

General Relation between Fluxes from Collimated Point and Plane Sources of Radiation*

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The radiation density (flux integrated over all directions) from a collimated point source is shown to be representable as a Fourier-Hankel transform of the density from a collimated plane oblique source. The spatial moments of a point-source density are similarly expressible as linear combinations of a finite number of moments of plane-source densities corresponding to different source obliquities. These results are valid for any type of radiation, provided that the medium is unbounded, homogeneous, and isotropic, and provided that a linear transport equation applies. The plane-geometry representation of densities and density-moments is equivalent to a "separation of variables." It allows one to solve a sequence of problems in one space variable (distance from a source plane) for different values of the source obliquity, and to use the information thus obtained for constructing the solution of a problem involving two space variables (e.g., the longitudinal and radial distances from a point source).

1. INTRODUCTION

THIS note is concerned with a separation of variables which facilitates the solution of the transport equation under the following conditions: (1) The interaction of the diffusing particles with each other is negligible compared to their interaction with the atoms of the medium, so that the transport equation is linear. (2) The medium is unbounded, homogeneous, and isotropic. The results of this note pertain to the spatial density distribution of particles, i.e., the flux integrated over all directions. The more complicated case of the joint spatial and directional flux distribution, for which an analogous separation of variables holds, is treated elsewhere.^{1,2}

A basic solution of the transport equation can be obtained by calculating the (cylindrically symmetric) density distribution $F(z, \rho)$ at a radial distance ρ and longitudinal distance z from a point source which emits particles in the z direction. The density distribution resulting from an arbitrary source can be represented as the sum of distributions from different point-collimated sources with different locations and directions of emission. The point-source density $F(z, \rho)$ is difficult to obtain because a two-variable transport problem in z and ρ must be solved. So far, almost all of the exact solutions of the linear transport equation have been achieved for the case of one space variable only.

An alternative basic solution of the transport equation is obtained by calculating the density distribution $G(z; \theta_0)$ at a distance z from a plane source which emits particles with obliquity θ_0 . It will be shown that the point-source distribution $F(z, \rho)$, and therefore the density distribution corresponding to an arbitrary source, can be represented as a superposition of density distributions from plane sources with different locations and obliquities of emission. This result may be regarded as

an analog and generalization of the well-known plane-to-point transformation for isotropic sources.³

2. PLANE-TO-POINT SUPERPOSITION OF DENSITIES

Let us consider the density $G(z; \omega_0)$ at a point P located at a distance z from a plane source that emits radiation in direction ω_0 . (See Fig. 1.) The source is assumed to lie in the plane $z=0$. The unit vector ω_0 is represented by spherical coordinates θ_0 and φ_0 , with the z axis as polar axis.

One can regard the plane source as consisting of many point sources, all of which emit in direction ω_0 , and which are distributed uniformly over the source plane. We now focus our attention on one of these point sources, say S , and denote the cylindrical coordinates of the point P relative to S as ρ , z and α . The contribution of S to the density $G(z; \omega_0)$ at P is equal to the density $F(\rho', z')$ from a point-collimated source emitting in direction ω_0 , where ρ' and z' are the radial and longitudinal distances, respectively, of P relative to S with respect to an axis passing through S in direction ω_0 . The contribution of the source point S depends only on the relative distances ρ' and z' , but not on the absolute positions of S and P or their relative orientation, because of the assumed homogeneity and isotropy of the medium.

Assuming one point source per unit area of the source plane, there are $\rho d\rho d\alpha$ point sources such that the relative coordinates of P with respect to them lie between ρ and $\rho+d\rho$ and between α and $\alpha+d\alpha$. By summing over the entire source plane, the following relation is obtained between plane-source and point-source densities:

$$G(z; \omega_0) = \int_0^\infty \rho d\rho \int_0^{2\pi} d\alpha F[\rho'(\rho, z, \alpha), z'(\rho, z, \alpha)]. \quad (1)$$

We want to invert this integral equation to obtain $F(\rho', z')$. The first step of the solution consists of applying a Fourier transform in z . We multiply (1) by e^{-iqz} and

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¹ Fano, Spencer, and Berger, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1959), Vol. 38, Part 2.

² The time- and energy-dependence of the flux are disregarded because they are irrelevant to the discussion.

³ See, e.g., R. Marshak, *Revs. Modern Phys.* **19**, 185 (1947).

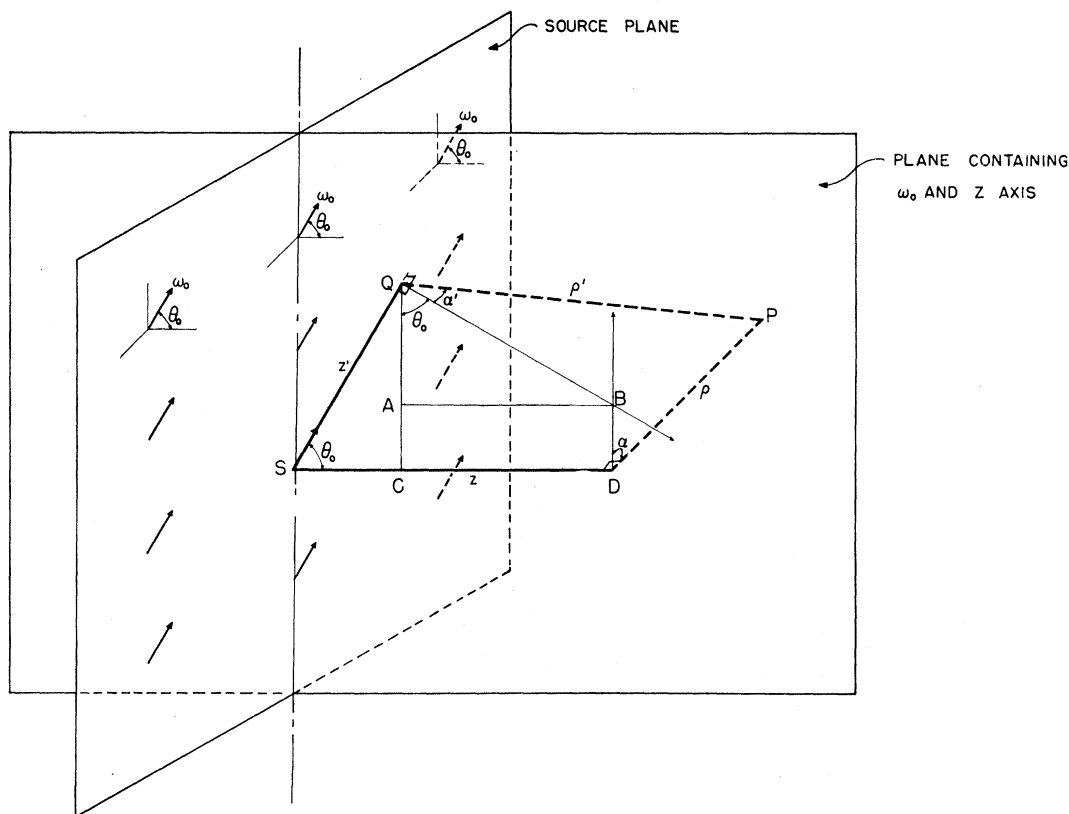


FIG. 1. Relation of collimated plane and point sources.

integrate with respect to z :

$$\int_{-\infty}^{\infty} dz e^{-iqz} G(z; \omega_0) = \int_0^{\infty} \rho d\rho \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\alpha e^{-iqz} F[\rho'(\rho, z, \alpha), z'(\rho, z, \alpha)]. \quad (2)$$

Note that in Fig. 1 the distance $SC = z' \cos \theta_0$, $QB = \rho' \sin \theta_0$ and $AB = CD = \rho' \sin \theta_0 \cos \alpha'$, so that one can make the substitution

$$z = \rho' \sin \theta_0 \cos \alpha' + z' \cos \theta_0 \quad (3)$$

on the right-hand side of (2). Furthermore, we may replace the volume element of integration $\rho d\rho dz d\alpha$ by $\rho' d\rho' dz' d\alpha'$ inasmuch as the Jacobian of the transformation from the unprimed to the primed coordinates is unity. The resulting expression is

$$\int_{-\infty}^{\infty} dz e^{-iqz} G(z; \omega_0) = \int_0^{\infty} \rho' d\rho' \int_{-\infty}^{\infty} dz' e^{-iqz' \cos \theta_0} F(\rho', z') \times \int^2 d\alpha' e^{-iq\rho' \sin \theta_0 \cos \alpha'}. \quad (4)$$

The right-hand side of this equation does not depend on φ_0 . Therefore $G(z; \omega_0)$ is also independent of φ_0 , and we shall write $G(z; \theta_0)$ from now on.

The integral with respect to α' yields a Bessel function:

$$\int_0^{2\pi} d\alpha' \exp(-iq\rho' \sin \theta_0 \cos \alpha') = 2\pi J_0(q\rho' \sin \theta_0). \quad (5)$$

After the replacement of q and θ_0 by new variables,

$$q_\rho = q \sin \theta_0, \quad q_z = q \cos \theta_0, \quad (6)$$

one can rewrite (4) as

$$(1/2\pi) \int_{-\infty}^{\infty} dz \exp[-iz(q_\rho^2 + q_z^2)^{1/2}] G(z; \theta_0) = \int_{-\infty}^{\infty} dz' e^{-iq_z z'} \int_0^{\infty} \rho' d\rho' J_0(q_\rho \rho') F(\rho', z'). \quad (7)$$

To this equation we apply inverse Fourier and Hankel⁴ transforms, i.e., we multiply (7) by $(1/2\pi) e^{iq_z z'} q_\rho J_0(q_\rho \rho')$ and integrate with respect to q_z and q_ρ . This leads to the

⁴For a description of Hankel transforms, see, e.g., I. N. Sneddon, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1955), Vol. 2, p. 298.

desired solution of the integral equation (1):

$$F(\rho', z') = (1/2\pi)^2 \int_{-\infty}^{\infty} dq_z e^{iq_z z'} \int_0^{\infty} q_\rho dq_\rho J_0(q_\rho \rho') \\ \times \int_{-\infty}^{\infty} dz \exp[-iz(q_\rho^2 + q_z^2)^{1/2}] G(z; \theta_0). \quad (8)$$

Note that the argument θ_0 in $G(z; \theta_0)$ is equal to $\tan^{-1}(q_\rho/q_z)$. As q_ρ and q_z assume different values, θ_0 runs through all angles between 0 and π . Thus the evaluation of $F(\rho', z')$ requires knowledge of plane-source densities for all obliquities.

Many methods of solving the transport equation in plane geometry yield as a direct result the Fourier transform of the density rather than the density itself.^{5,6} In connection with the evaluation of $F(\rho, z)$ this is actually an advantage because only the integrals with respect to q_ρ and q_z in (8) need then be carried out.

As a simple example of the application of (8), we consider the case of a purely absorbing medium, for which

$$G(z; \theta_0) = \frac{k}{|\cos\theta_0|} e^{-z a / \cos\theta_0} \quad \text{if } z \cos\theta_0 > 0 \\ = 0 \quad \text{if } z \cos\theta_0 < 0, \quad (9)$$

where k is a normalization constant and a is an attenuation coefficient. The Fourier transform of $G(z; \theta_0)$ is in this case

$$\int_{-\infty}^{\infty} e^{-iaz} G(z; \theta_0) dz = \frac{k}{a + iq \cos\theta_0} = \frac{ik}{ai - q_z}. \quad (10)$$

The inverse Fourier transformation yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iq_z z'} \frac{ik}{ai - q_z} dq_z = k e^{-az'} \quad \text{if } z' > 0 \\ = 0 \quad \text{if } z' < 0. \quad (11)$$

The inverse Bessel transform yields a delta function:

$$\frac{1}{2\pi} \int_0^{\infty} q_\rho dq_\rho J_0(q_\rho \rho') \\ = \frac{1}{4\pi^2} \int_0^{2\pi} d\alpha' \int_0^{\infty} q_\rho dq_\rho e^{-iq_\rho \rho' \cos\alpha'} \\ = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y e^{i(q_x x' + q_y y')} \\ = \delta(x') \delta(y') = \frac{\delta(\rho')}{2\pi\rho'}. \quad (12)$$

⁵ L. V. Spencer, Phys. Rev. **88**, 793 (1952).

⁶ Wilkins, Hellens, and Zweifel, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (United Nations, New York, 1956), Vol. 15.

Combining (11) and (12), we find the density from a point-collimated source in a purely absorbing medium to be

$$F(\rho', z') = k e^{-az'} \frac{\delta(\rho')}{2\pi\rho'} \quad \text{if } z' > 0 \\ = 0 \quad \text{if } z' < 0. \quad (13)$$

This result is of course obvious from first principles, because in a purely absorbing medium, i.e., in the absence of scattering, the radiation remains concentrated on a line corresponding to the direction of emission, and is attenuated exponentially along that line.

3. PLANE-TO-POINT SUPERPOSITION OF DENSITY MOMENTS

One of the most effective techniques for solving the transport equation is the moment-method, according to which one first determines the spatial moments of the flux, and then constructs flux (or density) distributions by moment-fitting. The moment method has been applied to the calculation of the diffusion of x-rays,^{1,7} electrons,^{8,9} neutrons⁶ and cosmic rays.¹⁰ The flux moments are computed with the use of recursion relations derived from the transport equation. These recursion relations are much simpler for plane-source densities than for point-source densities. It is therefore of practical importance that there exists a relation which is the analog of (8), according to which any finite moment of a point-source density can be obtained as a finite linear combination of plane-source density moments.

Plane-source density moments may be defined as

$$G_n(\theta_0) = \frac{1}{n!} \int_{-\infty}^{\infty} dz z^n G(z; \theta_0), \quad (14)$$

and point-source density moments as

$$F_{jm} = \frac{1}{j!m!} \int_{-\infty}^{\infty} dz \int_0^{2\pi} 2\pi\rho d\rho \rho^{jz} F(\rho, z). \quad (15)$$

We want to express the F_{jm} 's as functions of the G_n 's.

Upon multiplying (1) by $z^n/n!$ and integrating with respect to z , one obtains an expression for $G_n(\theta_0)$ which, with the use of (3), can be brought into the form

$$G_n(\theta_0) = \frac{1}{n!} \int_0^{\infty} \rho' d\rho' \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\alpha' \\ \times \{\rho' \sin\theta_0 \cos\alpha' + z' \cos\theta_0\}^n F(\rho', z'). \quad (16)$$

The right-hand side of (16) can be further transformed by expanding the binomial, carrying out the integration

⁷ L. V. Spencer and U. Fano, J. Research Natl. Bur. Standards **46**, 446 (1951).

⁸ H. W. Lewis, Phys. Rev. **78**, 526 (1950).

⁹ L. V. Spencer, Phys. Rev. **98**, 1597 (1955).

¹⁰ B. A. Chartres and H. Messel, Phys. Rev. **96**, 1651 (1954); B. A. Chartres, Phys. Rev. **105**, 707 (1957).

over α' , and making use of the moment definition (15). The resulting equation is

$$G_n(\theta_0) = \sum_{j=0}^{[\frac{1}{2}n]} \sin^{2j}\theta_0 \cos^{n-2j}\theta_0 \binom{2j}{j} 2^{-2j} F_{2j, n-2j}, \quad (17)$$

where the notation $[\frac{1}{2}a]$ indicates the largest integer $\leq \frac{1}{2}a$ and $\binom{a}{b}$ is the binomial coefficient $a!/(a-b)!b!$. Only even powers of $\sin\theta_0$ occur in (17), so that $G_n(\theta_0)$ is expressible as a polynomial in $\cos\theta_0$:

$$G_n(\theta_0) = \sum_{j=0}^{[\frac{1}{2}n]} a_{nj} \cos^{n-2j}\theta_0, \quad (18)$$

with

$$a_{nj} = \sum_{k=j}^{[\frac{1}{2}n]} (-1)^{j+k} \binom{k}{j} \binom{2k}{k} 2^{-2k} F_{2k, n-2k}. \quad (19)$$

It will now be shown that the inversion of (19) yields

$$F_{2j, m} = \frac{2^{2j}}{\binom{2j}{j}} \sum_{k=j}^{j+[\frac{1}{2}m]} \binom{k}{j} a_{2j+m, k}. \quad (20)$$

This result can be proved by inserting (20) into (19), whereby one obtains the equation

$$\begin{aligned} a_{nj} &= \sum_{k=j}^{[\frac{1}{2}n]} \sum_{i=k}^{[\frac{1}{2}n]} (-1)^{j+k} \binom{i}{k} \binom{k}{j} a_{ni} \\ &= \sum_{i=j}^{[\frac{1}{2}n]} a_{ni} \sum_{k=j}^i (-1)^{j+k} \binom{i}{k} \binom{k}{j}. \end{aligned} \quad (21)$$

The last sum in (21) can be evaluated with the use of a generating function:

$$\begin{aligned} &\sum_{k=j}^i (-1)^{j+k} \binom{i}{k} \binom{k}{j} \\ &= \frac{(-1)^j}{j!} \lim_{t \rightarrow 1} \frac{d^j}{dt^j} (1-t)^i = \binom{i}{j} \lim_{t \rightarrow 1} (1-t)^{i-j} \\ &= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i>j. \end{cases} \end{aligned} \quad (22)$$

Thus (21) is an identity, which proves (20).

In order to evaluate the point-source density moments according to (20), one must have the expansion coefficients a_{nj} of $G_n(\theta_0)$. The evaluation of $F_{2j, m}$ requires knowledge of the $[\frac{1}{2}m]+1$ coefficients $a_{2j+m, k}$ for $k=j, j+1, \dots, j+[\frac{1}{2}m]$. These coefficients can be determined from (14) after one has obtained $G_n(\theta_0)$ for $j+[\frac{1}{2}m]+1$ different values of the obliquity θ_0 .

There is another method of evaluating (20), which depends on a Legendre polynomial expansion of $G_n(\theta_0)$ and has the advantage of fitting very naturally into the

framework of the moment-method. Equation (18) can be rearranged as a Legendre series

$$G_n(\theta_0) = \sum_{j=0}^{[\frac{1}{2}n]} b_{nj} P_{n-2j}(\cos\theta_0), \quad (23)$$

with the expansion coefficients

$$b_{nj} = (n-2j+\frac{1}{2}) \int_0^\pi \sin\theta_0 d\theta_0 P_{n-2j}(\cos\theta_0) G_n(\theta_0), \quad (24)$$

where P_l is the l th Legendre polynomial. Legendre polynomials can be written as¹¹

$$P_l(\cos\theta) = \sum_{j=0}^{[\frac{1}{2}l]} c_{lj} \cos^{l-2j}\theta, \quad (25)$$

with

$$c_{lj} = \frac{(-1)^j}{2^l} \binom{l}{j} \binom{2l-2j}{l}. \quad (26)$$

On the basis of this representation, the coefficients a_{nj} in (18) can be expressed as linear combinations of the coefficients b_{nj} in (23). The insertion of these combinations into (20) yields the following equation:

$$F_{2j, m} = \frac{2^{2j}}{\binom{2j}{j}} \sum_{i=0}^{j+[\frac{1}{2}m]} b_{2j+m, i} \sum_{k=k^*}^{j+[\frac{1}{2}m]} \binom{k}{j} c_{2j+m-2i, k-i}, \quad (27)$$

where $k^*=i$ or j , whichever is larger.

The coefficients $b_{2j+m, i}$ in (27) can be interpreted in two different ways. One may consider them as the integrals over $G_n(\theta_0)$ computed according to (24). This computation can be done exactly if one knows $G_n(\theta_0)$ for $j+[\frac{1}{2}m]+1$ different values of the obliquity θ_0 . From another point of view, one may regard the coefficients $b_{2j+m, i}$ as the moments of the density from a plane source which emits particles according to an angular distribution $P_{2j+m-2i}(\cos\theta)$. Such Legendre-polynomial source distributions have no direct physical meaning (except for the polynomial of order zero) because they can assume negative values. But they constitute a useful system of functions in terms of which one can represent physical sources by superposition. Their usefulness derives from the circumstance that the chain of recursion relations by which the moments are calculated assumes its simplest form for Legendre-polynomial sources.¹ It can therefore be assumed that the coefficients $b_{2j+m, i}$ are directly available from a moment calculation, without further integration. Altogether $j+[\frac{1}{2}m]+1$ different Legendre-polynomial sources, of order $2j+m, 2j+m-2, \dots, 1$ (or 0), must be considered in a calculation of $F_{2j, m}$.

¹¹ E. T. Whittaker and G. N. Watson, *Modern Analysis* (Macmillan Company, New York, 1946), American edition, p. 302.

Note that (20) or (27) yields only *even* moments in ρ (with index $2j$). The reason for this lies in the structure of (16). In this equation, the binomial in ρ' and z' is of such a form that after expansion the odd powers of ρ' are accompanied by odd powers of $\cos\alpha'$. These odd powers of $\cos\alpha'$ vanish when one integrates with respect to α' , so that only even moments with index $2j$ are left. If one calculates moments directly from the recursion relation appropriate to a point source⁷ one also fails to

obtain odd moments in ρ . In order to obtain the missing moments, one would have to proceed indirectly, first evaluating the density distribution from the even moments, and then taking appropriate averages of the density distributions.

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Thermodynamics of Inhomogeneous Systems

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A self-consistent thermodynamic formalism is developed for the treatment of the equilibrium of systems, some of whose parameters vary continuously from place to place. The method is specially designed for the description of transition interfaces separating two phases. The energy per unit volume is assumed to depend explicitly on the space derivatives of the molecule densities. Equilibrium conditions are obtained for the appropriate internal variables of the system, and all externally measurable intensive variables are uniquely defined by a variational procedure.

INTRODUCTION

THE detailed description of the transition layer between two phases of a substance requires the treatment of a system in which some parameters of the system such as matter density and entropy density vary continuously from place to place in the system. The purpose of this paper is to provide a systematic formalism for the treatment of such systems within the framework of thermodynamics. The method developed here provides unique definitions of all externally controllable thermodynamic variables and consistently describes all physical operations that can be made on the system.

The power of a phenomenological description of inhomogeneous systems has been recently demonstrated by Cahn and Hilliard.¹ They represent the effect of inhomogeneity on the free energy of the system as a dependence of the free energy on the density gradients at each point as well as the density. The free energy is considered to be represented by an expansion in the various derivatives of the density close to the homogeneous state. The history of such types of treatments is well described by Bakker,² and I shall not present any further discussion of that method.

The basis for the present treatment is similarly an assumed pointwise dependence of the local energy density on the space derivatives of the densities at the point. This mode of description is introduced into the framework of a Gibbsian thermodynamic treatment.

¹ J. W. Cahn and J. E. Hilliard, *J. Chem. Phys.* **28**, 258 (1958).

² G. Bakker, in *Handbuch der Experimentalphysik*, edited by W. Wien and F. Harms (Akademische Verlagsgesellschaft, Leipzig, 1928), Vol. 6.

For simplicity a one-component system is treated, and the explicit dependence on only two derivative terms is included.

The method should also be useful in the treatment of critical-point fluctuations.

DEVELOPMENT OF THE THEORY

We shall take under consideration a one-component system in some infinitesimal state of strain relative to an assumed reference state. Each point x_i of the reference state is presumed to be continuously displaced to the corresponding point x'_i in the strained state such that x'_i is a differentiable function of the unprimed x 's. For fluid or fully isotropic systems, the choice of the reference state is relatively unimportant; for crystalline solids, however, the reference state must be selected in general with sufficient care so that the system is both conveniently and adequately specified. Since we shall restrict our present discussion to the fully equilibrated isotropic solid or fluid, we shall not be concerned with the latter problem. The energy of the system will be presumed to depend explicitly on the strains e_{ij} , defined by the relationship

$$e_{ij} = \partial x'_i / \partial x_j. \quad (1)$$

Another important parameter in the description of the state of strain is the dilatation e , the ratio of the volume of a small volume element v' in the strained state to the volume of the element v in the reference state from which it was deformed. The dilatation is given by the Jacobian of the transformation represented by the