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Revised Weizsäcker Semiempirical Formula for Diffuse Nuclear Surfaces*

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An attempt is made to modify the Weizsäcker semiempirical mass formula to include effects of the diffuse nuclear surface indicated by recent electron scattering experiments. Volume and surface effects are combined by integrating over an assumed trapezoidal density function similar to that found experimentally. Good fits to the experimental nuclear masses are achieved with r_0 (a radius parameter) and ϵ (half the surface depth) equal to 1.081×10^{-13} cm and 1.202×10^{-13} cm, respectively. These are in reasonably good agreement with the experimental values $(1.07\pm0.02)\times10^{-13}$ cm and $(1.50\pm0.20)\times10^{-13}$ cm found by Hahn, Ravenhall, and Hofstadter.

IN its original form,[1] the Weizsäcker mass formula attempts to reproduce the general trend of nuclear masses by assuming a uniform density throughout the nucleus and using a spherical drop model to calculate the various contributions to the binding energy. Recent experiments in electron scattering indicate that the nuclear surface is somewhat diffuse.[2] This study attempts to demonstrate the possibility of fitting a Weizsäcker formula to nuclear masses when this diffuseness is considered.

In recent literature on this subject, a reference surface was used to calculate the surface energy terms in the mass formula.[3] In this paper we have attempted to replace the volume, surface, and Coulomb energy terms by integrals over the nuclear volume assuming a trapezoidal distribution of charge and mass density. This method differs from previous Weizsäcker fits mainly in that it allows a *single* expression for both the volume and surface contributions to the binding energy.

The parameters used in describing the trapezoidal model are defined below and shown in Fig. 1. The parameter c is the radius to the point where the nuclear density drops to half of its original value. The parameter ϵ is one-half the distance between the point where the density starts to drop off and the point where it is zero.

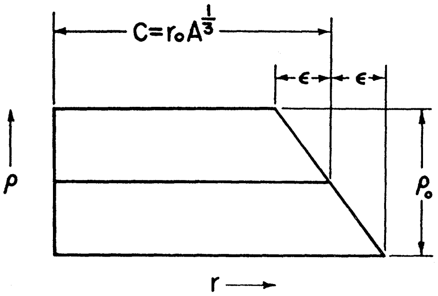

FIG. 1. Trapezoidal density showing meaning of parameters ϵ, c, ρ_0, and r_0.

Since internucleon forces drop off rapidly with distance, one may say with plausibility, though without certainty, that the energy due to these forces should be proportional to ρ^2.[4] Of course this cannot be completely correct since otherwise there would be no saturation effect. Nevertheless, this type of dependence has been used before and permits simple calculations. Hence we postulate that the total volume energy *including surface energy* is proportional to an integral

$$\int_V \rho^2 dV=\int_0^{c-\epsilon}\rho_0{}^2 4\pi r^2 dr+\int_{c-\epsilon}^{c+\epsilon}\rho_0{}^2\left(\frac{c+\epsilon-r}{2\epsilon}\right)^2 4\pi r^2 dr$$
$$=\tfrac{4}{3}\pi\rho_0{}^2[c^3-c^2\epsilon+c\epsilon^2-\epsilon^3/5],$$

where

$$\begin{aligned}&\rho=\rho_0 &&\text{for}\quad 0\leq r\leq c-\epsilon,\\ &\rho=\rho_0(c+\epsilon-r)/2\epsilon &&\text{for}\quad c-\epsilon\leq r\leq c+\epsilon,\\ &\rho=0 &&\text{for}\quad r>c+\epsilon.\end{aligned}$$

The calculation of the Coulomb energy is more straightforward but slightly more difficult. For a spherically symmetric charge distribution with density $\rho(r')$, it is easily verified that the potential at any field point r' is

$$\varphi(r')=\int_{r'}^{\infty}\frac{1}{r'^2}dr'\int_0^{r'}4\pi r^2\rho(r)dr,$$

whereas the electrostatic energy is

$$\tfrac{1}{2}\int_0^{r'}\varphi(r')\rho(r')dV=\tfrac{1}{2}\int_0^{c+\epsilon}\varphi(r')\rho(r')4\pi r'^2dr'.$$

By an integration over the nuclear volume, this yields the result

$$B_c=\frac{8\pi^2\rho_0{}^2}{3}\left[\frac{2}{5}c^5+\frac{1}{3}c^3\epsilon^2+\frac{1}{5}c^2\epsilon^3+\frac{1}{15}c\epsilon^4-\frac{\epsilon^5}{105}\right].$$

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[1] C. F. von Weizsäcker, Z. Physik **96**, 431 (1935).

[2] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956).

[3] A. G. S. Cameron, Chalk River Project Report CRP-690 (unpublished).

[4] W. A. Fowler (unpublished notes on Nuclear Physics, California Institute of Technology).

An obvious constraint on the values of ρ_0 for both the electrostatic and mass densities is

$$\int_0^{c+\epsilon} 4\pi r^2 \rho_e(r)dr = Ze, \quad \int_0^{c+\epsilon} 4\pi r^2 \rho_M(r)dr = A.$$

These yield

$$\rho_{0e} = \frac{3Ze}{4\pi(c^3+c\epsilon^2)}, \quad \rho_{0M} = \frac{3A}{4\pi(c^3+c\epsilon^2)},$$

or

$$\rho_{0e}{}^2 = \frac{9Z^2e^2}{16\pi^2c^6(1+\epsilon^2/c^2)^2}, \quad \rho_{0M}{}^2 = \frac{9A^2}{16\pi^2c^6(1+\epsilon^2/c^2)^2}.$$

An additional exchange correction term of the form $Z(Z-1)(1-0.767/Z^{\frac{2}{3}})$ was inserted in the foregoing in place of Z^2 as suggested by Cooper and Henley.[5]

Thus the complete functional form of the revised Weizsäcker formula which we shall use is

$$f = \frac{M-A}{A} = \frac{ZM_H+(A-Z)M_n-A}{A}$$
$$= -\frac{a_v}{r_0{}^3}\left[1-\frac{\epsilon}{c}+\left(\frac{\epsilon}{c}\right)^2-\frac{1}{5}\left(\frac{\epsilon}{c}\right)^3\right]$$
$$+\frac{a_c}{r_0}\frac{3Z(Z-1)(1-0.767Z^{-\frac{2}{3}})e^2}{5(1+\epsilon^2/c^2)^2A^{\frac{4}{3}}}\left[1+\frac{5}{6}\left(\frac{\epsilon}{c}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\epsilon}{c}\right)^3+\frac{1}{6}\left(\frac{\epsilon}{c}\right)^4-\frac{1}{42}\left(\frac{\epsilon}{c}\right)^5\right]+a_a\left(\frac{A-2Z}{A}\right)^2\pm\frac{\delta}{A},$$

where a_c, a_a, and a_v are constants which represent the relative strength of the Coulomb, asymmetry, and volume-surface contributions to the packing fraction of a given nucleus. Henceforth this expression will be referred to as the "f" function.

In this "f" function, we have assumed that the basic nuclear half-maximum radius c goes as the cube root of the mass number ($c=r_0A^{\frac{1}{3}}$), and that the width of the diffuse surface 2ϵ is considered to be a constant with respect to changes in nuclear mass number as indicated by Hahn, Ravenhall, and Hofstadter.[2] Furthermore, we have assumed that the factor of proportionality r_0 is the same for both the mass and electromagnetic size of the nucleus.

TABLE I. $\Delta f = f_{\text{calc}} - f_{\text{exp}}$ (in units of 10^{-4} amu).

ϵ/r_0	1.065	1.094	1.103	1.112	1.121	1.130	1.159
Co^{59}	0.152	0.153	0.147	0.144	0.136	0.132	0.141
Sr^{87}	0.092	0.005	0.035	0.063	0.095	0.124	0.223
Sn^{115}	0.104	0.080	0.076	0.069	0.065	0.053	0.029
Sm^{149}	0.026	0.052	0.065	0.076	0.090	0.102	0.139
Os^{189}	0.185	0.150	0.137	0.125	0.111	0.106	0.060
$\Sigma(\Delta f)^2$	0.0776	0.0550	0.0516	0.0509	0.0522	0.0572	0.0921

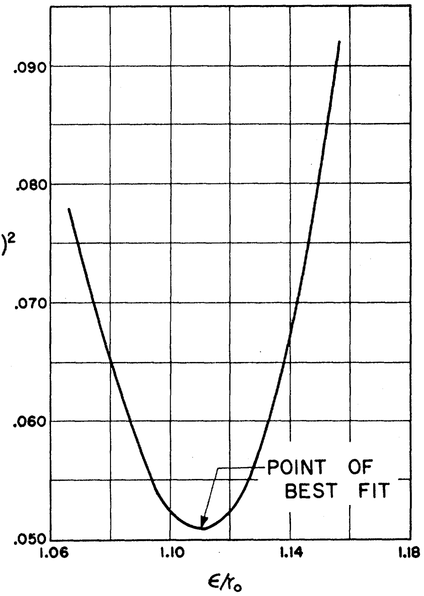

FIG. 2. Least-squares fit of the parameter ϵ/r_0 [$\Sigma(\Delta f)^2$ in units of 10^{-8}(amu)^2].

If we consider only odd-A nuclei, the pairing energy coefficient δ may be eliminated, and thus the "f" function contains only three unknowns—a_c, a_v, and a_a. The packing fraction function was thus fitted to the masses of three separate nuclei which were picked to be representative of the range in which the formula was expected to be valid. These were Cl^{37}, Mo^{95}, and U^{235}.

The main problem in accomplishing a fit is, of course, to solve three equations in three unknowns (a_c, a_a, and a_v) such that for particular values of the parameters ϵ and r_0 the "f" function passes through the points listed (Cl^{37}, Mo^{95}, U^{235}). For ease of calculation, it was noticed that except for the terms a_c/r_0 and $a_v/r_0{}^3$, "f" is a function of the ratio ϵ/c or essentially of the ratio ϵ/r_0. Thus we lumped a_c/r_0 and $a_v/(r_0)^3$ into new constants which would be determined when we fit the function to the three listed nuclei for a particular value of the ratio ϵ/r_0.

To determine the best fit, we defined a function Δf to be the difference between the predicted value of f and the actual value observed for a particular nucleus. The value of Δf was calculated for different nuclei as a function of the ratio ϵ/r_0 for different fits. The best fit was considered to be the particular ratio which gave a minimum to the sum of the squares of Δf. The values of $\Sigma(\Delta f)^2$ are given for the five representative nuclei used in this calculation in Table I and are also plotted in Fig. 2. As indicated on the graph, a best fit seems to occur for the ratio $\epsilon/r_0 = 1.111$.

To find the actual values of ϵ and r_0 for a particular fit, we noticed that if a_c is given some prescribed value, then from the value of a_c/r_0 determined from that fit we may determine the value of r_0. Since the expression for Coulomb energy in the "f" function is an essentially classical calculation for the assumed charge distribution,

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[5] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).

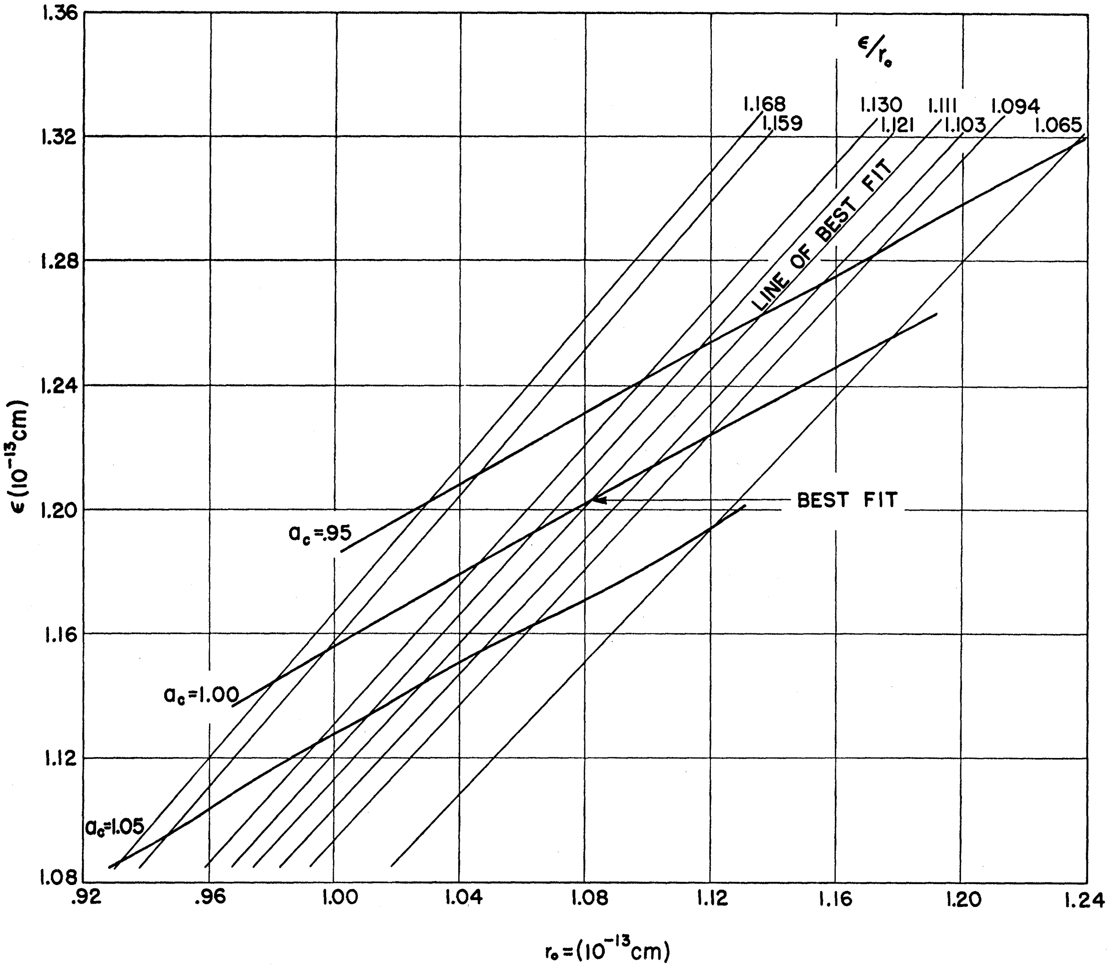

FIG. 3. The family of fits for various values of ϵ/r_0, showing the effect of various values of a_c and including the line of best fit and the point of best fit.

a reasonable value of a_c should be nearly unity. This is in agreement with many previous Weizsäcker fits which have essentially classical Coulomb energy terms.[6]

Figure 3 shows the actual values of ϵ and r_0 which occur for differing ratios of ϵ/r_0 and differing values of a_c near unity. The line of best fit is indicated on this graph. This is a line of $\epsilon/r_0=1.111$ and, as shown on the graph, the actual values of the parameters are somewhat sensitive to changes in the assumed value of a_c. Assuming that a_c equals unity, the best fit occurs for

$$r_0=1.081\times10^{-13}\text{ cm},\quad \epsilon=1.202\times10^{-13}\text{ cm}.$$

In comparison, the values obtained by scattering experiments (reference 1) are

$$r_0=1.07\pm0.02\times10^{-13}\text{ cm},\quad \epsilon=1.5\pm0.20\times10^{-3}\text{ cm}.$$

In Figs. 4 and 5, the "f" function is plotted for the value of ϵ/r_0 which gives the best fit and two extreme values on either side of the best fit. Also included are experimental points as given by Wapstra and Huizenga.[7] As is evident, our best fit follows the trend of experimental points better than either of the two extreme cases, although all three lie very close together.

We do not try to be too definite in giving probable error to the values of ϵ and r_0 since we required that all our solutions pass through the three listed points. Even though we have found the best fit subject to this constraint, it is possible that better fits might be obtained with solutions which do not pass through these particular points. Furthermore, as shown in Fig. 3, our best fit is sensitive to changes in a_c. A complete analysis of the data using our model might indicate a better fit at an a_c not exactly unity. This might give somewhat different results for the values of the nuclear parameters. Finally, it is obvious that the volume energy of the nucleus cannot be exactly dependent on ρ^2 because of saturation. Taking this effect into account

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[6] R. D. Evans, *The Atomic Nucleus* (McGraw-Hill Book Company, Inc., New York, 1955).

[7] A. H. Wapstra, Physica **21**, 367 and 385 (1955); J. R. Huizenga, Physica **21**, 410 (1955).

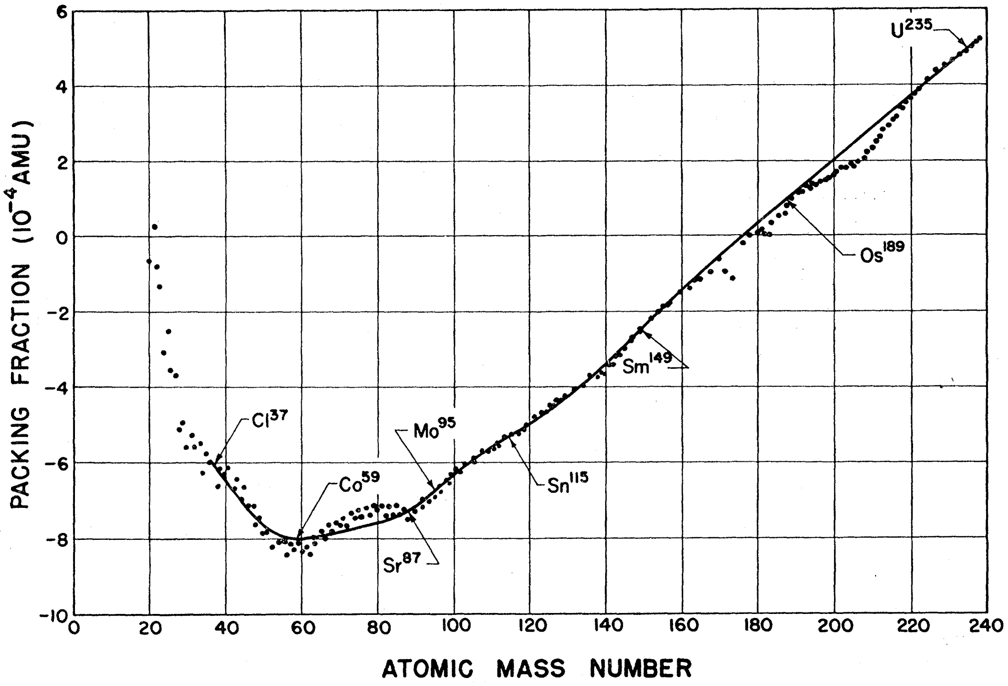

FIG. 4. Best fit (solid line) for packing fractions is shown with experimental points indicated.

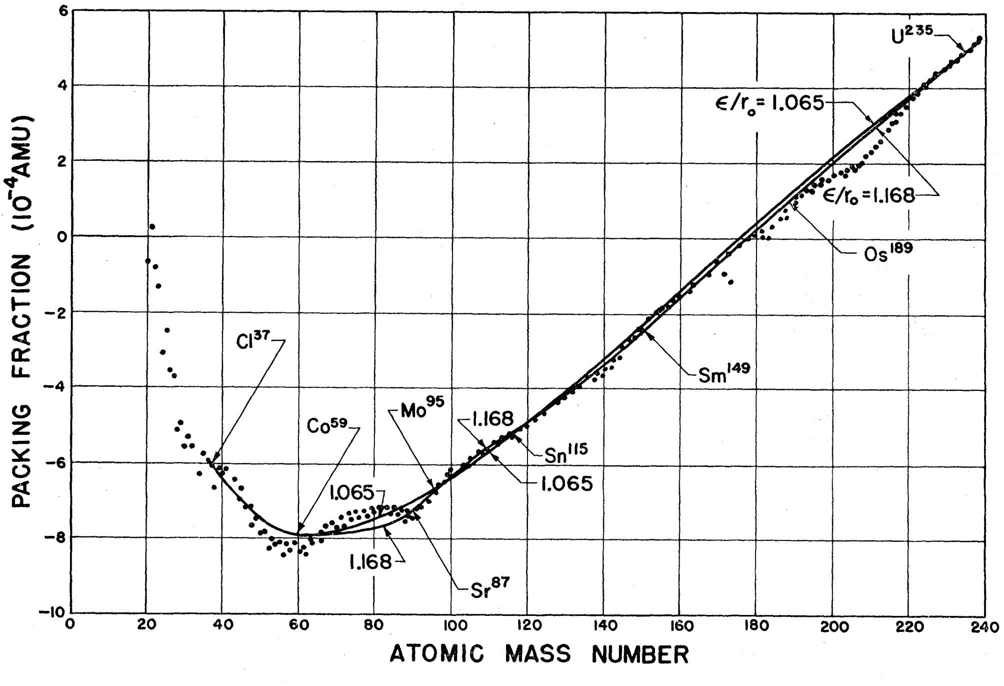

FIG. 5. Two less optimum fits (solid lines) are shown together with the experimental points.

would be very difficult, but as above, it might lead to a different set of parameter values for the best fit.

We have attempted in this paper to show phenomenlogically that good fits can be obtained with a trapezoidal nuclear model using a single expression for volume and surface-energy contributions. Furthermore we have attempted to show that a reasonable fit can occur at the values of ϵ and r_0 which agree with electron scattering data on the diffuse nuclear surface (reference 1). We have, however, not attempted a complete and detailed analysis of the trapezoidal model.

ACKNOWLEDGMENT

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