Proof of Dispersion Relations for the Production of Pions by Real and Virtual Photons and for Related Processes*†

REINHARD OEHME, *Enrico Fermi Institute for Nuclear Studies and Department of Physics, University of Chicago, Chicago, Illinois*

AND

JOHN G. TAYLOR,‡ *Department of Physics, University of Maryland, College Park, Maryland*

(Received August 27, 1958)

It is shown that the amplitudes for the production of pions by photons and electrons (virtual photons), as well as for elastic photon-proton and photon-deuteron scattering, have certain analytic properties as functions of energy and momentum transfer. These properties are proven on the basis of the axioms of field theory, especially local commutativity and the spectral conditions. They guarantee the validity of the usual dispersion relations for restricted values of the invariant momentum transfer. In the construction of these dispersion formulas the electromagnetic interaction is treated in lowest order. The residua of the poles arising from the single-particle intermediate states are related to the corresponding vertex functions. For fixed values of the total energy the absorptive parts of the amplitudes are analytic functions of momentum transfer; they are regular inside certain ellipses. These properties make it possible to continue the absorptive parts into the "unphysical region" appearing in the nonforward dispersion relations by means of partial-wave expansions.

In the Appendix a brief survey is given of the limitations in momentum transfer or the unphysical mass restrictions, which one encounters in the proof of dispersion relations for some elastic scattering processes.

1. INTRODUCTION

SEVERAL processes involving electromagnetic and strong interactions simultaneously can be studied to some extent on the basis of relativistic dispersion relations. In this approach the electromagnetic radiative corrections are neglected, but the strong interactions are treated exactly. With these restrictions one obtains dispersion formulas[1] for the amplitudes describing photoproduction of mesons,[2] the production of pions by electrons,[3] and the elastic scattering of photons by protons.[4]

It is the purpose of the present article to derive some analytic properties of the amplitudes mentioned above and to show that these properties guarantee the validity of the corresponding dispersion relations for restricted values of the momentum transfer. In addition they make it possible to continue the amplitudes into the "unphysical region" by means of expansions in terms of Legendre polynomials and their derivatives. In the case of photomeson-production and γ-p scattering, the coefficients in these expansions (multipoles) are physical quantities and in principle measurable. Also in the relations for pion production by electrons the convergence of the multipole expansion in the unphysical region is important. In the electromagnetic approximation considered, one is dealing practically with the production of pions by virtual photons.

The amplitude for a process like photoproduction of mesons is a matrix in spin and isotopic spin space. It is usually given in the center-of-mass system as an expansion in terms of a complete set of basic matrices which are compatible with all invariance properties. For the purpose of dispersion relations, it is convenient to use a corresponding expansion of the covariant amplitude with respect to a complete set of irreducible forms in spinor space. The coefficients of these forms are invariant functions of the momenta only, and they have simple symmetry properties if the matrices are properly chosen. In the present paper we shall not discuss these kinematical aspects[2–4]; they are unimportant for the analytic properties in which we are interested. It is sufficient for our purpose to work with spin-zero fields only, provided we observe all selection rules which are relevant for the spectral conditions. All steps of our proof can be directly applied to the invariant coefficients in the expansion of the general amplitude. These functions may be expressed in terms of the complete amplitude by means of projection operators and traces, and in this form the discussion of their analytic properties is completely analogous to the spin-zero case.[5]

The analytic properties of the photoamplitudes are obtained on the basis of general assumptions under-

* Work supported in part by the U. S. Atomic Energy Commission.

† A brief report of our results appears in the Proceedings of the Annual International Conference on High-Energy Physics at CERN, Geneva, Switzerland, 1958.

‡ On leave of absence from Christ's College, Cambridge, England.

[1] No proof of all these dispersion relations has been attempted till now.

[2] These relations have been considered by many authors; see for instance: Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957); E. Corinaldesi, Nuovo cimento 4, 1384 (1956); Logunov, Tavkhelidze, and Solovyov, Nuclear Phys. 4, 427 (1957).

[3] Fubini, Nambu, and Wataghin, Phys. Rev. **111**, 329 (1958); Logunov, Solovyov, Kukin, and Frenkin, Joint Institute for Nuclear Research, 1958 (to be published).

[4] T. Akiba and I. Sato, Progr. Theoret. Phys. Japan **19**, 83 (1958). For forward scattering, see also Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954). These papers contain further references.

[5] Compare Appendix I of the paper by Goldberger, Nambu, and Oehme, Ann. Phys. N. Y. **2**, 226 (1957).

lying relativistic quantum field theory.[6] The most important axioms are local commutativity (causality) and the spectral conditions. However, the information contained in these axioms has not been completely exhausted and hence the limitations in momentum transfer, which we encounter, are not necessarily characteristic for the assumptions we have made. In this respect the present situation is somewhat different from that of the mass restrictions for the vertex function and other processes.[7]

The problem of proving the analytic properties which are relevant for the dispersion relations is actually a special case of a more general task. This consists in a complete exploration of the consequences of causality and spectrum for the analytic properties of Green's functions. In the present paper we are dealing with the Fourier transforms of vacuum expectation values of fourfold retarded products. It follows from the axioms that these transforms can be characterized by invariant functions depending on the six inner products which can be formed from the three independent vectors.[8] As a consequence of local commutativity and the spectral conditions, these functions are analytic on a certain "primitive" domain D in the space of six complex variables. The problem is to characterize this domain, to compute its envelope of holomorphy $E(D)$, and to derive a representation for the most general function in the class of interest, which is analytic in $E(D)$ and has singularities everywhere on the boundary.[9]

Our present exploration is much less ambitious.[6] For the purpose of dispersion relations, we are mainly interested in the analytic properties of the amplitudes as functions of the "total energy variable" z_6 and the invariant "momentum transfer variable" z_5, with the other four variables $z_1\cdots z_4$ on the massshell (except for virtual photons). So we prove, for instance, that there is a certain region R in the space of the variables $z_1\cdots z_4$ such that for fixed, real z_5, the amplitude is analytic in $z_1\cdots z_4$ and z_6 for $(z_1\cdots z_4)$ in R and for z_6 in the whole complex plane except for a "physical cut" along the positive real axis. If this region R includes the points $z_1\cdots z_4$ on the mass shell for some physical values of the momentum transfer z_5, then we can prove the corresponding physical dispersion relation. But if the mass-shell points are outside R, we cannot guarantee that the amplitude is regular in the cut z_6 plane for physical values of $z_1\cdots z_5$. In fact, in some cases we know from examples that causality and spectrum are *not* sufficient to prove the desired dispersion relations.[7] In order to find the details of the region of analyticity in the z_6 plane for such amplitudes, a more general approach, along the lines outlined earlier, would be required.

[6] See Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958) for discussion and references. This paper will be quoted in the following as BOT.

[7] Res Jost, Helv. Phys. Acta **31**, 263 (1958); Reinhard Oehme Phys. Rev. **111**, 1430 (1958).

[8] D. Hall and A. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 5 (1957).

[9] At present the envelope $E(D)$ is known only for the three-point function, where no specific spectral conditions have been assumed. G. Källèn and A. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. (to be published). With the present mathematical method an extension of these results to include special mass spectra is quite involved.

2. STATEMENT OF RESULTS

We consider the S-matrix element for the reaction $k+p\rightarrow k'+p'$, where p and p' are the initial and final four-momenta of the target nucleon, k is the four-momentum of the initial real ($k^2=0$) or virtual ($k^2<0$) photon, and k' corresponds to the final pion ($k'^2=\mu^2$) or photon ($k'^2=0$). It is sufficient for our purpose to describe all particles by the corresponding spin-zero fields. In this framework we introduce the invariant, causal amplitude M_r by

$$S(k',p',k,p)=\langle k'|k\rangle\langle p'|p\rangle+i(2\pi)^4\delta(k'+p'-k-p)\times(16k_0'p_0'k_0p_0)^{-\frac{1}{2}}M_r(k+p,\,p',p),\tag{2.1}$$

where M_r may be considered as a function of the inner products,

$$p'^2=p^2=m^2,\ k'^2=\tau,\ k^2=\gamma,\ (p'-p)^2=(k-k')^2=-4\Delta^2,\quad (k+p)^2=(k'+p')^2=\sigma^2.\tag{2.2}$$

Let us concentrate first on the amplitude for the *production of pions by real and virtual photons*. Then we have $\tau=\mu^2$, $\gamma\leqslant 0$, and if $\mathbf{q}_1$ and $\mathbf{q}_2$ are the momenta of pion and photon in the center-of-mass frame of the final pion-nucleon system, we find

$$\begin{aligned}\mathbf{q}_1^2&=\tfrac{1}{4}\sigma^2[1+(m^2-\mu^2)/\sigma^2]^2-m^2,\\ \mathbf{q}_2^2&=\tfrac{1}{4}\sigma^2[1+(m^2-\gamma)/\sigma^2]^2-m^2,\\ \mathbf{q}_1\cdot\mathbf{q}_2&=[\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}\cos\vartheta=2[\Delta_0^2(\sigma,\gamma)-\Delta^2],\end{aligned}\tag{2.3}$$

where

$$\Delta_0^2(\sigma,\gamma)=\tfrac{1}{4}\{\mathbf{q}_1^2+\mathbf{q}_2^2-[(\mu^2-\gamma)/2\sigma]^2\},$$

and σ is the total energy of the system in the c.m. frame. The spectral conditions for the amplitude are determined by the properties

$$\begin{aligned}\langle 0|j(x)|n\rangle&=0\ \text{unless}\ p_n^2\geqslant(3\mu)^2,\\ \langle 0|J(x)|n\rangle&=0\ \text{unless}\ p_n^2\geqslant(2\mu)^2,\\ \langle 0|f(x)|n\rangle&=0\ \text{unless}\ p_n^2\geqslant(m+\mu)^2.\end{aligned}\tag{2.4}$$

Here $j(x)=(\square+\mu^2)\phi(x)$, $J(x)=\square A(x)$, $f(x)=(\square+m^2)\times\psi(x)$, and the spin-zero fields ϕ, A, and ψ correspond to "pions, photons and nucleons," respectively. The state $|n\rangle$ describes an eigenstate of the energy-momentum operator with the eigenvalue p_n; we have always $p_n^2\geqslant 0$ and $p_{n0}>0$. The intermediate states of the complete initial or final system consist of the discrete one-nucleon state with $\sigma=m$ and all allowed continuum states with $\sigma\geqslant m+\mu$.

Using the basic axioms of field theory, especially local commutativity and the spectral conditions described above, we have shown that the invariant amplitude $M_r(\gamma,\Delta^2,\sigma^2)$ is the boundary value of an analytic function $M(\gamma,\Delta^2,z)$. In the following we list only those properties of $M(\gamma,\Delta^2,z)$ which are important for the physical dispersion relations.

1. Consider γ and Δ^2 real, $\gamma\leqslant 0$ and $\Delta_{\text{th}}^2\leqslant\Delta^2<\Delta_{\max}^2$, where $\Delta_{\max}^2(\gamma)>\Delta_{\text{th}}^2(\gamma)$, and

$$\Delta_{\text{th}}^2=\frac{m}{m+\mu}\left(\frac{\mu^2-\gamma}{4}\right)$$

is the physical value of Δ^2 at threshold ($\sigma=m+\mu$). Then the function $M(\gamma,\Delta^2,z)$ is analytic in $z=x+iy$ and it may be written in the form

$$M(\gamma,\Delta^2,z)=F(\gamma,\Delta^2,z)+\bar{F}(\gamma,\Delta^2,z),$$

with (2.5)

$$\bar{F}(\gamma,\Delta^2,z)=F(\gamma,\Delta^2,\,-z+2m^2+\mu^2+\gamma+4\Delta^2).$$

The function F is regular in the z plane except for the cut $y=0$, $x\geqslant(m+\mu)^2$ and the pole $y=0$, $x=m^2$. It satisfies the relation

$$F^*(\gamma,\Delta^2,z^*)=F(\gamma,\Delta^2,z) \tag{2.6}$$

for all z in the cut plane. For $|y|>\delta>0$, one can find an integer N and constants $C_n(\gamma,\Delta^2,\delta)$ such that[6]

$$|F(\gamma,\Delta^2,z)|<\sum_{n=0}^{N}C_n|z|^n. \tag{2.7}$$

Note that the conventional variable $\nu=(k+k')(p+p')/4m$ is related to z by

$$2m\nu=z-m^2-\tfrac{1}{2}(\mu^2+\gamma)-2\Delta^2, \tag{2.8}$$

and the replacement $z\to -z+2m^2+\mu^2+\gamma+4\Delta^2$ corresponds to the change of ν into $-\nu$.

2. Let $\sigma_{\text{phys}}(\gamma,\Delta^2)$ be the threshold energy for the physical reaction with given $\gamma\leqslant 0$, and $\Delta_{\text{th}}^2\leqslant\Delta^2<\Delta_{\max}^2$. If $\sigma\geqslant\sigma_{\text{phys}}$, then the improper limit

$$\lim_{\epsilon\to 0+}M(\gamma,\Delta^2,\,\sigma^2+i\epsilon)=M_r(\gamma,\Delta^2,\sigma^2) \tag{2.9}$$

holds, where M_r is the invariant production amplitude defined in Eq. (2.1). The energy σ_{phys} is given by

$$\sigma_{\text{phys}}^2(\gamma,\Delta^2)=\tfrac{1}{2}(\mu^2+\gamma)+m^2+2\Delta^2$$
$$+2[(m^2+\Delta^2)\{\tfrac{1}{2}(\mu^2+\gamma)+\Delta^2+[(\mu^2-\gamma)/4\Delta]^2\}]^{\frac{1}{2}}. \tag{2.10}$$

3. The residuum of the function $F(\gamma,\Delta^2,z)$ at $z=m^2$ is the product of two *real* constants $mg(\mu^2)$ and $me(\gamma)$, where

$$mg(\tau)=(2p_0)^{\frac{1}{2}}\langle p|j(0)|p_n\rangle(2p_{n0})^{\frac{1}{2}}\quad\text{for}$$
$$p^2=p_n=m^2,\quad (p-p_n)^2=\tau, \tag{2.11}$$

and

$$me(\rho)=(2p_0)^{\frac{1}{2}}\langle p|J(0)|p_n\rangle(2p_{n0})^{\frac{1}{2}}\quad\text{for}$$
$$p^2=p_n^2=m^2,\quad (p-p_n)^2=\rho.$$

The functions $g(\tau)$ and $e(\rho)$ are analytic in a complex neighborhood of the real axis for

$$\tau<3\mu^2\left(\frac{m}{m-\mu}\right)\quad\text{and}\quad\rho<2\mu^2\left(\frac{2m}{2m-\mu}\right), \tag{2.12}$$

respectively.

In the general case of the invariant coefficients in the expansion of the covariant amplitude, the residua are determined in the same way as in the present model. There appear the Yukawa constant f, corresponding to our $g(0)$, and the charge e or the static anomalous nucleon moments μ_p and μ_n corresponding to $e(0)$ in our model.[1,2]

4. The limitation $\Delta_{\max}^2(\gamma)$ is given by

$$\Delta_{\max}^2(\gamma)=\min_{\sigma\geq m+\mu}\{\Delta_0^2(\sigma,\gamma)+A(\sigma,\gamma)\}, \tag{2.13}$$

where

$$A(\sigma,\gamma)=\tfrac{1}{2}\{[g_1g_2]^{\frac{1}{2}}+[(g_1+\mathbf{q}_1^2)(g_2+\mathbf{q}_2^2)]^{\frac{1}{2}}\} \tag{2.14}$$

and

$$g_1=\frac{8\mu^3(2m+\mu)}{\sigma^2-(m-2\mu)^2},\quad g_2=\frac{(4\mu^2-\gamma)(2m+\mu)\mu}{\sigma^2-(m-\mu)^2}$$

for

$$\gamma\geqslant\gamma_0=2\mu(m^2-\mu^2-\sigma^2)(m-\mu)^{-1},\quad\text{or}\quad g_2=N_2^2-\mathbf{q}_2^2,$$

with

$$N_2=\frac{\mu(2m+\mu)}{2\sigma}+\frac{1}{2\sigma}\{[(m+\mu)^2+\sigma^2-\gamma]^2-4(m+\mu)^2\sigma^2\}^{\frac{1}{2}},$$

for $\gamma<\gamma_0$.

The quantities Δ_0^2, $\mathbf{q}_1^2$, and $\mathbf{q}_2^2$ have been defined in Eq. (2.3). For real photons ($\gamma=0$) and for virtual photons with γ larger than $-2.7\mu^2$ ($\mu/m=$ experimental mass ratio), the minimum in Eq. (2.13) is at $\sigma=m+\mu$ and we find

$$\Delta_{\max}^2=\frac{m}{m+\mu}\frac{\mu^2-\gamma}{4}+\left[\frac{8\mu^2}{3}\frac{2m+\mu}{2m-\mu}\frac{4\mu^2-\gamma}{2}\left(1+\frac{\mu}{2m}\right)\right]^{\frac{1}{2}}$$
$$\times\frac{1}{2}\left\{1+\left[1+\frac{\mu^2-\gamma}{4\mu^2-\gamma}\frac{m}{(m+\mu)^2}\frac{(2m+\mu)^2-\gamma}{2m+\mu}\right]^{\frac{1}{2}}\right\}. \tag{2.15}$$

In the case of photoproduction of pions ($\gamma=0$), Eq. (2.15) leads to $\Delta_{\max}^2\approx 3\mu^2$. For virtual photons with $\gamma<-2.7\mu^2$ the minimum in Eq. (2.13) appears for $\sigma>m+\mu$. In Table I we give some values of $\Delta_{\max}^2(\gamma)$.

TABLE I. Maximum invariant momentum transfer $2\Delta_{\max}(\gamma)$ for which the dispersion relations have been proven in the case of virtual photons.[a]

$-\gamma/\mu^2$	3	5	7	8	9
$\Delta_{\max}^2(\gamma)/\mu^2$	5.00	6.00	7.03	7.50	7.95

[a] $\gamma=k_0^2-\mathbf{k}^2$, where k is the four-momentum of the photon. This table has been calculated for $\mu/m=140/940$.

5. Let us assume, for reasons of simplicity, that $M(\gamma,\Delta^2,z)$ vanishes like z^{-1} for $z\to\infty$. Then the properties described above may be summarized by the representation

$$M(\gamma,\Delta^2,z)=F(\gamma,\Delta^2,z)+F(\gamma,\Delta^2,\,-z+2m^2+\mu^2+\gamma+4\Delta^2)$$
$$=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma^2\,\phi(\gamma,\Delta^2,\sigma^2)\left\{\frac{1}{\sigma^2-z}+\frac{1}{\sigma^2+z-2m^2-\mu^2-\gamma-4\Delta^2}\right\}+e(\gamma)g(\mu^2)m^2\times\left\{\frac{1}{m^2-z}+\frac{1}{z-m^2-\mu^2-\gamma-4\Delta^2}\right\} \quad (2.16)$$

for $\gamma\leqslant 0$, $\Delta_{\rm th}^2\leqslant\Delta^2<\Delta_{\max}^2$. Here

$$\phi(\gamma,\Delta^2,\sigma^2)=\frac{1}{2i}\lim_{\epsilon\to 0+}\{F(\gamma,\Delta^2,\,\sigma^2+i\epsilon)-F(\gamma,\Delta^2,\,\sigma^2-i\epsilon)\} \quad (2.17)$$

is a real weight function which coincides with the absorptive part of the production amplitude for physical values of σ:

$$\phi(\gamma,\Delta^2,\sigma^2)=\operatorname{Im}M_r(\gamma,\Delta^2,\sigma^2)\quad\text{for}\quad\sigma\geqslant\sigma_{\rm phys}(\gamma,\Delta^2). \quad (2.18)$$

In the "unphysical region" $m+\mu\leqslant\sigma<\sigma_{\rm phys}$, the function ϕ may be obtained by explicit analytic continuation in Δ^2 with the help of expansions in terms of Legendre polynomials or their derivatives. This continuation is made possible by the following properties of ϕ:

6. For γ and σ fixed, ϕ is an analytic function in Δ^2. It is regular inside an ellipse, with foci at $\Delta^2=\Delta_0^2(\gamma,\sigma)\pm\frac{1}{2}[\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}$ and the boundary

$$\Delta^2=\Delta_0^2(\sigma,\gamma)-A(\sigma,\gamma)\cos\psi\pm iB(\sigma,\gamma)\sin\psi, \quad (2.19)$$

where

$$B(\sigma,\gamma)=\tfrac{1}{2}[4A^2(\sigma,\gamma)-\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}=\tfrac{1}{2}\{[g_1(g_2+\mathbf{q}_2^2)]^{\frac{1}{2}}+[g_2(g_1+\mathbf{q}_1^2)]^{\frac{1}{2}}\}; \quad (2.20)$$

$A(\sigma,\gamma)$, g_1, g_2 are defined in Eq. (2.14), and $\Delta_0^2(\sigma,\gamma)$, $\mathbf{q}_1^2$, $\mathbf{q}_2^2$ in Eq. (2.3). For $\sigma>m+\mu$ we may introduce the cosine of the c.m. angle ϑ as a new variable. Then ϕ is an analytic function of $\cos\vartheta$, which is regular inside an ellipse with foci at ± 1 and the boundary

$$\cos\vartheta=\xi\cos\psi\pm i[\xi^2-1]^{\frac{1}{2}}\sin\psi,$$
where
$$\xi=2A(\sigma,\gamma)[\mathbf{q}_1^2\mathbf{q}_2^2]^{-\frac{1}{2}}. \quad (2.21)$$

Hence we may expand ϕ in terms of Legendre polynomials in $\cos\vartheta=2(\Delta_0^2-\Delta^2)[\mathbf{q}_1^2\mathbf{q}_2^2]^{-\frac{1}{2}}$ or in terms of derivatives of these polynomials. The expansions are convergent inside the ellipse (2.21) and the coefficients are determined by the values of ϕ for $|\cos\vartheta|\leqslant 1$.[10,11] In this range the function ϕ is equal to the absorptive part of the physical amplitude. For practical applications the Legendre expansions may be chosen to coincide with the usual multipole expansions.

In the representation (2.16), and therefore also in the dispersion relations, the function ϕ is needed for real Δ^2 and all $\sigma\geqslant m+\mu$. We see from Eq. (2.21) that the Legendre expansions will converge in the whole unphysical region if Δ^2 is restricted by

$$\max_{\sigma\geq m+\mu}\{\Delta_0^2(\sigma,\gamma)-A(\sigma,\gamma)\}<\Delta^2<\min_{\sigma\geq m+\mu}\{\Delta_0^2(\sigma,\gamma)+A(\sigma,\gamma)\}=\Delta_{\max}^2(\gamma). \quad (2.22)$$

The range $\Delta_{\rm th}^2\leqslant\Delta^2<\Delta_{\max}^2$ is contained in (2.22). Preliminary calculations indicate that the maximum of the left-hand side of the inequality (2.22) appears for $\sigma\to\infty$ and has the value zero.

In the present paper we shall not discuss the analytic properties of $M(\gamma,\Delta^2,\,\sigma^2\pm i\epsilon)$ as a function of Δ^2 for fixed values of σ. We hope to come back to this problem in a later publication.

Let us briefly discuss the limitations we encounter in the proof of dispersion relations for *elastic γ-p scattering*. In this case we have $\tau=\gamma=0$ in Eq. (2.2) and the c.m. quantities are given by

$$\mathbf{q}^2=\mathbf{q}_1^2=\mathbf{q}_2^2=\tfrac{1}{4}\sigma^2(1-m^2/\sigma^2)^2,\ \Delta^2=\tfrac{1}{2}\mathbf{q}^2(1-\cos\vartheta). \quad (2.23)$$

The dispersion formulas can be proven for $0\leqslant\Delta^2<\Delta_{\max}^2$, where

$$\Delta_{\max}^2=\min_{\sigma\geq m+\mu}\{\mathbf{q}^2+g\},$$

with

$$g(\sigma)=\frac{4\mu^3(2m+\mu)}{\sigma^2-(m-\mu)^2}. \quad (2.24)$$

Using the experimental mass ratio, we find that the minimum is at $\sigma=m+\mu$, and

$$\Delta_{\max}^2=\mu^2\left\{\frac{(2m+\mu)^2}{4(m+\mu)^2}+\frac{2m+\mu}{m}\right\}\approx 3\mu^2. \quad (2.25)$$

The representation corresponding to Eq. (2.16) is of the form

$$M(\Delta^2,z)=F(\Delta^2,z)+F(\Delta^2,\,-z+2m^2+4\Delta^2)$$
$$=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma^2\,\phi(\Delta^2,\sigma^2)\left\{\frac{1}{\sigma^2-z}+\frac{1}{\sigma^2+z-2m^2-4\Delta^2}\right\}+e^2(0)m^2\left\{\frac{1}{m^2-z}+\frac{1}{z-m^2-4\Delta^2}\right\}, \quad (2.26)$$

where $\phi(\Delta^2,\sigma^2)$ is again an analytic function of Δ^2 for

[10] Gabor Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 238.

[11] Corresponding results for the case of pion-nucleon scattering have been obtained earlier by H. Lehmann, 1958 (to be published).

fixed $\sigma \geqslant m+\mu$. It is regular inside an ellipse[11] with the boundary

$$\Delta^2=\tfrac{1}{2}\mathbf{q}^2-\{\tfrac{1}{2}\mathbf{q}^2+g\}\cos\psi\pm i[g(g+\mathbf{q}^2)]^{\frac{1}{2}}\sin\psi. \quad (2.27)$$

We can express ϕ in the unphysical region $(m+\mu)\leqslant\sigma<\Delta+[m^2+\Delta^2]^{\frac{1}{2}}$ by Legendre expansions (multipole expansions) in $\cos\vartheta=1-2\Delta^2/\mathbf{q}^2$, provided Δ^2 is restricted to the range $0\leqslant\Delta^2<\Delta_{\max}{}^2$, where $\Delta_{\max}{}^2$ is given in Eq. (2.25).

So far we have treated the electromagnetic interaction only in lowest order. If we take into account all orders in the fine structure constant and introduce a small, auxiliary photon mass $\lambda>0$, then $\Delta_{\max}{}^2$ for γ-p scattering is given by

$$\Delta_{\max}{}^2=\min_{\sigma\geq m+\lambda}\{\mathbf{q}^2+g(\sigma,\lambda)\}, \quad (2.28)$$

with

$$g(\sigma,\lambda)=\frac{8\lambda^3(2m+\lambda)}{\sigma^2-(m-2\lambda)^2}.$$

The minimum is at $\sigma=m+\lambda$ and we find the value $\Delta_{\max}{}^2=(8/3)\lambda^2(2m+\lambda)(2m-\lambda)^{-1}$. Hence we can only prove the dispersion formulas for forward scattering and for the derivative amplitudes. The situation is the same for photon-electron scattering.

Dispersion relations for elastic *photon-deuteron scattering* can also be proven; they will be discussed briefly in the appendix.

3. DERIVATION OF DISPERSION RELATIONS

Since the following considerations are similar to those given in Sec. 2 of BOT, we may restrict ourselves here to a brief outline.[12] It is convenient to introduce a special Loretz frame where $p+p'=0$ and

$$p'=p_\Delta,\quad p=p_{-\Delta},\quad K=\tfrac{1}{2}(k+k')=\{\omega,\mathbf{K}\},$$

with

$$p_\Delta=\{E_\Delta,\mathbf{\Delta}\},\quad E_\Delta=(m^2+\Delta^2)^{\frac{1}{2}},$$
$$\mathbf{K}^2=\omega^2-\alpha-\Delta^2,\quad 2\mathbf{K}\cdot\mathbf{\Delta}=\beta.$$

Here we have introduced the variables $\alpha=\frac{1}{2}(\tau+\rho)$, $\beta=\frac{1}{2}(\tau-\rho)$, $\tau=k'^2$, $\rho=k^2$; and on the "mass shell" for the production amplitude, with $\tau=\mu^2$ and $\rho=\gamma\leqslant 0$, we write $\alpha_0=\frac{1}{2}(\mu^2+\gamma)$, $\beta_0=\frac{1}{2}(\mu^2-\gamma)$. We chose our space frame so that

$$\mathbf{\Delta}=\Delta\mathbf{e}_1,\ \mathbf{K}=(\beta/2\Delta)\mathbf{e}_1+[\omega^2-\alpha-\Delta^2-\beta^2/4\Delta^2]^{\frac{1}{2}}\mathbf{e}_2, \quad (3.1)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal unit vectors. Note that the quantity

$$\omega_{\text{phys}}(\gamma,\Delta^2)=[\alpha_0+\Delta^2+\beta_0{}^2/4\Delta^2]^{\frac{1}{2}} \quad (3.2)$$

is the physical threshold energy in the special system. By the usual methods we obtain Fourier representations for the invariant amplitude M_r and the corresponding advanced amplitude M_a. In the special frame these are of the form

$$M_{r,a}(\alpha,\beta,\Delta^2,\omega)=D(\alpha,\beta,\Delta^2,\omega)\pm iA(\alpha,\beta,\Delta^2,\omega)$$
$$=\pm 2E_\Delta i\int d^4x\, e^{iK\cdot x}\theta(\pm x_0)\langle p_\Delta|[j(\tfrac{1}{2}x),J(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle+\sum_{n=0}^{N}C_n(\alpha,\beta,\Delta^2)\omega^{2n}, \quad (3.3)$$

where the dispersive part D, the absorptive part A, and the coefficients C_n are real quantities; this may be shown using invariance under space-time inversion or charge conjugation. Let us assume now that $\alpha<-\Delta^2-\beta^2/4\Delta^2$. Then the absorptive part is defined by Eq. (3.3) for all real ω, and by decomposition of the matrix element into intermediate states we obtain the spectral representation

$$A(\alpha,\beta,\Delta^2,\omega)=\int d\sigma^2\,\rho(\alpha,\beta,\Delta^2,\sigma^2)\times\{\delta(\sigma^2-\alpha-m^2-2\Delta^2-2E_\Delta\omega)-\delta(\sigma^2-\alpha-m^2-2\Delta^2+2E_\Delta\omega)\}, \quad (3.4)$$

where

$$\rho(\alpha,\beta,\Delta^2,\sigma^2)=\pi m^2g(\alpha+\beta)e(\alpha-\beta)\delta(\sigma^2-m^2)+\phi(\alpha,\beta,\Delta^2,\sigma^2), \quad (3.5)$$

and $\phi=0$ for $\sigma<m+\mu$. The functions $g(\tau)$ and $e(\rho)$ have been defined in Eq. (2.11) and they will be discussed later.

We see from Eqs. (3.3) and (3.1) that for α, β, Δ^2 fixed and $\alpha+\Delta^2+\beta^2/4\Delta^2<0$, the functions M_r and M_a, taken together, define an analytic function in ω which is regular in the cut ω plane. There are no branch points on the imaginary axis because M_r and M_a are symmetric under $\mathbf{e}_2\to-\mathbf{e}_2$. If we require in addition that $\alpha<-2\Delta^2$, then there is a finite gap on the real axis where $A=(1/2i)(M_r-M_a)=0$, and we have one analytic function $M(\alpha,\beta,\Delta^2,\omega)$. We prefer to introduce the variable

$$z=2E_\Delta\omega+\alpha+m^2+2\Delta^2, \quad (3.6)$$

and consider $M(\alpha,\beta,\Delta^2,z)$ in the cut z plane. Assuming sufficient boundedness at infinity, we obtain for M a representation of the form

$$M(\alpha,\beta,\Delta^2,z)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma^2\,\phi(\alpha,\beta,\Delta^2,\sigma^2)\times\left\{\frac{1}{\sigma^2-z}+\frac{1}{\sigma^2+z-2m^2-2\alpha-4\Delta^2}\right\}$$
$$+g(\alpha+\beta)e(\alpha-\beta)m^2\left\{\frac{1}{m^2-z}+\frac{1}{z-m^2-2\alpha-4\Delta^2}\right\}, \quad (3.7)$$

where ϕ has been defined in Eq. (3.5).

[12] See also Bogoliubov, Medvedev, and Polivanov, *Problems of the Theory of Dispersion Relations* [Gostechnisdat, Moscow (to be published)]; N. N. Bogoliubov and D. V. Shirkov, *Introduction into the Theory of Quantized Fields* (Gostechnisdat, Moscow, 1957), Chap. 9.

Let us discuss first the functions $g(\tau)$ and $e(\rho)$, which are special cases of the meson-nucleon and photon-nucleon vertex functions. We are interested here in the analytic properties of $g(\tau)$ and $e(\rho)$ in the neighobrhood of the real axis for $\tau\leqslant\mu^2$ and $\rho\leqslant\gamma\leqslant 0$. These properties may be obtained from the representation of the vertex function which has been described earlier by one of us (R.O.).[13] Here we use this representation in the form

$$V(m^2,z_2,z_3)=\int_{(m+\mu)^2}^{\infty}d\sigma^2\int_0^1 d\xi\int_{\xi-1}^{1-\xi}d\eta\int_{\kappa_0}^{\infty}d\kappa\,\frac{\chi(\kappa^2,\xi,\eta,\sigma^2)}{\sigma^2-z_3}\times\{[2\kappa^2+\tfrac{1}{2}(1+\xi^2-\eta^2)\sigma^2-(m^2+z_2)+\eta(m^2-z_2)]^2-\xi^2\lambda(m^2,z_2,\sigma^2)\}^{-1},\quad(3.8)$$

where χ is a real generalized function, $\lambda(z_1z_2z_3)=z_1^2+z_2^2+z_3^2-2z_1z_2-2z_1z_3-2z_2z_3$, and

$$\kappa_0=\min\{0,\ m+\mu-\tfrac{1}{2}\sigma[(1+\eta)^2-\xi^2]^{\frac{1}{2}},\ b-\tfrac{1}{2}\sigma[(1-\eta)^2-\xi^2]^{\frac{1}{2}}\}.$$

We have $b=3\mu$, $z_2=\tau$, $z_3=m^2$ for the pion vertex $g(\tau)$ and $b=2\mu$, $z_2=\rho$, $z_3=m^2$ for the photon vertex $e(\rho)$; the weight functions χ are, of course, different in both cases. A discussion of the denominator in Eq. (3.8) shows that $g(\tau)$ and $e(\rho)$ are analytic in a strip along the real axis provided

$$\tau<3\mu^2\left(\frac{2m+\mu}{2m-\mu}\right)\quad\text{and}\quad\rho<2\mu^2\left(\frac{2m+\mu}{2m}\right),\quad(3.9)$$

respectively. Both functions are *real* for real values of τ and ρ which satisfy Eq. (3.9). We will encounter the restriction (3.9) again in Sec. 4 when we discuss the analytic properties of $\phi(\alpha,\beta,\Delta^2,\sigma^2)$ as a function of $\tau=\alpha+\beta$ and $\rho=\alpha-\beta$, for fixed Δ^2 and *all* $\sigma\geqslant m+\mu$.

It may be of interest to note that the limitations (3.9) have been obtained from a representation which is an analytic function in the cut z_3 plane in addition to its regularity properties as a function of z_2 (τ or ρ). At the special points $z_3=m^2$, we have regularity also for somewhat larger real values of τ or ρ. This can be seen from a representation of the form

$$\langle p|j(0)|q\rangle(4p_0q_0)^{\frac{1}{2}}=\int d\kappa^2d^4u\,\frac{\chi(\kappa^2,u,p)}{\kappa^2-(\tfrac{1}{2}p-q-u)^2},\quad(3.10)$$

where $p^2=q^2=m^2$ and where χ is a real weight function which vanishes outside the region

$$|u_0|+|\mathbf{u}|\leqslant\tfrac{1}{2}m,$$
$$\kappa\geqslant\max\{0,\ a-[(\tfrac{1}{2}m+u_0)^2-\mathbf{u}^2]^{\frac{1}{2}},\ b-[(\tfrac{1}{2}m-u_0)^2-\mathbf{u}^2]^{\frac{1}{2}}\},$$
$$a=m+\mu,\quad b=3\mu\ \text{ or }\ 2\mu,\quad\text{respectively.}$$

The formula (3.10) is a direct application of the Dyson-Jost-Lehmann representation.[14,15] In the Lorentz frame for which $\mathbf{p}=0$ and $q=\{\omega,\ \mathbf{e}[\omega^2-m^2]^{\frac{1}{2}}\}$, with $\mathbf{e}$ being some fixed unit vector, the denominator in Eq. (3.10) cannot vanish for $\omega>\omega_0=(m^2+ma-ab)(m+a-b)^{-1}$. Hence the functions $g(\tau)$ and $e(\rho)$, with $\rho=(q-p)^2$, are analytic in a strip about the real axes provided

$$\tau<3\mu^2\left(\frac{m}{m-\mu}\right)\quad\text{and}\quad\rho<2\mu^2\left(\frac{2m}{2m-\mu}\right),\quad(3.11)$$

respectively.

In the following section we shall show that the function $\phi(\alpha,\beta,\Delta^2,\sigma^2)$ has analytic properties in α, β, and Δ^2 for fixed $\sigma\geqslant m+\mu$. Here we are interested in the fact that there is a $\delta>0$ such that for all $\sigma\geqslant m+\mu$, $\Delta_{\text{th}}^2\leqslant\Delta^2<\Delta_{\max}^2$, $\beta=\beta_0\leqslant\frac{1}{2}\mu^2$, ϕ is an analytic function of α for $\alpha\epsilon S$, where S is the strip[6]

$$S=[\alpha:\ |\text{Im}\alpha|<\delta,\ \text{Re}\alpha<\alpha_0+\delta],$$

and $\alpha_0=\frac{1}{2}(\mu^2+\gamma)$, $\gamma\leqslant 0$; by "$[a:\ \cdots]$" we denote "the set of all a which satisfy the condition$\cdots$." From Eq. (3.9) we see that the functions $g(\alpha+\beta_0)$ and $e(\alpha-\beta_0)$ are also analytic for $\alpha\epsilon S$, and hence we may continue $M(\alpha,\beta_0,\Delta^2,z)$ in terms of the right-hand side of Eq. (3.7) from $\alpha<\Delta^2+\beta_0/4\Delta^2$, $\alpha<-4\Delta^2$ to the physical value $\alpha=\alpha_0$. For the M thus continued, it remains to be shown that the improper limits

$$\lim_{\epsilon\to 0+}M(\alpha_0,\beta_0,\Delta^2,\ \sigma^2\pm i\epsilon)=M_{r,a}(\alpha_0,\beta_0,\Delta^2,\sigma^2)$$

hold for $\sigma\geqslant\sigma_{\text{phys}}(\gamma,\Delta^2)$, where $M_{r,a}$ are the amplitudes defined in Eq. (3.3) in terms of Fourier representations.

These representations define an analytic function of ω and α for

$$|\text{Im}\omega|>|\text{Im}[\omega^2-\alpha-\Delta^2-\beta_0^2/4\Delta^2]^{\frac{1}{2}}|,\quad(3.12)$$

and the right-hand side of Eq. (3.7) is certainly regular for

$$\alpha\epsilon S,\quad\text{and}\quad|y|=2E_\Delta|\text{Im}\omega|>|\text{Im}\alpha|.\quad(3.13)$$

Now we choose $\alpha=\alpha_0\pm 2ia\epsilon$ and $\omega=\omega_r\pm i\epsilon$, $\omega_r\geqslant\omega_{\text{phys}}(\gamma,\Delta^2)$, which corresponds to $z=\sigma^2\pm 2i(E_\Delta+a)\epsilon$, $\sigma\geqslant\sigma_{\text{phys}}(\gamma,\Delta^2)$. Then (ω,α) satisfy the conditions (3.12) and (3.13) provided

$$|a-\omega_r|<[\omega_r^2-\alpha_0-\Delta^2-\beta_0^2/4\Delta^2]^{\frac{1}{2}}\quad\text{and}\quad|a|<E_\Delta.$$

At least for $\Delta^2\geqslant\Delta_{\text{th}}^2$ such an a can always be found and we have

$$\lim_{\epsilon\to 0+}M(\alpha_0,\beta_0,\Delta^2,\ \sigma^2\pm i\epsilon)=\lim_{\epsilon\to 0+}M(\alpha_0\pm 2ia\epsilon,\ \beta_0,\Delta^2,\ \sigma^2\pm 2i(E_\Delta+a)\epsilon)=M_{r,a}(\alpha_0,\beta_0,\Delta^2,\sigma^2).\quad(3.14)$$

[13] Reinhard Oehme, reference 5. See Eqs. (3) and (4).
[14] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[15] R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957); L. Gårding and A. Wightman, 1958 (to be published).

This gives the required dispersion relations. If the function $M(\alpha,\beta_0,\Delta^2,z)$ should be less bounded for $z\to\infty$, then we have to make "substractions" in Eq. (3.7) and there will be a polynomial in z with coefficients depending on α and Δ^2. The proof that these coefficients are analytic functions of α for $\alpha\epsilon S$ may be given using the method described in Sec. 2 of BOT.

4. ANALYTIC PROPERTIES OF THE ABSORPTIVE PART

In this section we wish to prove some analytic properties of $\phi(\alpha,\beta,\Delta^2,\sigma^2)$ as a function of α, β, and Δ^2. The regularity in α for $\alpha\epsilon S$ and fixed $\beta=\beta_0$, $\Delta_{\text{th}}^2\leqslant\Delta^2<\Delta_{\max}^2$, $\sigma\geqslant m+\mu$ has been used in Sec. 3 in order to derive the dispersion relations. Analytic properties in Δ^2 make it possible to express ϕ in the unphysical region in terms of physical quantities via Legendre expansions.

Let us introduce the variables

$$q_1=\tfrac{1}{2}(p'-k'),\quad q_2=\tfrac{1}{2}(p-k),\quad q_3=\tfrac{1}{2}(k+p),\tag{4.1}$$

and the function

$$G(q_1q_2q_3)=(\sigma^2-m^2)\phi(\alpha,\beta,\Delta^2,\sigma^2).\tag{4.2}$$

In the c.m. system we have then

$$\begin{aligned}&\mathbf{q}_3=0,\quad q_{30}=\tfrac{1}{2}\sigma,\\ &q_{10}=(m^2-\tau)/2\sigma,\quad q_{20}=(m^2-\rho)/2\sigma,\\ &\qquad\qquad\tau=\alpha+\beta,\quad \rho=\alpha-\beta,\\ &\mathbf{q}_1^2=\tfrac{1}{4}\sigma^2[1+(m^2-\tau)/\sigma^2]^2-m^2,\\ &\mathbf{q}_2^2=\tfrac{1}{4}\sigma^2[1+(m^2-\rho)/\sigma^2]^2-m^2,\\ &(\mathbf{q}_1-\mathbf{q}_2)^2=4\Delta^2+\beta^2/\sigma^2.\end{aligned}\tag{4.3}$$

By standard methods we obtain for G a Fourier representation of the form

$$\begin{aligned}&G_{r,a}(q_1q_2q_3)\\ &=(m^2-4q_3^2)\int d^4x_1\,d^4x_2\,e^{i(q_1x_1+q_2x_2)}\theta(x_{10})\theta(-x_{20})\\ &\times\pi\sum_{n,\,p_n=2q_3}\langle 0|[f(\tfrac{1}{2}x_1),j(-\tfrac{1}{2}x_1)]|n\rangle\\ &\times\langle n|[f^\dagger(-\tfrac{1}{2}x_2),J(\tfrac{1}{2}x_2)]|0\rangle\\ &\qquad\qquad+\text{degenerate terms},\end{aligned}\tag{4.4}$$

where we have assumed that $\tau<(3\mu)^2$ and $\rho<(2\mu)^2$. The "degenerate terms" contain equal-time commutators; they do not alter the analytic properties of G. In Eq. (4.4) we have introduced the subscripts r and a of G in order to indicate the retarded and advanced character of the integrand in the variables x_1 and x_2, respectively. We introduce in addition to G_{ra} the three other functions G_{rr}, G_{aa} and G_{ar} by replacing in Eq. (4.4) $\theta(-x_{20})$ by $-\theta(x_{20})$, $\theta(x_{10})$ by $-\theta(-x_{10})$, and $\theta(x_{10})$ by $-\theta(-x_{10})$ as well as $\theta(-x_{20})$ by $-\theta(x_{20})$, respectively. Then we have the four functions $G_{ij}(q_1q_2q_3)$, $i=r$, a and $j=r$, a, which, on the basis of the spectral conditions (2.4), have the properties

$$\begin{aligned}&G_{rj}-G_{aj}=0\quad\text{for}\quad (q_1+q_3)^2<(m+\mu)^2\quad\text{and}\\ &\qquad\qquad\qquad\qquad (q_1-q_3)^2<(3\mu)^2,\\ &G_{ir}-G_{ia}=0\quad\text{for}\quad (q_2+q_3)^2<(m+\mu)^2\quad\text{and}\\ &\qquad\qquad\qquad\qquad (q_2-q_3)^2<(2\mu)^2,\\ &G_{ij}=0\quad\text{unless}\quad q_3^2\geqslant(m+\mu)^2\quad\text{and}\quad q_{30}>0.\end{aligned}\tag{4.5}$$

From Eq. (4.4) we see that the functions G_{ij} are analytic in q_1 and q_2 in certain tube domains.[6] We may take these functions as one function $G(q_1q_2q_3)$ which is then analytic in the region $(q_1,q_2)\epsilon W\times W$ for each real q_3, where

$$W=[q:|\mathrm{Im}q_0|>|\mathrm{Im}\mathbf{q}|].\tag{4.6}$$

Let us denote the "joining up region" of Eq. (4.6) by $(q_1,q_2)\epsilon S_3\times S_2$, where S_n is the set of real points

$$\begin{aligned}S_n=[q:\mathrm{Im}q=0,\ (\mathrm{Re}q+q_3)^2<(m+\mu)^2,\\ (\mathrm{Re}q-q_3)^2<(n\mu)^2].\end{aligned}$$

The function G satisfies all the conditions of the *edge of the wedge theorem* of BOT and hence it may be continued, for every fixed q_3, into the region $(W\cup N_3)\times(W\cup N_2)$, where N_n is some complex neighborhood of the set S_n. This continuation is sufficient to prove the dispersion relations for $\Delta^2=\Delta_{\text{th}}^2(\gamma)$. Writing

$$\begin{aligned}&\mathbf{q}_1=R_1\mathbf{e}_1+D\mathbf{e}_2,\quad \mathbf{q}_2=R_2\mathbf{e}_1-D\mathbf{e}_2,\\ &R_{1,2}=\frac{(\mathbf{q}_1+\mathbf{q}_2)^2\pm(\mathbf{q}_1^2-\mathbf{q}_2^2)}{2[(\mathbf{q}_1+\mathbf{q}_2)^2]^{\frac{1}{2}}},\quad D=\left[\frac{\mathbf{q}_1^2\mathbf{q}_2^2-(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{(\mathbf{q}_1+\mathbf{q}_2)^2}\right]^{\frac{1}{2}},\end{aligned}\tag{4.7}$$

with q_{10}, q_{20}, $\mathbf{q}_1^2$, $\mathbf{q}_2^2$, and $(\mathbf{q}_1-\mathbf{q}_2)^2$ given by Eqs. (4.3) we have

$$\begin{aligned}&(q_1+q_3)^2=(q_2+q_3)^2=m^2,\quad (q_1-q_3)^2=\alpha+\beta,\\ &(q_2-q_3)^2=\alpha-\beta,\quad (q_1-q_2)^2=-4\Delta^2,\quad 4q_3^2=\sigma^2;\end{aligned}\tag{4.8}$$

for $\beta=\beta_0$ and $\alpha\leqslant\alpha_0$ the vectors (q_1,q_2) are lying in the region of analyticity provided $|\Delta^2-\Delta_{\text{th}}^2|<\delta$ with δ positive and sufficiently small. Hence, for the present case, the edge of the wedge theorem proves the required analyticity of the ϕ as a function of α.

A straightforward method to extend the proof to larger values of Δ^2 would be the following. We map the domain $(W\cup N_3)\times(W\cup N_2)$ (or its envelope of holomorphy) into the space of q_{10}, q_{20}, and the inner products $\mathbf{q}_1^2$, $\mathbf{q}_2^2$, $(\mathbf{q}_1-\mathbf{q}_2)^2$, and compute the complete envelope of homomorphy. However, the present function-theoretic methods for the construction of envelopes are very impractical; we prefer to use a general representation of functions which have properties like those expressed in Eqs. (4.5) and (4.6). The existence of such a representation has been proven by Dyson,[14] who reduced the problem to the simpler case of symmetric spectral conditions which has been treated by Jost and Lehmann.[15] With these tools we obtain for G a representation of the form[11]

$$G(q_1q_2q_3)=(\sigma^2-m^2)\phi(\alpha,\beta,\Delta^2,\sigma^2)$$
$$=\int\frac{d^4u_1d^4u_2d\kappa_1^2d\kappa_2^2\,\chi(\kappa_1^2,\kappa_2^2,u_1,u_2,q_3)}{[\kappa_1^2-(q_1-u_1)^2][\kappa_2^2-(q_2-u_2)^2]},\quad(4.9)$$

where χ is an arbitrary real weight function which vanishes for $\sigma<m+\mu$ and also outside the region

$$|\mathbf{u}_i|\leqslant\tfrac{1}{2}\sigma-|u_{i0}|,\quad i=1,2,$$
$$\kappa_i\geqslant\max\{0,\,a_i-[(\tfrac{1}{2}\sigma+u_{i0})^2-\mathbf{u}_i^2]^{\frac{1}{2}},\quad(4.10)$$
$$b_i-[(\tfrac{1}{2}\sigma-u_{i0})^2-\mathbf{u}_i^2]^{\frac{1}{2}}\}.$$

In Eq. (4.10) we have used the special Lorentz frame where $\mathbf{q}_3=0$ and $q_{30}=\frac{1}{2}\sigma$. From the spectral conditions in Eq. (4.6) it follows that

$$a_1=a_2=m+\mu,\quad b_1=3\mu\quad\text{and}\quad b_2=2\mu.\quad(4.11)$$

The reality of χ may be demonstrated using the representations (4.4) for the G_{ij} and invariance under inversion of motion. The function χ is also invariant under space rotations, and in the c.m. frame we may write

$$\chi=\bar{\chi}(\kappa_1^2,\kappa_2^2,u_{10},u_{20},\mathbf{u}_1^2,\mathbf{u}_2^2,\mathbf{u}_1\cdot\mathbf{u}_2(|\mathbf{u}_1|\,|\mathbf{u}_2|)^{-1},\sigma).\quad(4.12)$$

If we now introduce (4.12) into the representation (4.9) and perform the redundant angle integration, then we are left with a function depending only on q_{10}, q_{20}, and the inner products $\mathbf{q}_1^2$, $\mathbf{q}_2^2$, and $\mathbf{q}_1\cdot\mathbf{q}_2$. We may take these variables complex and discuss the points of regularity of this representation for every fixed value of the generalized variable $\sigma\geqslant m+\mu$. The set of all points where the function is analytic is then the envelope of holomorphy which we have discussed earlier. For the purpose of dispersion relations we are interested in the analytic properties of ϕ as a function of α, β, and Δ^2, especially for α and β in the neighborhood of the real axis.

First we discuss the representation (4.9) for the special case where the parameters $\mathbf{u}_1$ and $\mathbf{u}_2$ are zero. Then the integrand depends only upon q_{10}, q_{20}, $\mathbf{q}_1^2$, and $\mathbf{q}_2^2$, and it is independent of Δ^2. We are interested in those real points τ and ρ (or α and β) for which the denominator does not vanish for *any* $\sigma\geqslant m+\mu$, and we find the conditions

$$\tau<3\mu^2\left(\frac{2m+\mu}{2m-\mu}\right)\quad\text{and}\quad\rho<2\mu^2\left(\frac{2m+\mu}{2m}\right).\quad(4.13)$$

The restrictions (4.13) are the same as those obtained with the representation (3.8) for the corresponding vertex functions.

Let us now consider the cases $\mathbf{u}_i\neq0$. We introduce polar coordinates and perform one angle integration. Then we have

$$(\sigma^2-m^2)\phi(\alpha,\beta,\Delta^2,\sigma^2)=2\pi\int u_1du_1\,u_2du_2\,d\kappa_1^2d\kappa_2^2$$
$$\times\int_0^{2\pi}d\psi\int_0^{\pi}d\varphi_1\int_0^{\pi}d\varphi_2\frac{\bar{\chi}(\kappa_1^2,\kappa_2^2,u_{10},u_{20},\mathbf{u}_1^2,\mathbf{u}_2^2,\cos\psi\,\sin\varphi_1\sin\varphi_2+\cos\varphi_1\cos\varphi_2,\sigma)}{N_1N_2+[(N_1^2-\mathbf{q}_1^2)(N_2^2-\mathbf{q}_2^2)]^{\frac{1}{2}}-[\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}\cos(\vartheta-\psi)}$$
$$\times\left\{\frac{N_1}{[N_1^2-\mathbf{q}_1^2]^{\frac{1}{2}}}+\frac{N_2}{[N_2^2-\mathbf{q}_2^2]^{\frac{1}{2}}}\right\},\quad(4.14)$$

where

$$N_i=\frac{\kappa_i^2+\mathbf{u}_i^2+\mathbf{q}_i^2-(q_{i0}-u_{i0})^2}{2|\mathbf{u}_i|\,\sin\varphi_i},\quad(4.15)$$

and ϑ is the angle between the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. In order to discuss the limitations in τ, ρ, and Δ^2, it is useful to have the minima of the N_i as functions of the parameters κ_i^2, $\mathbf{u}_i^2$, and u_{i0}, where these variables are restricted to the region given in Eq. (4.10). We find, for real τ, ρ and $\sigma\geqslant m+\mu$, with $\varphi_i=\frac{1}{2}\pi$,

$$\min N_1=[g_1(\tau)+\mathbf{q}_1^2]^{\frac{1}{2}},$$
$$\min N_2=[g_2(\rho)+\mathbf{q}_2^2]^{\frac{1}{2}},\quad(4.16)$$

where

$$g_1(\tau)=(a_1^2-m^2)(b_1^2-\tau)/[\sigma^2-(a_1-b_1)^2],$$
$$g_2(\rho)=(a_2^2-m^2)(b_2^2-\rho)/[\sigma^2-(a_2-b_2)^2]\quad(4.17)$$

for

$$(\tau,\rho)\geqslant(\tau_0,\rho_0)=b_i[a_i-\sigma^2/(a_i-b_i)]$$

and

$$g_1(\tau)=N_1^2(\tau)-\mathbf{q}_1^2,\quad g_2(\rho)=N_2^2(\rho)-\mathbf{q}_2^2$$

with

$$N_i(\lambda)=\frac{a_i^2-m^2}{2\sigma}+\frac{1}{2\sigma}\{(a_i^2+\sigma^2-\lambda)^2-4a_i^2\sigma^2\}^{\frac{1}{2}},\quad(4.17\text{a})$$

for

$$(\tau,\rho)<(\tau_0,\rho_0),\quad(\tau_0,\rho_0)\leqslant(0,0).§$$

The masses a_i and b_i are given in Eq. (4.11), and in Eqs. (4.3) the quantities $\mathbf{q}_1^2$ and $\mathbf{q}_2^2$ have been expressed in terms of τ, σ and ρ, σ, respectively. The corresponding minimum of the expression $N_1N_2+[(N_1^2-\mathbf{q}_1^2)$

§ *Note added in proof*.—We would like to thank Professor A. A. Logunov and Professor V. S. Vladimirov for bringing this second possibility to our attention.

$\times(N_2^2-\mathbf{q}_2^2)]^{\frac{1}{2}}$ becomes then

$$2A(\sigma,\rho,\tau)=\{[(g_1+\mathbf{q}_1^2)(g_2+\mathbf{q}_2^2)]^{\frac{1}{2}}+[g_1g_2]^{\frac{1}{2}}\}. \quad (4.18)$$

Let us consider first the last factor in the representation (4.14). It is independent of Δ^2 and leads only to the restrictions $\tau<(3\mu)^2$ and $\rho<(2\mu)^2$ for all values of $\sigma\geqslant m+\mu$. These limitations are due to the onset of the static cuts in the complex τ and ρ planes.

For $\sigma\geqslant m+\mu$ and real τ and ρ, the first denominator in Eq. (4.14) leads to a region of analyticity in Δ^2 which is the inside of an ellipse with foci at

$$\Delta^2=\Delta_0^2(\sigma,\rho,\tau)\pm\tfrac{1}{2}[\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}, \quad (4.19)$$

and the boundary

$$\Delta^2=\Delta_0^2(\sigma,\rho,\tau)-A(\sigma,\rho,\tau)\cos\psi\pm iB(\sigma,\rho,\tau)\sin\psi. \quad (4.20)$$

Here B is given by

$$\begin{aligned}B(\sigma,\rho,\tau)&=\tfrac{1}{2}[4A^2-\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}\\&=\tfrac{1}{2}\{[g_1(g_2+\mathbf{q}_2^2)]^{\frac{1}{2}}+[g_2(g_1+\mathbf{q}_1^2)]^{\frac{1}{2}}\}, \quad (4.21)\end{aligned}$$

and we have

$$[\mathbf{q}_1^2\mathbf{q}_2^2]^{\frac{1}{2}}\cos\vartheta=2(\Delta_0^2-\Delta^2),$$

with

$$\begin{aligned}\Delta_0^2(\sigma,\rho,\tau)&=\tfrac{1}{4}\{\mathbf{q}_1^2+\mathbf{q}_2^2-(q_{10}-q_{20})^2\}\\&=\tfrac{1}{8}\{\sigma^2-2m^2-\tau-\rho+(m^2-\tau)(m^2-\rho)/\sigma^2\}. \quad (4.22)\end{aligned}$$

We note that in the region of interest; i.e., for $\sigma\geqslant m+\mu$, $\tau\leqslant\mu^2$, and $\rho\leqslant\gamma\leqslant0$, the functions A, B, and Δ_0^2 are increasing with decreasing values of τ and/or ρ. Then we find from Eqs. (4.13) and (4.20) that for $\beta=\beta_0=\frac{1}{2}(\mu^2-\gamma)$ and real Δ^2 the function ϕ has the required analytic properties in $\alpha=\frac{1}{2}(\tau+\rho)$ provided

$$\max_{\sigma\geq m+\mu}\{\Delta_0^2(\sigma,\gamma,\mu^2)-A(\sigma,\gamma,\mu^2)\}<\Delta^2<\min_{\sigma\geq m+\mu}\{\Delta_0^2(\sigma,\gamma,\mu^2)+A(\sigma,\gamma,\mu^2)\}. \quad (4.23)$$

This leads to the limitations in momentum transfer which we have discussed in Sec. 2. There we have also described the implications of Eq. (4.20) for the continuation of the absorption part into the unphysical region by means of Legendre expansions.

It should be noted that those analytic properties of ϕ, which we have derived earlier in this section using the edge of the wedge theorem, follow also from the representation (4.14). We have discussed here both methods in order to show that the dispersion relations for $\Delta^2=\Delta_{\text{th}}^2$ follow already from the very general, and in principle quite simple, edge of the wedge theorem. In addition we wanted to point out the connection between the use of the Dyson representation and the function theoretical methods which we have employed in BOT.

ACNKOWLEDGMENTS

During part of this work the second-named author was at the Institute for Advanced Study, Princeton, New Jersey. He would like to express his thanks to Professor Robert Oppenheimer for his kind and stimulating hospitality at the Institute, and to thank the Commonwealth Fund of New York for a grant.

We are much indebted to Mr. Chieh Ho, Mr. John Curtis, and Mr. Wayles Browne for the computation of Table I.

APPENDIX

We give here a brief survey of the limitations in the invariant momentum transfer 2Δ, which we encounter in the proof of dispersion relations for some elastic scattering processes. As has been explained in the Introduction, these restrictions need not be characteristic for the assumptions (causality, spectrum etc.) we have made. For the elastic amplitudes the quentity $\Delta_{\max}^2$ is given by an expression of the form[16]

$$\Delta_{\max}^2=\min_{\sigma\geq c}\{\mathbf{q}^2(\sigma,x_1,x_2)+g(\sigma,x_1,x_2)\}, \quad (\text{A1})$$

where

$$\left.\begin{aligned}\mathbf{q}^2(\sigma,x_1,x_2)&=\tfrac{1}{4}\sigma^2-\tfrac{1}{2}(x_1+x_2)+(x_1-x_2/2\sigma)^2,\\g(\sigma,x_1,x_2)&=(a^2-x_1)(b^2-x_2)/[\sigma^2-(a-b)^2],\end{aligned}\right\}\quad(\text{A2})$$

$$a\geqslant b,$$
$$x_2\geqslant b[a-\sigma^2/(a-b)],$$

and $\sigma^2=(k+p)^2$, $x_1=p^2$, $x_2=k^2$; the masses a, b, and c are determined by the spectral conditions. For fixed σ, the absorptive part of the corresponding amplitude is an analytic function of Δ^2, regular inside the ellipse given by[11]

$$\Delta^2=\tfrac{1}{2}\mathbf{q}^2-\{\tfrac{1}{2}\mathbf{q}^2+g\}\cos\psi\pm i[(\mathbf{q}^2+g)g]^{\frac{1}{2}}\sin\psi. \quad (\text{A3})$$

These analytic properties can be used in order to express the absorptive part in the "unphysical region" in terms of physical quantities by means of Legendre expansions. The dispersive part is also an analytic function of Δ^2; it is regular inside a smaller ellipse given by[11]

$$\Delta^2=\tfrac{1}{2}\mathbf{q}^2-\tfrac{1}{2}[\mathbf{q}^2(\mathbf{q}^2+g)]^{\frac{1}{2}}\cos\psi\pm i\tfrac{1}{2}[\mathbf{q}^2g]^{\frac{1}{2}}\sin\psi. \quad (\text{A4})$$

The following relations have been proven:

Pion-pion scattering.—We have $x_1=x_2=\mu^2$; $a=b=3\mu$ and $c=2\mu$. The minimum in Eq. (A1) appears at $\sigma=4\mu$ and we find $\Delta_{\max}^2=7\mu^2$.

Pion-nucleon scattering.—Here we have $x_1=m^2$, $x_2=\mu^2$; $a=m+\mu$, $b=3\mu$, and $c=m+\mu$. The minimum is at $\sigma=m+\mu$ and

$$\Delta_{\max}^2=\frac{8\mu^2}{3}\frac{2m+\mu}{2m-\mu}.\text{[17]}$$

[16] See the footnote in Sec. 4 of BOT, where Eqs. (A1) and (A2) have been given in a somewhat different form.

[17] BOT, Sec. 4. Compare also N. N. Bogoliubov and V. S. Vladimirov, 1957 (to be published); and V. S. Vladimirov, Joint Institute for Nuclear Research, 1958 (unpublished). In these papers the original proof of Bogoliubov has been extended to larger values of Δ^2. In the first preprint the limitation $\Delta^2<2\mu^2$ is given and in the second preprint the proof is extended to $\Delta^2<2.56\mu^2$ (with $m=7\mu$).

Photon-deuteron scattering.—We may either consider the deuteron as an elementary particle and describe it by a separate field operator which satisfies the requirement of local commutativity, or we may treat it as a composite particle. In the latter case, we make use of the formalism developed by several authors,[18] who have shown that it is possible to define a local deuteron field-operator in terms of the operators for neutron and proton. The S-matrix element for γ-d scattering may then be expressed in terms of retarded products of photon and deuteron operators in the same way as in the case of "elementary" particles only. We obtain, from Eq. (A1) with $x_1=M_d^2=(2m-B)^2$, $x_2=0$, and $a=2m$, $b=3\mu$, $c=2m$,

$$\Delta_{\max}{}^2=B\mu(4m-B)/(2m-\mu)+B^2(1-B/4m)^2.$$

Here the electromagnetic interaction has been treated in lowest order.

Photon-proton scattering.—This case has been discussed at the end of Sec. 2.

There are some important elastic scattering processes for which the dispersion relations cannot be proven on the basis of causality and spectral conditions alone. Even for forward scattering, the analytic properties of the absorptive part,

$$\phi(p'^2,p^2,k'^2,k^2,\ (k-k')^2,\ (k+p)^2) =\phi(x_1,x_1,\tau,\tau,\ -4\Delta^2,\ \sigma^2),$$

as a function of τ are not sufficient to include the points on the mass shell $(\tau=x_2)$ for every $\sigma\geqslant c$.

In special cases we obtain the following limitations:

Nucleon-nucleon scattering.—For $x_1=m^2$ and $\Delta^2=0$, the real points τ belong to the region of analyticity for all $\sigma\geqslant 2\mu$ provided $\tau<\frac{1}{2}(m+\mu)^2$. The mass shell $(\tau=m^2)$ could be reached only under the unphysical condition $\mu>(\sqrt{2}-1)m$.[6] For fixed $\sigma>(m+\mu)+[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}$ and τ on the mass shell, the absorptive part ϕ is analytic in Δ^2; it is regular inside the ellipse (A3) with the parameters $x_1=x_2=m^2$, $a=b=m+\mu$ and $c=2\mu$. Correspondingly, for fixed $\sigma>2m$, the dispersive part is regular inside the ellipse (A4).

In a perturbation theory based upon the usual pion-nucleon interaction, it can be shown that the dispersion relations for nucleon-nucleon scattering hold in every finite order provided $\Delta^2<\frac{1}{4}\mu^2$.[19]

K-meson-nucleon scattering.—The parameters are $x_1=m^2$, $x_2=\mu_K{}^2$, $a=m+\mu$, $b=\mu_K+2\mu$, and $c=m_\Lambda+\mu$. For $x_1=m^2$, $\Delta^2=0$, and all $\sigma\geqslant m_\Lambda+\mu$, a real point τ is inside the region of analyticity of the absorptive part provided

$$\tau<\tau_0=\min_{\sigma\geq c}\frac{2ab+\sigma(\sigma-a-b)-x_1[1-(a-b)/\sigma]}{1+(a-b)/\sigma}. \quad \text{(A5)}$$

With the parameters given above, the minimum is at $\sigma=c=m_\Lambda+\mu$, and it is smaller than $\mu_K{}^2$ by a narrow margin.[6] Note that the condition $\mathbf{q}^2(\sigma,m^2,\tau)+g(\sigma,m^2,\tau)>0$ is fulfilled for $\tau<\tau_0$ and all $\sigma\geqslant m_\Lambda+\mu$. We have again analytic properties of the absorptive and the dispersive part as functions of Δ^2. On the mass shell the absorptive part is regular in (A3) for fixed $\sigma>\sigma_0$, where $m_\Lambda+\mu<\sigma_0<m+\mu_K$ and $\mathbf{q}^2(\sigma_0,m^2,\mu^2)+g(\sigma_0,m^2,\mu^2)=0$; the dispersive part is regular (A4) for fixed $\sigma>m+\mu_K$.

Pion-deuteron scattering.—As in the case of γ-d scattering, we treat the deuteron as an elementary particle. We have the parameters $x_1=M_d{}^2$, $x_2=\mu^2$, and $a=2m$, $b=3\mu$, $c=2m$. The limitation in τ becomes

$$\tau<\tau_0=3\mu B(4m-B)/(4m-3\mu), \quad \text{(A6)}$$

and is far below the mass shell $\tau=\mu^2$. Again the absorptive and the dispersive part have the corresponding analytic properties as functions of Δ^2.

Let us recall briefly the implications of a restriction like (A6). According to the method of our proof, it says that the amplitude is an analytic function of $z=x+iy=(k+p)^2$ in the complex z plane except for the cut $y=0$, $x\geqslant c^2=(2m)^2$ *provided* $\tau<\tau_0$. For $\tau>\tau_0$, causality and spectrum do not guarantee the absence of additional singularities in the z plane. In the present case of π-d scattering, it is possible that, even in a perturbation theory with intermediate lines corresponding to deuterons, nucleons, and pions only, we do not have the cut $y=0$, $x\geqslant(2m)^2$ *alone* if $\tau=\mu^2$. There appear additional singularities on the real axis below $x=(2m)^2$.[20]

---

[18] K. Nishijima, Phys. Rev. **111**, 995 (1958); W. Zimmermann, 1958 (unpublished); R. Haag, Phys. Rev. **112**, 669 (1958). These papers contain further references.

[19] K. Symanzik, 1958 (to be published).

[20] Y. Nambu (private communication).