}_k-\mathbf{v}_{k'})|^2$$
$$\times\delta(E_{k'}-E_k\mp\hbar\omega_\lambda\mp\hbar\omega_i)\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i}. \quad \text{(II2)}$$

In (II2) the superscript $(\pm)$ represents the absorption or emission of an *impressed* phonon, the subscript $(\pm)$ the same for a *thermal* phonon (of the $\lambda$th mode).

For the utilization of (II2) in transport theory—in particular, in the computation of mean energy transfer via an expression of the type of (4.4)—it is convenient to express the $N_\lambda$ dependence of (II2) explicitly (while at the same time treating the impressed displacement amplitude as a classical field). Thus [see (2.35) and (2.36)]

$$P_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')=\left\{\begin{matrix}N_\lambda\\N_\lambda+1\end{matrix}\right\}W_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}'),\quad\text{(II3)}$$

where

$$W_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')=w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}|m\mathbf{u}_i\cdot(\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'})|^2\times\delta(E_{\mathbf{k}'}-E_\mathbf{k}\mp\hbar\omega_\lambda\mp\hbar\omega_i)\delta_{\mathbf{k}',\,\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i},\quad\text{(II4)}$$

$w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}$ being given by (2.37).

A further feature of significance in the application to transport theory is that, in writing down the reversibility relation—the analog of (3.2)—one must reverse the impressed-phonon process, as well as that of the thermal phonon; thus

$$W_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')=W_{(\mp)(\lambda)}{}^{(\mp)}(\mathbf{k}'\rightarrow\mathbf{k}).\quad\text{(II5)}$$

It is then readily shown [by appropriate generalization of (4.3), use of (II5), and replacement of $f(\mathbf{k})$ by $f_0(E_\mathbf{k})$] that the analog of (4.4) is

$$\left\langle\left(\frac{\partial E}{\partial t}\right)_{\text{coll}}\right\rangle_{\text{Av}}=V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}\sum_{(\pm)}(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)W_{(+)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')\times\{N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]-(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_\mathbf{k})]\}$$

$$=V^{-1}\sum_{\lambda,\mathbf{k},\mathbf{k}'}\sum_{(\pm)}\hbar\omega_i W_{(+)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')\times\{N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]-(N_\lambda\pm1)f_0(E_{\mathbf{k}'})[1-f_0(E_\mathbf{k})]\},\quad\text{(II6)}$$

where $N_\lambda$ is given by Planck's distribution, and where the notation $\sum_{(\pm)}$ indicates summation with respect to the two processes of absorption and emission of an impressed phonon. The last equality of (II6) is based on conservation of energy, as expressed by the delta function in $W_{(+)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')$.

One may now take advantage of the smallness of $\hbar\omega_i$ and $q_i$ to develop (II6) as a power series in these parameters. This is done most conveniently by expanding $\delta(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda\mp\hbar\omega_i)$ and $\delta_{\mathbf{k}',\,\mathbf{k}+\mathbf{q}_\lambda\pm\mathbf{q}_i}$ in powers of $\hbar\omega_i$ and $\mathbf{q}_i$ [with the understanding that the Kronecker delta is to be treated as a three-dimensional delta function, in the same way as done previously in connection with (4.11)]. It is then seen that if this expansion is limited to terms linear in $\hbar\omega_i$ or $\mathbf{q}_i$, only the term in $\hbar\omega_i$, containing the derivative, $\delta'(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)$, of the energy delta function, contributes; since all other terms contain $\delta(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)$, and since the curly-bracket factor of (II6) vanishes for $E_{\mathbf{k}'}=E_\mathbf{k}+\hbar\omega_\lambda$, the energy dependence of such terms is of the form $x\delta(x)=0$. One is thus left with

$$\left\langle\left(\frac{\partial E}{\partial t}\right)_{\text{coll}}\right\rangle_{\text{Av}}=-2\sum_{\lambda,\mathbf{k},\mathbf{k}'}\hbar^2\omega_i{}^2\delta'(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)\delta_{\mathbf{k}',\,\mathbf{k}+\mathbf{q}_\lambda}w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}\times[m\mathbf{u}_i\cdot(\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'})/\hbar]^2\{N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]-(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_\mathbf{k})]\}$$

$$=-\sum_{\lambda,\mathbf{k},\mathbf{k}'}\delta(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)\delta_{\mathbf{k}',\,\mathbf{k}+\mathbf{q}_\lambda}w_{\mathbf{k},\,\mathbf{k}'}{}^{(\lambda)}\times\langle[m\mathbf{v}_i\cdot(\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'})]^2\rangle_{\text{Av}}N_\lambda'\{f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]-f_0(E_{\mathbf{k}'})[1-f_0(E_\mathbf{k})]\},\quad\text{(II7)}$$

where $N_\lambda'$ is the derivative of $N_\lambda$ with respect to $\hbar\omega_\lambda$. In obtaining the last equality of (II7), use was made of (4.14) and (2.18); from the latter relation one has

$$\langle v_{ix}{}^2\rangle_{\text{Av}}=2\omega^2u_{ix}{}^2\quad\text{(II8)}$$

(the subscript $x$ denoting an arbitrary Cartesian component) which is used to replace $\mathbf{u}_i$ by $\mathbf{v}_i$.

Utilizing the fact that

$$N_\lambda'=-(1/\kappa T)N_\lambda(N_\lambda+1),$$

and that

$$N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]=(N_\lambda+1)f_0(E_\mathbf{k})[1-f_0(E_\mathbf{k})],$$

when the argument of the delta function vanishes (i.e., when $E_{\mathbf{k}'}=E_\mathbf{k}+\hbar\omega_\lambda$), one may rewrite (II7)

$$\left\langle\left(\frac{\partial E}{\partial t}\right)_{\text{coll}}\right\rangle_{\text{Av}}=\sum_{\lambda,\mathbf{k},\mathbf{k}'}\delta(E_{\mathbf{k}'}-E_\mathbf{k}-\hbar\omega_\lambda)\delta_{\mathbf{k}',\,\mathbf{k}+\mathbf{q}_\lambda}w_{\mathbf{k},\,\mathbf{k}'}{}^{(\lambda)}\times\langle[m\mathbf{v}_i\cdot(\mathbf{v}_\mathbf{k}-\mathbf{v}_{\mathbf{k}'})]^2\rangle_{\text{Av}}\times\{(1/\kappa T)N_\lambda f_0(E_\mathbf{k})[1-f_0(E_{\mathbf{k}'})]\}.\quad\text{(II9)}$$

Equation (II9) plays an important role in the discussion of energy transfer, given in Sec. 4 of this paper. For this reason, it is desirable to extend its domain of applicability beyond that indicated by the above derivation. In particular, it is of interest to investigate its validity in the region of $l_e/\lambda_i\ll1$, for which the transition-probability expressions (II4) no longer hold.

The method to be used here is a simplified version of the wave-packet approach employed in Sec. 2 of the text. The simplification consists in that no spatial localization is required; hence, the basic electronic states may be taken to be plane waves, as in the con-

ventional treatment given above. However, in order to avoid the multiple-scattering problem, the artifice of limiting the interaction time to an interval, $T_c$, small compared to the mean time between collisions, will be employed (as in Sec. 2). The result of such a limitation is that (II4) will be modified by the replacement

$$\delta(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda\mp\hbar\omega_i)\rightarrow \frac{1}{2\pi\hbar T_c}\Omega(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda\mp\hbar\omega_i), \quad \text{(II10)}$$

where

$$\Omega(x)\equiv\frac{4\sin^2(xT_c/2\hbar)}{x^2/\hbar^2} \quad \text{(II11)}$$

[see (2.23) and (2.24)].

Despite this modification, it is seen that (II5) and, hence, the first equality of (II6) are still valid. Furthermore, if $(\partial E/\partial t)_{\text{coll}}$ is defined as the net loss of ultrasonic energy (rather than as the gain in electron-phonon energy), so that the ultrasonic quantum energy, $\pm\hbar\omega_i$, occurs *ab initio* in place of $(E_{\mathbf{k}'}-E_{\mathbf{k}}\pm\hbar\omega_\lambda)$, the second equality of (II6) becomes superfluous.

At this point it is desirable to take advantage of the invariance of $W_{(\pm)(\lambda)}{}^{(\pm)}(\mathbf{k}\rightarrow\mathbf{k}')$, $E_{\mathbf{k}}$, $\hbar\omega_\lambda$, and $N_\lambda$ with respect to changing the signs of the wave vectors $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{q}_\lambda$, and $\mathbf{q}_i$. [This invariance is obvious for $E_{\mathbf{k}}$, $\hbar\omega_\lambda$, and $N_\lambda$, and can be verified for the collision operator *in toto* by inspection of (II4).] The first step is to reverse the signs of $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{q}_\lambda$, in (II6); this reversal is not a symmetry operation but a simple rearrangement of the terms of the summation over $\mathbf{k}$, $\mathbf{k}'$, and $\lambda$. Next, one reverses the signs of all the wave vectors, invoking thereby the above-stated invariance properties. The net result is an expression of the same form as (II6), the only change being the sign reversal of $\mathbf{q}_i$ in the Kronecker-delta factor of the collision operator. Upon adding this expression to the right-hand side of (II6) and dividing by 2, one obtains

$$\langle(\partial E/\partial t)_{\text{coll}}\rangle_{\text{Av}} = \sum_{\lambda,\mathbf{k},\mathbf{k}'}\sum_{(\pm)}(\pm\hbar\omega_i)W_{(+)(\lambda)}{}^{(\pm),\text{sym}}(\mathbf{k}\rightarrow\mathbf{k}') \times\{N_\lambda f_0(E_{\mathbf{k}})[1-f_0(E_{\mathbf{k}'})] -(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_{\mathbf{k}})]\}, \quad \text{(II12)}$$

where

$$W_{(\pm),(\lambda)}{}^{(\pm),\text{sym}}(\mathbf{k}\rightarrow\mathbf{k}') \equiv w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}|m\mathbf{u}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})|^2\frac{\Omega(E_{\mathbf{k}'}-E_{\mathbf{k}}\mp\hbar\omega_\lambda\mp\hbar\omega_i)}{2\pi\hbar T_c} \times\tfrac{1}{2}[\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda\pm\mathbf{q}_i}+\delta_{\mathbf{k}',\mathbf{k}\pm\mathbf{q}_\lambda\mp\mathbf{q}_i}]. \quad \text{(II13)}$$

From the form of (II13), in particular its symmetry with respect to the sign of $\mathbf{q}_i$, it is clear that the superscript "$(\pm)$," which was originally introduced to denote simultaneous change of sign of $\mathbf{q}_i$ and $\omega_i$, may now be taken to refer to the change of sign of $\omega_i$, alone.

One now expands the energy-resonance function in powers of $\hbar\omega_i$, *viz.*,

$$\Omega(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda\mp\hbar\omega_i) =\Omega(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\mp\hbar\omega_i\Omega'(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda). \quad \text{(II14)}$$

Upon substituting (II14) into (II13), one notes that the contribution of the first term of the right-hand side of (II14) vanishes, leaving

$$\left\langle\left(\frac{\partial E}{\partial t}\right)_{\text{coll}}\right\rangle_{\text{Av}} =\frac{1}{2\pi\hbar T_c}\sum_{\lambda,\mathbf{k},\mathbf{k}'}\sum_{(\pm)}(-\hbar\omega)^2 w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}|m\mathbf{u}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})| \times\Omega'(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}_\lambda}\{N_\lambda f_0(E_{\mathbf{k}})[1-f_0(E_{\mathbf{k}'})] -(N_\lambda+1)f_0(E_{\mathbf{k}'})[1-f_0(E_{\mathbf{k}})]\}, \quad \text{(II15)}$$

where $\mathbf{q}_i$ has been set equal to zero in the Kronecker deltas (since it is now obviously a small correction).

At this point, one takes cognizance of the fact that the square bracket of (II15) is of the form $(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)F_{(\lambda)}(\mathbf{k}',\mathbf{k})$, where $F_{(\lambda)}(\mathbf{k}',\mathbf{k})$ is "slowly varying," i.e., its variation is small over energy intervals $\ll\kappa T$. Then, if the collision time interval, $T_c$, determining the energy breadth, $\hbar/T_c$, of $\Omega(x)$ be chosen so that

$$\hbar/T_c\ll\kappa T, \quad \text{(II16)}$$

$F_{(\lambda)}(\mathbf{k}',\mathbf{k})$ may be considered constant, and hence equal to the derivative, with respect to $\hbar\omega_\lambda$, of the curly-bracket factor of (II15) (evaluated at $\hbar\omega_\lambda=E_{\mathbf{k}'}-E_{\mathbf{k}}$). (As far as the other "nonresonant" factors of (II15) are concerned, namely $w_{\mathbf{k}\mathbf{k}'}{}^{(\lambda)}$ and $|m\mathbf{u}_i\cdot(\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}'})|^2$, these may certainly be considered as constant over the energy interval $\hbar/T_c$.) One is thus left with the factor

$$(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)\Omega'(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda),$$

which, through an integration by parts, may be shown to be equivalent to $-\Omega(E_{\mathbf{k}'}-E_{\mathbf{k}}-\hbar\omega_\lambda)$. One thus obtains for $\langle(\partial E/\partial t)_{\text{coll}}\rangle_{\text{Av}}$ an expression of the same form as (II7), except for the replacement indicated by (II10). Then, in view of the slow variation of $N_\lambda'$ and the curly bracket of (II7) in comparison to $\Omega(x)$, one may finally dispense with (II10) altogether [i.e., consider $\Omega(x)$ equivalent to $\delta(x)$], so that (II7) and (II9) are valid for $l_e/\lambda_i\ll1$, as was to be demonstrated.

## APPENDIX III

In this appendix, the restrictions on the wave-vector spread, $\Delta k$, of the electron wave packets, introduced in Sec. 2, will be discussed. For the wave-vector conservation rule (2.1) to be valid, it is required that

$$\Delta k\ll q_\lambda, \quad \text{(III1)}$$

$q_\lambda$ being of the order of $\kappa T/\hbar c_s$ (where $c_s$ is the velocity of sound and $\kappa$ Boltzmann's constant). Since $\Delta k$ has also to be large compared to $q_i$, it is necessary that

$$q_i \ll q_\lambda \sim \kappa T/\hbar c_s. \tag{III2}$$

This requirement is easily satisfied, even at temperatures as low as 1°K.

A more severe requirement is imposed by the energy conservation law (2.2). For this relation to be applicable, it is necessary that

$$v_0 \Delta k \ll \hbar\omega_\lambda \sim \kappa T, \tag{III3}$$

where $v_0$ is the average velocity at the Fermi surface. This inequality, together with $\Delta k \gg q_i$, leads to

$$q_i \ll \kappa T/\hbar v_0. \tag{III4}$$

In discussing the significance of this limit, it will be convenient to consider two cases:

*Case I.*—The electron mean free path, $l_e$, is small compared to the impressed wavelength. In this case, (III4) is less restrictive than the corresponding condition

$$1/\tau_e = v_0/l_e \ll \kappa T/\hbar, \tag{III5}$$

which is required for the applicability of the standard first-order electron-lattice theory to transport phenomena at low temperatures.[24] According to reference 24, this condition is generally fulfilled at low temperatures (i.e., low compared to the Debye temperature), so that (III4) also holds.

In the case of impurity scattering (which, although not treated in this paper, is of interest to include in the present discussion), the situation is different in that the collisions are elastic; arguments based on that fact are given in the cited material of reference 24 for replacing (III5) by the much less restrictive condition

$$1/\tau_e \ll \zeta/\hbar, \tag{III6}$$

where $\zeta$ is the Fermi energy. (III6) is equivalent to the statement that the uncertainty of wave packets is to be small compared to $\zeta$, rather than to $\kappa T$; in that event, a similar modification should hold for (III4), i.e., $q_i \ll \zeta/\hbar v_0$, which is always satisfied.

[24] See reference 14, p. 124, Eq. (6.41), and pp. 139–142.

*Case II.*—The electron mean free path is large compared to the ultrasonic wavelength ($l_e \gg 1/q_i$). In this case, it is first of interest to estimate the limitation on ultrasonic frequency, imposed by (III4). The latter is equivalent to

$$\nu_i \ll \nu_{\max}, \tag{III7}$$

where

$$\nu_{\max} = (c_s/v_0)\kappa T/2\pi\hbar. \tag{III8}$$

In a typical case ($c_s/v_0 \sim 10^{-3}$, $T = 4$°K), one estimates

$$\nu_{\max} \sim 80 \text{ megacycles/sec.}$$

This frequency, though numerically higher, is of the order of magnitude of those attained in the most recent experimental work.[25,26]

It should, however, be realized that, from the standpoint of the general theoretical interpretation of ultrasonic absorption experiments, the above-indicated limitation on the applicability of the present treatment is actually not very serious. The reason is as follows. It has been shown, e.g., by Pippard[1] and by Akhieser *et al.*,[27] that, throughout the domain of mean free path large compared to ultrasonic wavelength, absorption results mainly from the interaction of electrons with the first-order impressed deformation field (as given by its intraband matrix elements, $(n\mathbf{k}'|V_i|n\mathbf{k})$. In such a situation one may, to a good approximation, compute the absorption by ignoring collisions entirely[28]; the effects treated in the present paper would then constitute a small higher-order correction to such a computation. The domain in which collisions become of paramount importance is that of mean free path *small* compared to ultrasonic wavelength; for this case, as shown above, the treatment of Sec. 2 does apply.

[25] W. P. Mason and H. E. Bömmel, J. Acoust. Soc. Am. **28**, 930 (1956).

[26] R. W. Morse and H. V. Bohm, Phys. Rev. **108**, 1094 (1957).

[27] Akhieser, Kaganov, and Liubarskii, J. Exptl. Theoret. Phys. U.S.S.R. **32**, 837 (1957) [translation: Soviet Phys. JETP **5**, 685 (1957)].

[28] See C. Kittel, Acta Met. **3**, 295 (1955), who, for the case of $l_e \gg 1/q_i$, computes the probabilities of transitions $\mathbf{k} \rightarrow \mathbf{k} \pm \mathbf{q}_i$ resulting from the action of the first-order deformation field, without regard for collisions of electrons with thermal phonons or impurities.