(1); and where initial atomic P states are concerned, all compete with the $2P \rightarrow 1S$ radiative transition. The latter goes with a rate of order 10^{12} sec^{-1}. We now argue that reactions (3) and (4) cannot be significant for initial P states; the $2P \rightarrow 1S$ rate certainly is much larger than the rate for (4),[4] and very likely also for (3). From initial S states, (3) and (4) are expected to be significant only when channel (1) is forbidden and channel (2) is either energetically forbidden or strongly suppressed. As one sees from Table I, these conditions could be met only in the case of even cascade parity, for then the $^3S_1$ initial state can deplete only through channels (3) and (4). The observation of an appreciable probability for reaction (3), according to this argument, would in itself be strong evidence that the cascade parity is even. Corroborative evidence in such a circumstance would be provided by studying the correlation function P discussed above for reaction (1). For now we would expect that (1) proceeds mainly through initial $^1S_0$ states and hence $P \simeq 1-\alpha^2 \mathbf{p}_1 \cdot \mathbf{p}_2$.

As for the relative importance of reactions (1) and (2), we can say nothing beyond what is implied by the selection rules of Table I. One doesn't even known if (2) is energetically allowed. Moreover, for initial $2P$ states we cannot reliably estimate the importance of either process relative to $2P \rightarrow 1S$ radiative transitions. Nevertheless, one pertinent observation can be made. This concerns the case of odd cascade parity. Here, assuming that production of final D states in reaction (1) is negligible, we see that among all initial P states only the $^3P_0$ state can react according to (1). If the P states are populated statistically this represents a maximum probability of one-twelfth for reaction (1) to occur from a P state. The remaining P states either react according to (2) or undergo radiative transitions to the $1S$ states. We can therefore conclude the following: If it is found experimentally that reaction (2) does *not* occur with appreciable probability, then either the cascade parity is even; or, if odd, the capture reactions (1) occur predominantly from S states. But for the latter case one has stringent restrictions on the correlation function of the Λ-decay pions. In particular, Eq. (16) must hold in good approximation. In this way one could distinguish between the two possibilities.

Finally, suppose reaction (2) occurs with large probability. If the energy release is nevertheless small, less than a few Mev, one would argue on the basis of centrifugal barrier effects that this implies a preponderance of captures from the S states—for reaction (2) and therefore also for reaction (1). In this case the parity of Ξ^- can again be determined by studying the correlation function P.

It is clear then that for many circumstances the parity of Ξ^- could be rather cleanly established. Only if reaction (2) is prominent and proceeds with a large energy release would the situation become highly ambiguous.

[4] One believes that the Ξ lifetime is of order 10^{-10} or greater; see a summary talk by I. Alvarez, *Proceedings of Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957), p. VI-1.

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Invariants of General Relativity and the Classification of Spaces*

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A unique equivalence is established between the Riemann curvature tensor and two spinors. The fourteen invariants which can be constructed from the curvature tensor are listed in terms of the spinors. The vanishing of the invariants for several different types of spaces is described. A classification of Einstein spaces is made together with a few additional remarks concerning classification of spaces.

I. INTRODUCTION

The general theory of relativity deals with the metric tensor, g_{pq}, and its first and second derivatives with respect to space-time. It has long been known[1] that fourteen independent differential invariants can be constructed from the second derivative of the metric tensor and that these invariants can be expressed in terms of g_{pq} and the Riemann curvature tensor, R^{pqrs}. The fourteen invariants have been constructed[2]; it is obvious by inspection of these invariants that when the contracted curvature tensor R^{pq} vanishes (Einstein's equations for empty space), ten of the invariants also vanish leaving four of them which may

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[1] C. N. Haskins, Trans. Am. Math. Soc. **3**, 71 (1902).

[2] J. Geheniau and R. Debever, Bull. acad. roy. Belg. **42**, 114 (1956).

not be equal to zero. It becomes now pertinent to consider the possible significance of the nonvanishing invariants and the possible significance of there being four of these invariants (the same number as the dimensionality of space-time). Regarding the latter point it becomes proper to ask how many independent invariants do not vanish when space is not empty but is rather filled with electromagnetic fields. We shall show later that the number is nine for a non-null electromagnetic field. For a null electromagnetic field it will turn out that ten invariants vanish, the same ten as for empty space-time. A null electromagnetic field is one in which the two electromagnetic invariants are both equal to zero. We also explicitly exhibit a space where there may be at most five surviving independent invariants. Further, one asks whether the invariants can be usefully used in classifying geometric spaces. From the above remark on null electromagnetic fields and from our later analysis, it is clear that the vanishing of ten invariants is not sufficient to guarantee that $R^{pq}=0$; in fact all fourteen invariants can vanish and still R^{pq} may have nonzero components.

This paper is concerned with the above issues. In Sec. II, we show how the curvature tensor R^{pqrs} can be equivalently (uniquely) represented by means of two spinors of special types. In Sec. III, we construct the fourteen invariants from these two spinors; we then show that at most four independent invariants are not zero when space-time is empty and that at most nine independent invariants are not zero when space-time is filled with a non-null electromagnetic field and four when it is filled with a null electromagnetic field. In Sec. IV we show how the invariants can be used to classify the geometry when $R_{pq}=0$ and in Sec. V we make one or two general remarks concerning classification of spaces. The specific calculations in these last four sections could, of course, be done without the use of spinors but using only tensor algebra; however, we thought it interesting to see how the matter went in spinor language.

II. THE RIEMANN CURVATURE TENSOR AND ITS EQUIVALENCE TO TWO SPINORS

Spinor fields[3,4] have been treated in general relativity from several points of view. It is known that to each real tensor one can assign a spinor. In this section we assign a spinor to the Riemann curvature tensor and see what the symmetries of the tensor require of the assigned spinor, which turns out to be a unique spinor.

Regarding notation, Latin tensor indices (as in g^{pq}) are given values 0, 1, 2, 3; the signature of the metric tensor is $+,-,-,-$; Greek spinor indices (as in $g^p{}_{\dot{\alpha}\beta}$) are given two values ($\dot{\alpha}=\dot{1},\dot{2}$; $\alpha=1, 2$, etc.). The spin matrices, $g^p{}_{\dot{\alpha}\beta}$, transform like tensors for the index p and like spinors of the proper type for each index $\dot{\alpha}$, β. We use a representation in which the g matrices are Hermitean, $\bar{g}^p{}_{\dot{\alpha}\beta}=g^p{}_{\dot{\beta}\alpha}$ (bar denotes complex conjugate) and define the spin matrices by

$$g^{p\dot{\alpha}\beta}g^q{}_{\dot{\alpha}\gamma}+g^{q\dot{\alpha}\beta}g^p{}_{\dot{\alpha}\gamma}=2g^{pq}\delta^\beta{}_\gamma. \tag{1}$$

The Latin index in $g^p{}_{\dot{\alpha}\beta}$, $g_{p\dot{\alpha}\beta}$ can be lowered or raised by using the metric tensor g_{pq} or g^{pq}. The Greek indices $\dot{\alpha}$, β can be raised or lowered by using the antisymmetric fundamental spinors $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ which are equal to 0 when $\alpha=\beta$ and equal $+1$ if $\alpha=1$, $\beta=2$, and equal to -1 if $\alpha=2$, $\beta=1$.

An Hermitean second-rank spinor $\phi_{\dot{\alpha}\beta}$ is determined by four real numbers and can therefore be made to correspond to a real world vector x^p by the relation

$$\phi_{\dot{\alpha}\beta}=g_{p\dot{\alpha}\beta}x^p, \tag{2}$$

which can be inverted by multiplying by $g^{q\dot{\alpha}\beta}$ to give

$$x^q=\tfrac{1}{2}g^q{}_{\dot{\alpha}\beta}\phi^{\dot{\alpha}\beta}. \tag{3}$$

For the more complicated Riemann tensor,

$$R_{pqrs}=g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\kappa\lambda\rho\sigma}. \tag{4}$$

The reality of R_{pqrs} requires $\bar{\phi}^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\kappa\lambda\rho\sigma}=\phi^{\dot{\kappa}\dot{\lambda}\dot{\rho}\dot{\sigma}\alpha\beta\mu\nu}$. The antisymmetry relation $R_{pqrs}=-R_{qprs}$ imposes the relation $\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\kappa\lambda\rho\sigma}=-\phi^{\dot{\beta}\dot{\alpha}\mu\nu\lambda\kappa\rho\sigma}$, so that

$$R_{pqrs}=\tfrac{1}{2}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}(\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\kappa\lambda\rho\sigma}-\phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda\kappa\rho\sigma}). \tag{5}$$

This can be rewritten as

$$R_{pqrs}=\tfrac{1}{2}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}\{\epsilon^{\kappa\lambda}\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma}+\epsilon^{\dot{\alpha}\dot{\beta}}\psi^{\kappa\lambda\rho\sigma\dot{\mu}\dot{\nu}}\}, \tag{6}$$

where we have defined

$$2\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma}=\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_\lambda{}^{\lambda\rho\sigma}+\phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}}{}_\lambda{}^{\lambda\rho\sigma}. \tag{7}$$

ψ is symmetric in the indices $\dot{\alpha}\dot{\beta}$.

To verify (6), consider (5) in the cases $\dot{\alpha}=\dot{\beta}$, $\kappa=\lambda$; $\dot{\alpha}=\dot{\beta}$, $\kappa\neq\lambda$; $\dot{\alpha}\neq\dot{\beta}$, $\kappa=\lambda$; and $\dot{\alpha}\neq\dot{\beta}$, $\kappa\neq\lambda$. The antisymmetry relation $R_{pqrs}=-R_{pqsr}$ imposes the relation

$$\epsilon^{\kappa\lambda}\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma}+\epsilon^{\dot{\alpha}\dot{\beta}}\psi^{\kappa\lambda\rho\sigma\dot{\mu}\dot{\nu}}=-[\epsilon^{\kappa\lambda}\psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}\sigma\rho}+\epsilon^{\dot{\alpha}\dot{\beta}}\psi^{\kappa\lambda\sigma\rho\dot{\nu}\dot{\mu}}], \tag{8}$$

so that

$$R_{pqrs}=\tfrac{1}{4}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}[\epsilon^{\kappa\lambda}(\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma}-\psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}\sigma\rho}) +\epsilon^{\dot{\alpha}\dot{\beta}}(\psi^{\kappa\lambda\rho\sigma\dot{\mu}\dot{\nu}}-\psi^{\kappa\lambda\sigma\rho\dot{\nu}\dot{\mu}})]. \tag{9}$$

By considering the cases $\dot{\mu}=\dot{\nu}$, $\rho=\sigma$; $\dot{\mu}=\dot{\nu}$, $\rho\neq\sigma$; $\dot{\mu}\neq\dot{\nu}$, $\rho=\sigma$; $\dot{\mu}\neq\dot{\nu}$, $\rho\neq\sigma$; and defining $2\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}\equiv\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_\rho{}^\rho+\psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}}{}_\rho{}^\rho$, $2\phi^{\dot{\alpha}\dot{\beta}\rho\sigma}\equiv\psi^{\dot{\alpha}\dot{\beta}}{}_\lambda{}^{\lambda\rho\sigma}+\psi^{\dot{\alpha}\dot{\beta}}{}_\lambda{}^{\lambda\sigma\rho}$, one sees that

$$R_{pqrs}=\tfrac{1}{4}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}[\epsilon^{\kappa\lambda}\epsilon^{\rho\sigma}\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}+\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\mu}\dot{\nu}}\psi^{\kappa\lambda\rho\sigma} +\epsilon^{\kappa\lambda}\epsilon^{\dot{\mu}\dot{\nu}}\phi^{\dot{\alpha}\dot{\beta}\rho\sigma}+\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\rho\sigma}\phi^{\kappa\lambda\dot{\mu}\dot{\nu}}]. \tag{10}$$

$\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}$ is symmetric in $\dot{\alpha}\dot{\beta}$ and in $\dot{\mu}\dot{\nu}$; $\phi^{\dot{\alpha}\dot{\beta}\rho\sigma}$ is symmetric in $\dot{\alpha}\dot{\beta}$ and in $\rho\sigma$.

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[3] W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953), review spinor analysis and have references to the previous literature.

[4] E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations* (Hafner Publishing Company, New York, 1953). Chapter 2 discusses spinor algebra and uses it in special relativity.

The symmetry requirement $R_{pqrs}=R_{rspq}$ means that

$$\epsilon^{\kappa\lambda}\epsilon^{\rho\sigma}\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}+\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\mu\dot\nu}\psi^{\kappa\lambda\rho\sigma}+\epsilon^{\kappa\lambda}\epsilon^{\dot\mu\dot\nu}\phi^{\dot\alpha\dot\beta\rho\sigma}+\epsilon^{\dot\alpha\dot\beta}\epsilon^{\rho\sigma}\phi^{\kappa\lambda\dot\mu\dot\nu}$$
$$=\epsilon^{\kappa\lambda}\epsilon^{\rho\sigma}\psi^{\dot\mu\dot\nu\dot\alpha\dot\beta}+\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\mu\dot\nu}\psi^{\kappa\lambda\rho\sigma}$$
$$+\epsilon^{\dot\alpha\dot\beta}\epsilon^{\rho\sigma}\phi^{\dot\mu\dot\nu\kappa\lambda}+\epsilon^{\dot\mu\dot\nu}\epsilon^{\kappa\lambda}\phi^{\rho\sigma\dot\alpha\dot\beta}. \quad (11)$$

Equation (11) is satisfied if $\psi^{\alpha\beta\mu\nu}=\psi^{\mu\nu\alpha\beta}$ and if $\bar\phi^{\dot\alpha\dot\beta\mu\nu}=\phi^{\dot\mu\dot\nu\alpha\beta}$ which conditions we now impose. There is still one other restriction that the tensor R_{pqrs} must satisfy before it can be considered a curvature tensor. This is the cyclic constraint

$$R_{pqrs}+R_{prsq}+R_{psqr}=0. \quad (12)$$

Tedious but very straightforward calculations starting with (10) and using the symmetries of ψ and ϕ will show that (12) is satisfied if and only if

$$\psi^{\dot\alpha}{}_{\dot\beta}{}^{\dot\beta}{}_{\dot\alpha}=\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}. \quad (13)$$

Equation (10) thus represents the Riemann curvature tensor by means of two independent spinors $\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}$ and $\phi^{\dot\alpha\dot\beta\rho\sigma}$. These two spinors are symmetric in an interchange of their two first indices or of their two last indices, moreover $\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}=\psi^{\dot\mu\dot\nu\dot\alpha\dot\beta}$, and $\bar\phi^{\dot\alpha\dot\beta\rho\sigma}=\phi^{\dot\rho\dot\sigma\alpha\beta}$; also $\psi^{\dot\alpha}{}_{\dot\beta}{}^{\dot\beta}{}_{\dot\alpha}=\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}$. Let us look at the number of independent components of each spinor. First $\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}$: the pair $\dot\alpha\dot\beta$ can take three independent values, and the pair $\dot\mu\dot\nu$ can take three more, making nine altogether. $\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}=\psi^{\dot\mu\dot\nu\dot\alpha\dot\beta}$ impose three constraints; there are now six complex or twelve real components left, which number is reduced to eleven because of condition (13). $\phi^{\dot\alpha\dot\beta\rho\sigma}$ can have three choices for $\dot\alpha\dot\beta$ and three more for $\rho\sigma$, making again nine choices. $\phi^{\dot\alpha\dot\beta\rho\sigma}$ can have so far nine complex and eighteen real components. The Hermitean requirement $\bar\phi^{\dot\alpha\dot\beta\rho\sigma}=\phi^{\dot\rho\dot\sigma\alpha\beta}$ imposes one constraint for each choice of indices, leaving nine independent real parameters. Thus $\psi^{\alpha\beta\lambda\nu}$ has eleven independent real parameters and $\phi^{\dot\alpha\dot\beta\mu\nu}$ has nine, for a total of twenty. There are twenty independent components in R_{pqrs} so that the results are compatible.

It is instructive to look at the contracted curvature tensor $R_{qr}=R^{p}{}_{qrp}$. Using the spinor relation $g^{p}{}_{\dot\alpha\rho}g_{p\dot\beta\sigma}=2\epsilon_{\dot\alpha\dot\beta}\epsilon_{\rho\sigma}$, one has

$$R_{qr}=\tfrac{1}{2}g_{q\dot\beta\rho}g_{r\dot\mu\sigma}[\epsilon^{\sigma\rho}\psi^{\dot\alpha\dot\beta\dot\mu}{}_{\dot\alpha}+\epsilon^{\dot\mu\dot\beta}\psi^{\alpha\rho\sigma}{}_{\alpha}+2\phi^{\dot\beta\dot\mu\rho\sigma}]. \quad (14)$$

Using the spinor relation

$$g_{p}{}^{\dot\beta}{}_{\alpha}g_{q\dot\beta\mu}=g_{pq}\epsilon_{\alpha\mu}+\tfrac{1}{2}g^{-\frac{1}{2}}\epsilon_{pqrs}g^{r\dot\beta}{}_{\alpha}g^{s}{}_{\dot\beta\mu}, \quad (15)$$

and the fact that $\psi^{\dot\alpha\dot\beta\dot\mu}{}_{\dot\alpha}$ is antisymmetric in $\dot\beta$ and $\dot\mu$, one finds

$$R_{pq}=g_{pq}\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}+g_{p\dot\beta\rho}g_{q\dot\mu\sigma}\phi^{\dot\beta\dot\mu\rho\sigma}. \quad (16)$$

Contracting again,

$$R=4\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}. \quad (17)$$

Let $S_{pq}=R_{pq}-g_{pq}R/4$ so that $S_{pq}\equiv R_{pq}$ if $R=0$. Then

$$S_{pq}=g_{p\dot\beta\rho}g_{q\dot\mu\sigma}\phi^{\dot\beta\dot\mu\rho\sigma}. \quad (18)$$

III. CURVATURE INVARIANTS

The fourteen invariants that can be derived from the curvature tensor can now all be expressed in terms of the two spinors $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot\alpha\dot\beta\mu\nu}$. Any expression involving these two spinors can be re-expressed in terms of geometric elements. Obviously, from (18),

$$\phi^{\dot\beta\dot\mu\rho\sigma}=\tfrac{1}{4}g_{p}{}^{\dot\beta\rho}g_{q}{}^{\dot\mu\sigma}(R^{pq}-\tfrac{1}{4}g^{pq}R). \quad (19)$$

Now we must construct a geometric expression from $\psi^{\alpha\beta\mu\nu}$. Define the double dual $\tilde R_{pqrs}$ of the Riemann curvature tensor

$$\tilde R_{pqrs}\equiv\tfrac{1}{4}g\,\epsilon_{pqab}\epsilon_{cdrs}R^{abcd}. \quad (20)$$

Now let $E_{pqrs}=R_{pqrs}+\tilde R_{pqrs}$. Define the dual

$$\bar E_{pqrs}=\tfrac{1}{2}g^{\frac{1}{2}}\epsilon_{pqab}E^{ab}{}_{rs}. \quad (21)$$

$\bar E_{pqrs}$ is purely imaginary if E_{pqrs} is real. It can be readily seen that

$$\psi^{\kappa\lambda\rho\sigma}=\tfrac{1}{8}g^{p\dot\beta\kappa}g^{q}{}_{\dot\beta}{}^{\lambda}g_{r}{}^{\dot\alpha\rho}g_{s\dot\alpha}{}^{\sigma}[E_{pq}{}^{rs}+\bar E_{pq}{}^{rs}]. \quad (22)$$

E_{pqrs} and $\bar E_{pqrs}$ are purely geometric tensors. Obviously any scalar invariant which can be constructed from $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot\alpha\dot\beta\mu\nu}$ can be written, by use of (19) and (22), in terms of geometric tensors alone.

To list the invariants, there is first the scalar corresponding to R:

$$\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}. \quad (23)$$

This is a real expression and corresponds to one invariant. There are three independent real invariants which can be constructed from $\phi_{\dot\alpha\dot\beta\mu\nu}$ alone:

$$\phi_{\dot\alpha\dot\beta\mu\nu}\phi^{\dot\alpha\dot\beta\mu\nu},\quad \phi_{\dot\alpha\dot\beta\mu\nu}\phi^{\dot\beta\dot\kappa\nu\rho}\phi_{\dot\kappa}{}^{\dot\alpha}{}_{\rho}{}^{\mu},\quad \phi_{\dot\alpha\dot\beta\mu\nu}\phi^{\dot\beta\dot\kappa\nu\rho}\phi_{\dot\kappa\dot\lambda\rho\sigma}\phi^{\dot\lambda\dot\beta\sigma\mu}. \quad (24)$$

There are two independent complex or four independent real invariants which can be constructed from $\psi^{\alpha\beta\mu\nu}$ alone:

$$\psi_{\alpha\beta\mu\nu}\psi^{\alpha\beta\mu\nu},\quad \psi_{\alpha\beta\mu\nu}\psi^{\mu\nu}{}_{\rho\sigma}\psi^{\rho\sigma\alpha\beta}. \quad (25)$$

We have so far found eight real invariants. The construction of the other invariants involves products of ψ's with ϕ's. There are six real invariants or three complex invariants of this type:

$$\phi_{\rho\sigma\dot\alpha\dot\beta}\psi^{\dot\alpha\dot\beta}{}_{\dot\mu\dot\nu}\phi^{\rho\sigma\dot\mu\dot\nu},\quad \phi^{\gamma\delta}{}_{\dot\alpha\dot\beta}\psi^{\dot\alpha\dot\beta}{}_{\dot\rho\dot\sigma}\psi^{\dot\rho\dot\sigma\dot\kappa\dot\lambda}\phi_{\gamma\delta\dot\kappa\dot\lambda},$$
$$\phi^{\gamma\delta\dot\alpha\dot\beta}\phi_{\rho\sigma\dot\alpha\dot\beta}\psi^{\rho\sigma\kappa\lambda}\phi_{\kappa\lambda\dot\mu\dot\nu}\phi_{\gamma\delta}{}^{\dot\mu\dot\nu}. \quad (26)$$

This completes the listing of the invariants. Suppose now space is empty and $R_{pq}=0$. By (17) and (18) this requires $\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}$ and $\phi^{\dot\alpha\dot\beta\mu\nu}$ both to vanish. Hence all the invariants vanish except those labeled (25). Thus in empty space there are at most four nonzero curvature invariants.

Suppose that space-time is not empty but is filled with electromagnetic fields. In this case the stress-energy-momentum tensor relation takes the form

$$R_{pq}=f_{pr}f_{q}{}^{r}-\tfrac{1}{4}g_{pq}f_{rs}f^{rs}. \quad (27)$$

$f_{pq}\equiv(2G)^{\frac{1}{2}}F_{pq}/c^{2}$; G is the gravitational constant, c is the velocity of light, and F_{pq} is the usual antisymmetric electromagnetic tensor. In flat space and Cartesian coordinates F_{12}= the Z component of the magnetic field; F_{14}= the X component of the electric field multiplied by c, etc. Define the dual tensor of f by

$$\tilde{f}^{pq}\equiv\tfrac{1}{2}g^{-\frac{1}{2}}\epsilon^{pqrs}f_{rs}, \quad f_{rs}=\tfrac{1}{2}g^{\frac{1}{2}}\epsilon_{rspq}\tilde{f}^{pq}. \tag{28}$$

ϵ^{pqrs} and ϵ_{pqrs} are tensor densities which take the values $+1$ when $pqrs$ is an even permutation of 1230; -1 when $pqrs$ is an odd permutation of 1230; and 0 otherwise. The second of relations (28) follows from the first and from the properties of the ϵ's. Now define the self-dual antisymmetric tensor,

$$\omega_{pq}\equiv f_{pq}+\tilde{f}_{pq}. \tag{29}$$

It is well known[5] that a self-dual antisymmetric tensor of the second rank can be represented uniquely by a symmetric second rank spinor:

$$\omega_{pq}=g_p{}^{\dot{\alpha}}{}_{\mu}g_{q\dot{\alpha}\nu}\phi^{\mu\nu}, \quad \phi^{\mu\nu}=\tfrac{1}{8}g_p{}^{\dot{\alpha}\mu}g_{q\dot{\alpha}}{}^{\nu}\omega^{pq}. \tag{30}$$

Moreover, the equation (27) becomes

$$\begin{aligned} R_{pq}&=\omega_{pr}\bar{\omega}_q{}^r=g_p{}^{\dot{\alpha}}{}_{\beta}g_{r\dot{\alpha}\mu}g_{q\dot{\nu}}{}^{\rho}g^r{}_{\dot{\sigma}\rho}\phi^{\beta\mu}\phi^{\dot{\nu}\dot{\sigma}},\\ R_{pq}&=-2g_{p\dot{\nu}\beta}g_{q\dot{\sigma}\mu}\phi^{\beta\mu}\phi^{\dot{\nu}\dot{\sigma}}. \end{aligned} \tag{31}$$

Consequently consider the two spinors $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ which represent the curvature tensor. It is clear that in the case of space filled with electromagnetic energy the spinors are constrained to satisfy the conditions

$$\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}=0, \quad \phi^{\dot{\alpha}\dot{\beta}\mu\nu}=-2\phi^{\dot{\alpha}\dot{\beta}}\phi^{\mu\nu}. \tag{32}$$

For electromagnetic fields, invariant (23) thus vanishes. Of the three invariants (24), the first one survives in the form (neglecting in the following constant factors)

$$\phi_{\dot{\alpha}\dot{\beta}}\phi_{\mu\nu}\phi^{\dot{\alpha}\dot{\beta}}\phi^{\mu\nu}. \tag{33}$$

Using relations such as

$$\phi^{\alpha\beta}\phi_{\beta}{}^{\gamma}=\tfrac{1}{2}\epsilon^{\alpha\gamma}\phi^{\mu\nu}\phi_{\mu\nu}, \tag{34}$$

one sees that the second invariant of the series (24) vanishes and that the third is expressed as the square of (33). The four real invariants of (25) are unaffected by the change in the controlling equation for R_{pq}. Consider the three expressions (26), the first two remain independent and are expressed by

$$\phi_{\alpha\beta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\dot{\mu}\dot{\nu}}{}_{\dot{\rho}\dot{\sigma}}\phi^{\alpha\beta}\phi^{\dot{\rho}\dot{\sigma}}, \quad \phi^{\alpha\beta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\dot{\mu}\dot{\nu}}{}_{\dot{\rho}\dot{\sigma}}\psi^{\dot{\rho}\dot{\sigma}\dot{\kappa}\dot{\lambda}}\phi_{\alpha\beta}\phi_{\dot{\kappa}\dot{\lambda}}. \tag{35}$$

The third expression (26) becomes

$$\phi^{\alpha\beta}\phi^{\dot{\mu}\dot{\nu}}\phi_{\gamma\delta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\gamma\delta\eta\zeta}\phi_{\eta\zeta}\phi_{\dot{\rho}\dot{\sigma}}\phi_{\alpha\beta}\phi^{\dot{\rho}\dot{\sigma}}.$$

This is not independent, being the product of (33) and the complex conjugate of the first expression in (35). *Thus in the case of electromagnetic radiation there may in general be nine independent nonvanishing invariants; only one invariant must vanish; only nine of the remaining thirteen are independent.* From the electromagnetic tensor one can construct the complex invariant, $\omega_{pq}\omega^{pq}$. In case this vanishes we shall call the field a null field and the tensor ω_{pq} a null tensor. It is easy to establish[5] that for a null antisymmetric tensor, ω_{pq}, the corresponding spinor, $\phi^{\alpha\beta}$, takes the form $\phi^{\alpha}\phi^{\beta}$. With $\phi^{\alpha\beta}=\phi^{\alpha}\phi^{\beta}$ one sees that all of the invariants (33) and (35) vanish. *Thus if space-time is filled with electromagnetic radiation of the null-field type, the same ten invariants vanish as vanish for empty space-time, leaving four possible nonvanishing invariants.*

Another interesting case to consider is one where the field contains a fluid with no pressure or internal stresses. In this case

$$T^{pq}=\rho_0 v^p v^q, \quad v^p v_p=1. \tag{36}$$

T^{pq} is the stress-energy momentum tensor, ρ_0 is the proper density of the fluid, and v^p is the proper velocity (dx^p/ds), v^p, being a real tensor, can be represented by a spinor $\phi^{\dot{\alpha}\beta}$ with $\bar{\phi}^{\dot{\alpha}\beta}=\phi^{\dot{\beta}\alpha}$,

$$\bar{v}^p=\tfrac{1}{2}g^p{}_{\dot{\alpha}\beta}\phi^{\dot{\alpha}\beta}, \quad \phi^{\dot{\alpha}p}\phi_{\dot{\alpha}\beta}=2. \tag{37}$$

Since $T^p{}_p=\rho_0$, and $R^{pq}-\frac{1}{2}g^{pq}R=-T^{pq}$, we see that [from (16) and (17)] $\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}=\frac{1}{4}\rho_0$ and $\phi^{\dot{\alpha}\dot{\mu}\nu\rho}=\frac{1}{8}\rho_0\epsilon^{\dot{\alpha}\dot{\mu}}\epsilon_{\dot{\lambda}}{}^{\rho}-\frac{1}{4}\rho_0\phi^{\dot{\alpha}\nu}\phi^{\dot{\mu}\rho}$. Leaving out constant factors, the invariant (23) thus equals ρ_0; the invariants (24) become ρ_0^2, ρ_0^3, ρ_0^4, respectively; the four real invariants (25) are still free; the six real invariants (26) become

$$\begin{gathered} \rho_0^2\phi_{\dot{\alpha}\gamma}\phi_{\dot{\beta}\delta}\psi^{\dot{\alpha}\dot{\beta}}{}_{\dot{\mu}\dot{\nu}}\phi^{\dot{\mu}\gamma}\phi^{\dot{\nu}\delta}\approx\rho_0^3,\\ \rho_0^2\phi^{\gamma}{}_{\dot{\alpha}}\phi^{\delta}{}_{\dot{\beta}}\psi^{\dot{\alpha}\dot{\beta}}{}_{\dot{\kappa}\dot{\lambda}}\psi^{\dot{\kappa}\dot{\lambda}\dot{\mu}\dot{\nu}}\phi_{\dot{\mu}\gamma}\phi_{\dot{\nu}\delta}\approx\rho_0^4,\\ \rho_0^4(\tfrac{1}{2}\epsilon^{\gamma\delta}\epsilon^{\dot{\alpha}\dot{\beta}}-\phi^{\dot{\alpha}\gamma}\phi^{\dot{\beta}\delta})(\phi_{\dot{\alpha}\eta}\phi_{\dot{\beta}\zeta})\\ \times\psi^{\eta\zeta\xi\epsilon}\phi_{\dot{\rho}\xi}\phi_{\dot{\sigma}\epsilon}(\tfrac{1}{2}\epsilon^{\gamma\delta}\epsilon^{\dot{\rho}\dot{\sigma}}-\phi^{\dot{\rho}}{}_{\gamma}\phi^{\dot{\sigma}}{}_{\delta})\approx\rho_0^5. \end{gathered} \tag{38}$$

The equalities in (38) are true up to constant factors because of the symmetries of $\psi^{\alpha\beta\mu\nu}$ and because $\phi^{\dot{\alpha}\beta}\phi_{\dot{\alpha}}{}^{\gamma}=\epsilon^{\beta\gamma}$. *Thus in the case of a fluid with no pressure or internal stresses there are at most five independent nonvanishing invariants.*

From the above analysis concerning null electromagnetic fields, it is obvious that one cannot tell whether space is empty by counting the number of invariants which are not equal to zero. To show this more strikingly, we make the following remark: *It is possible that all fourteen invariants may vanish and still R^{pq} will not vanish.*

An example of the vanishing of the fourteen invariants with $R_{pq}\neq 0$ is the case of the null field above with the additional requirement that the spinor $\psi^{\alpha\beta\mu\nu}$ vanish or at least that $\psi^{\alpha\beta\mu\nu}$ be so chosen that the invariants (21) vanish. There are other choices of $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ which will make the fourteen invariants vanish without requiring $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ to vanish, or without requiring $R_{pq}\neq 0$. Suppose for example $\psi^{\alpha\beta\mu\nu}$ is identically zero. Then only the three invariants (24) will not be zero. These three can be made equal to zero, leaving still many nonzero components in the spinor $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ or correspondingly in the tensor R_{pq}.

[5] See reference 4, p. 31.

IV. CLASSIFICATION OF EINSTEIN SPACES

Calling a space for which $R_{pq}=0$ an Einstein space, we see that for an Einstein space the curvature tensor R_{pqrs} is completely determined by the spinor $\psi^{\alpha\beta\mu\nu}$ with $\psi^{\alpha\beta}{}_{\alpha\beta}=0$. Such a space allows the existence of four nonzero differential invariants of the type we are considering. It becomes reasonable to attempt to classify the space by the behavior of these invariants. Let us define an eigenspinor $\psi^{\alpha\beta}$ of $\psi^{\alpha\beta\mu\nu}$ by the following equation:

$$\psi^{\alpha\beta\mu\nu}\psi_{\alpha\beta}=\lambda\psi^{\mu\nu}. \tag{39}$$

Here λ is obviously an invariant and we ask whether any eigenspinors exist. Obviously $\psi^{\mu\nu}$ is symmetric in μ and ν if it exists. Expanding equation (39) into components, recalling that $\psi_{11}=\psi^{22}$, $\psi_{22}=\psi^{11}$, $\psi_{12}=-\psi^{21}$, and that ψ must be symmetric, one gets a set of three algebraic linear homogeneous equations in $\psi^{\mu\nu}$ which permit a nontrivial solution if, and only if, the following determinant, D, vanishes:

$$D\equiv\begin{vmatrix}\psi^{2211}-\lambda & -2\psi^{1211} & \psi^{1111}\\ \psi^{2212} & -2\psi^{1212}-\lambda & \psi^{1112}\\ \psi^{2222} & -2\psi^{1222} & \psi^{1122}-\lambda\end{vmatrix}=0. \tag{40}$$

This is a third order algebraic equation in λ; however, by virtue of $\psi^{\alpha\beta}{}_{\alpha\beta}=\psi^{1111}-2\psi^{1212}+\psi^{1122}=0$, the sum of the three roots must equal zero. $\lambda_1+\lambda_2+\lambda_3=0$. Thus we can find three symmetric eigenspinors with the properties

$$\begin{aligned}\psi^{\alpha\beta\mu\nu}\psi_{\alpha\beta}&=\lambda_1\psi^{\mu\nu},\\ \psi^{\alpha\beta\mu\nu}\xi_{\alpha\beta}&=\lambda_2\xi^{\mu\nu},\\ \psi^{\alpha\beta\mu\nu}\eta_{\alpha\beta}&=\lambda_3\eta^{\alpha\beta}.\end{aligned} \tag{41}$$

λ_1, λ_2, and λ_3 may all be complex; they represent by virtue of the vanishing of their sum four independent invariant quantities which may be represented by the group (25). Each symmetric spinor can be made to correspond to a self-dual antisymmetric tensor of the second rank.

$$\begin{aligned}M_{pq}&=g_p{}^{\dot\mu}{}_\alpha g_{q\dot\mu\beta}\psi^{\alpha\beta},\\ N_{pq}&=g_p{}^{\dot\mu}{}_\alpha g_{q\dot\mu\beta}\xi^{\alpha\beta},\\ P_{pq}&=g_p{}^{\dot\mu}{}_\alpha g_{q\dot\mu\beta}\eta^{\alpha\beta}.\end{aligned} \tag{42}$$

These self-dual tensors (each of which is complex and hence contains two real tensors) are eigenbitensors of the curvature tensor. That is, they satisfy relations like

$$R^{pqrs}M_{rs}=\lambda M^{pq}. \tag{43}$$

The eigenspinors for nondegenerate λ's are orthogonal as can be proved easily; for degenerate λ's they can be chosen orthogonal; moreover they can be normalized so that

$$\begin{aligned}\psi^{\alpha\beta}\psi_{\alpha\beta}&=\xi^{\alpha\beta}\xi_{\alpha\beta}=\eta^{\alpha\beta}\eta_{\alpha\beta}=1,\\ \psi^{\alpha\beta}\xi_{\alpha\beta}&=\psi^{\alpha\beta}\eta_{\alpha\beta}=\xi^{\alpha\beta}\eta_{\alpha\beta}=0.\end{aligned} \tag{44}$$

The second equation defines orthogonality of the eigenspinors. There is the important possibility, which we shall not investigate further here, that an eigenbivector may be null, $\omega_{\alpha\beta}\omega^{\alpha\beta}=0$, and therefore not normalizable.

Degenerate cases can now be considered. One of the six real quantities represented by λ_1, λ_2, and λ_3 may be equal to zero, in which case one of them, say λ_1, is either purely real or purely imaginary. In this case $\psi^{\alpha\beta\mu\nu}\psi_{\alpha\beta}\psi_{\mu\nu}=\pm\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}\psi_{\dot\alpha\dot\beta}\psi_{\dot\mu\dot\nu}$, the positive sign being chosen for purely real λ_1 and the negative for purely imaginary λ_1. In either case there remain still three independent eigenspinors. Suppose two of the six quantities in the λ's are zero. Here we can still have three independent eigenspinors if, for example, λ_1 is real and λ_2 is purely imaginary. On the other hand, it might be, for example, that one of the λ's, say λ_1, equals zero. In this case there are only two independent eigenspinors, $\xi^{\alpha\beta}$ and $\eta^{\alpha\beta}$, which satisfy

$$\psi^{\alpha\beta\mu\nu}\xi_{\alpha\beta}=\lambda_2\xi^{\mu\nu},\quad \psi^{\alpha\beta\mu\nu}\eta_{\alpha\beta}=-\lambda_2\eta^{\mu\nu}. \tag{45}$$

There is, however, a $\psi_{\alpha\beta}$ orthogonal to $\xi^{\alpha\beta}$ and $\eta^{\alpha\beta}$ such that

$$\psi^{\alpha\beta\mu\nu}\psi_{\alpha\beta}=0. \tag{46}$$

Again we may have the case that λ_2 is either purely real or purely imaginary. This does not change the remarks just made. If all four independent invariants vanish so that $\lambda_1=\lambda_2=\lambda_3=0$, Eq. (39) degenerates completely to Eq. (46). There are thus three important cases to consider.

Case 1. $\lambda_1\neq0$, $\lambda_2\neq0$, $\lambda_3=-(\lambda_1+\lambda_2)\neq0$. In this case there are three eigenspinors or three self-dual tensors (six real tensors) which satisfy (43). λ_1 may be purely real or purely imaginary; so may λ_2. There are thus two, three, or four nonvanishing real invariants. If there are fewer than four, additional constraints are imposed on the eigenspinors.

Case 2. $\lambda_1=0$, $\lambda_2\neq0$, $\lambda_3=-\lambda_2$. (The role of the λ's can of course be interchanged). In this case there are two eigenspinors or two self-dual eigenbitensors. One can also find an eigenspinor, $\psi_{\alpha\beta}$, that satisfies (46). It is important to see that one can require also the $\psi^{\alpha\beta}$ that satisfies (46) to be null in the sense that $\psi^{\alpha\beta}\psi_{\alpha\beta}=0$; in this case $\psi^{\alpha\beta}$ can be represented by a single component spinor $\psi^{\alpha\beta}=\psi^\alpha\psi^\beta$ such that

$$\psi^{\alpha\beta\mu\nu}\psi_\alpha\psi_\beta=0. \tag{47}$$

The eigenbitensor constructed from ψ^α by $M_{pq}=g_p{}^{\dot\beta}{}_\alpha g_{q\dot\beta\mu}\psi^\alpha\psi^\mu$ is null in the sense that its complex invariant $M_{pq}M^{pq}=0$. (This requires two real invariants to vanish.) λ_2 may be purely real or purely imaginary which will impose additional constraints on the eigenbispinors.

Case 3. $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=0$. Here there are no spinor relations of the type (39). All eigenspinors satisfy (46). It is probably true for this case where all invariants are equal to zero that, if $\psi^{\alpha\beta\mu\nu}$ has nonvanishing components, one can find a particular coordinate system where a noncovariant relation that looks like (39) is satisfied. It is probably true also in case 2 that in a particular coordinate system under certain conditions one can satisfy a noncovariant relation that looks like (39).

The cases we have enumerated would seem to correspond to the "types" of Petrov and Pirani[6]: case 1 to type 1, case 2 to type 2, and case 3 to type 3.

V. GENERAL REMARKS ON THE CLASSIFICATION OF SPACES

In the previous section we have discussed the classification of spaces for which $R_{pq}=0$. It is important to see that this discussion actually concerns a much larger variety of spaces. Suppose we require only that $R=0$, or $\psi^{\alpha\beta}{}_{\alpha\beta}=0$. Call such a space a traceless space. Construct from $\psi^{\alpha\beta\mu\nu}$ the tensor G^{pqrs};

$$G_{pqrs}=\tfrac{1}{4}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}[\epsilon^{\kappa\lambda}\epsilon^{\rho\sigma}\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}+\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\mu}\dot{\nu}}\psi^{\kappa\lambda\rho\sigma}]. \tag{48}$$

When $R_{pq}=0$ this tensor equals the Riemann curvature tensor. However, it can be constructed even when $R_{pq}\neq 0$. In the case of a traceless space ($R=0$), the classification of the previous section can be applied to G_{pqrs} and there are three possible cases for G_{pqrs}. If the energy-momentum tensor is built up of the electromagnetic field and the field of the two-component neutrino and nothing else, the space is traceless; thus this type of space is not without physical significance. One may thus make remarks like the following: A space being investigated is traceless, contains electromagnetic fields, and is represented by case 1; or it contains the two-component neutrino field and is represented by case 2.

For a traceless field $R_{pq}=S_{pq}$, where S_{pq} has been defined in Eq. (18), we have

$$R_{pq}=g_{p\dot{\alpha}\mu}g_{q\dot{\beta}\nu}\phi^{\dot{\alpha}\dot{\beta}\mu\nu}. \tag{49}$$

We wish to classify these fields by looking for eigenspinors defined by

$$\phi^{\dot{\alpha}\dot{\beta}\mu\nu}\phi_{\dot{\alpha}\mu}=\lambda\phi^{\dot{\beta}\nu}. \tag{50}$$

Obviously $\bar{\phi}^{\dot{\beta}\nu}=\phi^{\dot{\nu}\beta}$, so the vector l_p defined by $l_p=\tfrac{1}{2}g_{p\dot{\alpha}\beta}\phi^{\dot{\alpha}\beta}$ is real and is an eigenvector of R_{pq},

$$R_p{}^q l_q=\lambda l_p. \tag{51}$$

It can be shown easily that (50) can be satisfied if, and only if, the determinant D' vanishes,

$$D'\equiv\begin{vmatrix} \phi^{\dot{2}\dot{1}21}-\lambda & -\phi^{\dot{1}\dot{1}21} & -\phi^{\dot{2}\dot{1}11} & \phi^{\dot{1}\dot{1}11} \\ \phi^{\dot{2}\dot{2}21} & -\phi^{\dot{1}\dot{2}21}-\lambda & -\phi^{\dot{2}\dot{2}11} & \phi^{\dot{1}\dot{2}11} \\ \phi^{\dot{2}\dot{1}22} & -\phi^{\dot{1}\dot{1}22} & -\phi^{\dot{2}\dot{1}12}-\lambda & \phi^{\dot{1}\dot{1}12} \\ \phi^{\dot{2}\dot{2}22} & -\phi^{\dot{1}\dot{2}22} & -\phi^{\dot{2}\dot{2}12} & \phi^{\dot{1}\dot{2}12}-\lambda \end{vmatrix}=0. \tag{52}$$

Because of the symmetries of $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$, the four λ's defined by (52) are real; moreover $\lambda_1+\lambda_2+\lambda_3+\lambda_4=0$.

There are thus in general four independent vectors that satisfy (51). The space can now be additionally classified by investigating the possibilities that one, two, three, or four of the invariants vanish. If they do, Eq. (51) takes the special form

$$R_p{}^q l_q=0. \tag{53}$$

Classification along these lines has been implicitly discussed many times.[7]

One can of course classify spaces for which R_{pq} is completely arbitrary; this amounts to "diagonalizing" by the process of equations (37), (40), (50), and (52) the "matrices" $\psi^{\alpha\beta\mu\nu}$, $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ which will involve a classification in terms of the behavior of the invariants (23), (24), and (25). One should also define something resembling a "simultaneous diagonalization" of both $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ which would introduce the "mixed" invariants (26). In this general case, instead of dealing with the spinor $\psi^{\alpha\beta\mu\nu}$ it might be convenient to deal with a new spinor $\xi^{\alpha\beta\mu\nu}$ defined by

$$\xi^{\alpha\beta\mu\nu}\equiv\psi^{\alpha\beta\mu\nu}-\tfrac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\mu\nu}\psi^{\epsilon}{}_{\rho}{}^{\rho}{}_{\epsilon}. \tag{54}$$

Construct now the tensor F_{pqrs}:

$$F_{pqrs}\equiv\tfrac{1}{4}g_{p\dot{\alpha}\kappa}g_{q\dot{\beta}\lambda}g_{r\dot{\mu}\rho}g_{s\dot{\nu}\sigma}[\epsilon^{\kappa\lambda}\epsilon^{\rho\sigma}\xi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}+\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\mu}\dot{\nu}}\xi^{\kappa\lambda\rho\sigma}]. \tag{55}$$

The tensor F_{pqrs} has the same symmetries as R_{pqrs}. It is equal to R_{pqrs} for the case of an Einstein field and in general can be classified as was R_{pqrs} for the Einstein field.

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