Nuclear Structure Correction to the Hyperfine Structure in Hydrogen

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In previous papers, corrections to the hyperfine structure (hfs) in hydrogen of relative order $\alpha m/M$ have been calculated by treating the proton as a point particle with an anomalous magnetic moment in addition to its Dirac moment. In this paper the proton is treated as a particle with structure by making use of the high-energy electron-proton-scattering data. Corrections of the previous work in which only a point particle was considered are calculated by using the Feynman formulation of quantum electrodynamics. It is also shown that exactly the same terms may be obtained by using the covariant Bethe-Salpeter equation.

The calculated shift of -35 parts per million (including the "recoil corrections") is not in agreement with the combined results of several experiments (-1.4 ± 18 parts per million). A possible source of this difference is meson corrections to a two-photon form-factor which is taken here as the product of two (Hofstadter) single-photon form-factors.

INTRODUCTION

ALTHOUGH the hyperfine structure splitting of atomic hydrogen has been measured precisely[1] to one part in 10^7, the theoretical value is not as accurately known. However, within the past ten years a relatively extensive series of calculations have been performed which correct the Fermi formula for the hyperfine structure, hereafter called hfs:

$$\text{hfs}=\frac{2\pi\alpha\mu_p}{3mM}(\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b)|\phi(0)|^2, \tag{1}$$

where α is the fine structure constant and is equal to e^2 in units in which $\hbar$ and c are each equal to 1; μ_p is the proton moment in nuclear magnetons, and m and M are the electron and proton masses, respectively; $|\phi(0)|^2$ is the magnitude of the nonrelativistic Schrödinger wave function at the origin. The hfs is corrected by higher order terms in α and in m/M. Most recently, Arnowitt,[2] used the Schwinger-Tomonaga formulation of quantum electrodynamics, and Newcomb and Salpeter,[3] hereafter referred to as N.S., using the covariant Bethe-Salpeter equation,[4] have calculated corrections up to and including terms of order $\alpha m/M$(hfs). The corrections involving powers of m/M may be considered to be recoil corrections. These calculations, which would be exact to the required order of magnitude, are complicated because of the anomalous moment and the finite size of the proton. The only presently available method for including the interaction of the proton with the electromagnetic field is to treat the proton as a point Dirac particle and to introduce a Pauli interaction for the anomalous part of the magnetic moment. All the corrections for mass are produced by processes which involve the interchange of two intermediate virtual photons between the electron and the proton. The terms involving two Pauli interactions are logarithmically divergent; this accounts for the appearance of the rather arbitrary sharp cutoff, Λ_0 in the results of N.S. and of Arnowitt. The cutoff, of course, corresponds crudely to a spreading of the proton of order Λ_0^{-1}. We propose to treat the cutoff empirically by using an experimental form factor for the charge and for the magnetic moment of the proton, obtained from the data on high-energy electron-proton scattering.[5]

Unfortunately, a study of the corrections due to the structure of the nucleon indicates that what is essential is the amplitude for the rapid exchange between the electron and nucleon of two photons (of high but nearly opposite momenta). The Hofstadter results give only the coupling for a single photon.

In the present paper we have made the crude approximation that the two-photon amplitude is given by the product of two single photons acting alone. The amplitude which is really needed is the forward-scattering amplitude of virtual photons on nucleons at high momenta; at present we are investigating other contributions to this amplitude by dispersion-theory techniques. Although the results of this paper are only approximate, they do indicate explicitly how one might find the corrections to the hfs if the exact scattering amplitude were known.

We assume, in accord with the Hofstadter results,[5] that the absorption of a virtual photon of four-momentum k_μ and polarization i is characterized by the proton interaction operator

$$\frac{e\Lambda^4}{(\Lambda^2-k^2)^2}\left[\gamma_i+\frac{\mu}{4M}(k\gamma_i-\gamma_i k)\right]=eF_4(k^2)\Delta_i(k), \tag{2}$$

where $\Lambda=0.91M$, $\mu=\mu_p-1=1.79$, $k\equiv k_\mu\gamma_\mu$, $\Delta_i(k)$ is the term in brackets on the left-hand side of (2), $k_\mu^2=k_4^2-k_1^2-k_2^2-k_3^2$, and the poles of F_4 are handled in the usual way.

We shall compute only the difference between the

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[1] A. G. Prodell and P. Kusch, Phys. Rev. **79**, 1009 (1950).
[2] R. Arnowitt, Phys. Rev. **92**, 1002 (1953).
[3] W. A. Newcomb and E. E. Salpeter, Phys. Rev. **97**, 1146 (1955).
[4] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[5] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

corrections to the hfs that results from using the form factor $F_4(k^2)$ times $\Delta_i(k)$ [as in (2)] as opposed to using Δ_i alone with a sharp cutoff on the logarithmically divergent terms. That is, we compute all terms $\Delta E'$ of order $\alpha m/M$(hfs) by using "$(F_4-1)\Delta_i$" as a perturbation. In the second section we obtain this shift in terms of a set of four-dimensional integrals. In the third section, these integrals are tabulated, and in the fourth section these same results are calculated using the Bethe-Salpeter equation. The fifth section contains a numerical evaluation of the results for various values of Λ/M and some remarks on the validity of this method.

EXPRESSION FOR $\Delta E'$

We consider the corrections arising from the processes indicated by the three graphs of Fig. 1. The change of energy caused by the process shown in Fig. 1(a) is given by associating the change of amplitude of the wave function with a series expansion of an exponential,

$$e^{-i\Delta E t}\psi=(1-i\Delta E t+\cdots)\psi \tag{3}$$

in the same manner as given in Secs. 3 and 4 of Feynman's paper.[6] For the diagram in Fig. 1(a) we find[7] that

$$i\Delta E_1=4e^4\int\frac{d^4k}{(2\pi)^2k'^2k^2}[F_4(k^2)F_4(k'^2)-1]$$
$$\times[\bar{u}_b M_{\mu\nu}(kk')u_b][\bar{u}_a N_{\mu\nu}(kk')u_a], \tag{4}$$

where

$$M_{\mu\nu}=\Delta_\mu(k')(\not{p}_b+\not{k}-M)^{-1}\Delta_\nu(k), \tag{5}$$

$$N_{\mu\nu}=\gamma_\mu(\not{p}_a-\not{k}-m)^{-1}\gamma_\nu. \tag{6}$$

The terms $[F_4(k^2)F_4(k'^2)-1]$ arise because we want to find the difference between the result obtained with the form factor and that obtained by N.S. ($F_4=1$) for a point proton. Using $(F_4-1)\approx k^2/\Lambda^2$ for $k^2\ll\Lambda^2$, the single-photon exchange, which would ordinarily give the hfs, will now give a contribution of the order of $\alpha^2(m^2/M^2)$(hfs) (if $|k|$ is taken to be of the order of the Bohr momentum). Since this is of higher order than the corrections due to Fig. 1(a), the single-photon terms may be dropped. That these terms should not contribute is quite reasonable since only high-momentum photons can "see" the difference between the form factor and one. However, such high-energy single scatterings leave the final state approximately orthogonal to the hydrogen wave function. In the case of Figs. 1(a) and 1(b), if the high-momentum photons are to contribute, they must leave the final state in approximately the same condition as the initial state (both particles nearly at rest); that is, they must have nearly equal and opposite momenta. It can also be shown that, to the required order, the motion of the proton and electron may be neglected. Thus, $\boldsymbol{k}'$ is taken as $-\boldsymbol{k}$, p_a as $(m,0,0,0)$, and m/M and $m/|k|$ are neglected compared with one.

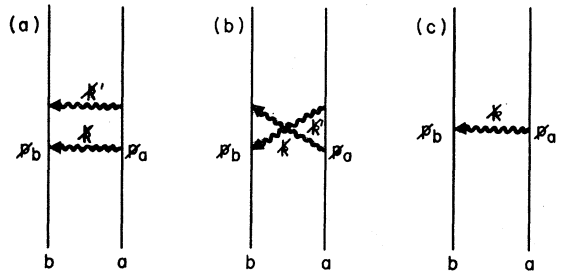

FIG. 1. Feynman diagrams for corrections to hfs splitting. Fermions are denoted by solid lines, bosons by wavy ones. (The slashed symbols p and k represent $\not{p}\equiv p_\mu\gamma_\mu$ and $\not{k}\equiv k_\mu\gamma_\mu$.)

[6] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[7] We have used the notation of reference 6 throughout, except that $d^4k=dk_1dk_2dk_3dk_4$.

The hyperfine splitting is the difference in the expectation values of ΔE_1 in the triplet and singlet states. Inserting the spin projection operators for $3S$ and $1S$,

$$\begin{aligned}3S&=\tfrac{1}{4}(3+\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b),\\ 1S&=\tfrac{1}{4}(1-\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b),\end{aligned} \tag{7}$$

and summing over all polarizations of the electron and proton (three triplet states and one singlet state) we obtain

$$\langle M_{\mu\nu}N_{\mu\nu}\rangle_{3S}-\langle M_{\mu\nu}N_{\mu\nu}\rangle_{1S}=\sum_{\text{spins}}\langle\tfrac{1}{3}M_{\mu\nu}N_{\mu\nu}\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b\rangle. \tag{8}$$

Since the final matrix element is to be a scalar under rotations, we may equate the terms involving different Cartesian components, taking

$$\langle M_{\mu\nu}N_{\mu\nu}\sigma_x{}^a\sigma_x{}^b\rangle=\langle M_{\mu\nu}N_{\mu\nu}\sigma_y{}^a\sigma_y{}^b\rangle=\langle M_{\mu\nu}N_{\mu\nu}\sigma_z{}^a\sigma_z{}^b\rangle.$$

If we insert positive-energy projection operators and sum over a complete set of states, the matrix element becomes a trace and we obtain

$$\begin{aligned}&\langle M_{\mu\nu}N_{\mu\nu}\rangle_{3S}-\langle M_{\mu\nu}N_{\mu\nu}\rangle_{1S}\\ &=\mathrm{Tr}\left[M_{\mu\nu}\frac{(1+\gamma_t{}^b)}{2}(i\gamma_x{}^b\gamma_y{}^b)\right]\\ &\qquad\times\mathrm{Tr}\left[N_{\mu\nu}\frac{(1+\gamma_t{}^a)}{2}(i\gamma_x{}^a\gamma_y{}^a)\right]\\ &=\frac{-8\eta}{(\tilde{k}^2+2\tilde{\omega})\tilde{k}^2M^2},\end{aligned} \tag{9}$$

where

$$\begin{aligned}\eta=\tilde{\omega}^2&-\tfrac{2}{3}\tilde{K}^2-\tfrac{1}{2}\mu(2\tilde{k}^2\tilde{\omega}+\tfrac{4}{3}\tilde{K}^2)\\ &-\tfrac{1}{4}\mu^2(2\tilde{\omega}\tilde{k}^2+\tfrac{4}{3}\tilde{K}^2\tilde{\omega}-2\tilde{\omega}^3-\tfrac{1}{3}\tilde{\omega}^2\tilde{k}^2-\tfrac{2}{3}\tilde{k}^4),\end{aligned} \tag{10}$$

in which $\tilde{k}$ is a dimensionless momentum, k/M, $\tilde{K}$ is the three-dimensional part of $\tilde{k}$, and we have replaced the squares of Cartesian components by their averages in solid angle, e.g., $\tilde{k}_x{}^2=\frac{1}{3}\tilde{K}^2$.

In the approximation $\tilde{k}_\mu{}'=-\tilde{k}_\mu$, $m/M\ll1$, $\tilde{k}^2\approx M^2$, Fig. 1(b) shows a contribution equal to that in Fig.

1(a), as is obvious upon writing out the matrix element. The energy shift in hydrogen is then equal to that due to Figs. 1(a) and 1(b), multiplied by $|\phi(0)|^2$, the density of the electron at the proton in the atom. The final expression for the energy now becomes

$$+i\langle\Delta E'\rangle_{3S\text{-}1S}=-\frac{16\phi^2e^4}{M^2\pi^2}\int\frac{d^4\tilde{k}\eta}{\tilde{k}^6(2\tilde{\omega}+\tilde{k}^2)}[F_4^2(\tilde{k}^2)-1]. \quad (11)$$

EVALUATION OF $\Delta E'$

If we write

$$[F_4^2(\tilde{k}^2)-1]=\mathrm{Op}[\tilde{k}^2/(\tilde{k}^2-a^2)], \quad (12)$$

where

$$\mathrm{Op}=[-1+a^2\partial-\tfrac{1}{2}a^4\partial^2+\tfrac{1}{6}a^6\partial^3], \quad (13)$$

in which

$$a^2=\Lambda^2/M^2, \quad \partial\equiv\partial/\partial a^2,$$

and if we write η as

$$\eta=\{[\tfrac{1}{3}(1+\mu)-\tfrac{1}{2}(1+\mu)^2]\tilde{\omega}\tilde{k}^2+\tfrac{2}{3}(1+\mu)\tilde{k}^2 -\tfrac{1}{4}\mu^2(2\tilde{\omega}+\tilde{k}^2)(\tfrac{2}{3}\tilde{K}^2-\tilde{\omega}^2) +(\tfrac{1}{3}-\tfrac{2}{3}\mu)(\tfrac{1}{2}\tilde{\omega})(2\tilde{\omega}+\tilde{k}^2)\}, \quad (14)$$

then the integral for $\Delta E'$ may be split up into four integrals. The last term yields zero since it is odd in ω, and the first two terms may be integrated by the method given in reference 6:

$$\int\frac{d^4\tilde{k}(\tilde{\omega},1)}{(2\pi)^2(\tilde{k}^2-a^2)(\tilde{k}^2)(\tilde{k}^2+2\tilde{\omega})} =\frac{1}{8i}\int_0^1\frac{dx[-(1-x^2),+2(1+x)]}{(1-x)^2+a^2x}. \quad (15)$$

The third term is integrated first by taking residues in the plane and then integrating over solid angles. The remaining integral,

$$\int\frac{(\frac{2}{3}\tilde{K}^2-\tilde{\omega}^2)d^4\tilde{k}}{\tilde{k}^4(\tilde{k}^2-a^2)(2\pi)^2} =\frac{1}{3ia^2}\int_0^{\Lambda_0/M}d\tilde{K}\left[\frac{\tilde{K}^3}{a^2}+\frac{5\tilde{K}}{2} -\frac{\tilde{K}^4}{a^2(K^2+a^2)^{\frac{1}{2}}}-\frac{3K^2}{(K^2+a^2)^{\frac{1}{2}}}\right], \quad (16)$$

is logarithmically infinite because the interaction (2) is finite while that of N.S. ($F_4=1$) is infinite. Thus the difference is infinite, but this infinity will cancel against a similar term in the work of N.S. and give a finite answer for the over-all energy shift. Since the N.S. term was integrated up to Λ_0 in (three-dimensional) momentum space in the center-of-mass Lorentz frame, we must also impose such a cutoff on our three-momentum vector in order to make the cancellation unambiguous.

$$\Delta E'(3S)-\Delta E'(1S) =-\frac{e^4\phi^2}{M^2}\mathrm{Op}\left\{-8\int\frac{2D(1+x)-C(1-x^2)}{(1-x)^2+a^2x}dx +\frac{8}{3}\mu^2\left[-\frac{17}{16}+\frac{9}{4}\ln\left(\frac{\Lambda_0}{2}\right)\right]\right\}, \quad (17)$$

where

$$C=\tfrac{1}{3}(1+\mu)-\tfrac{1}{2}(1+\mu)^2,$$

$$D=\tfrac{2}{3}(1+\mu).$$

The integrations with respect to x and the differentiations with respect to a^2 are straightforward:

$$\Delta E'(3S)-\Delta E'(1S) =+\frac{8\alpha^2\phi^2}{M^2}\left\{-C+4D\,\mathrm{Op}I+2(D+C) \times\left[-\ln a+\frac{11}{12}-\frac{a^8}{12}I'''\right] -C\left[\frac{a^2}{6}-\frac{a^8I''}{4}-\frac{a^8(a^2-2)}{12}I'''\right] -\frac{\mu^2}{3}\left[-1-\frac{9}{4}\ln\frac{2\Lambda_0}{M}+\frac{9}{4}\ln a\right]\right\}, \quad (18)$$

$$I(a^2)=\frac{2}{(4a^2-a^4)^{\frac{1}{2}}}\tan^{-1}\left[\left(\frac{4-a^2}{a^2}\right)^{\frac{1}{2}}\right]. \quad (19)$$

Primes indicate differentiations with respect to a^2.

COMPARISON WITH THE BETHE-SALPETER EQUATION

As an alternative to the preceding perturbation theory, we have considered the correction to the hfs due to the finite size of the proton by means of the covariant Bethe-Salpeter equation.

We follow the procedure of N.S. exactly and attempt to make all notation used in this section agree with theirs.

The starting point is the Bethe-Salpeter equation, which is [after we separate out the center-of-mass motion and adopt a coordinate system where the momentum of the center-of-mass K_μ is (0,0,0,E)]

$$F(p_\mu)\psi(p_\mu)=-\frac{1}{2\pi i}\int d^4k\,G'(k_\mu)\psi(p_\mu+k_\mu), \quad (20)$$

where

$$F(p_\mu)=[\mu_aE-H_a(p)+p_4][\mu_bE-H_b(p)-p_4], \quad (21)$$

in which

$$H_a(p)=(m\beta^a+\mathbf{p}\cdot\boldsymbol{\alpha}^a),$$
$$H_b(p)=(M\beta^b-\mathbf{p}\cdot\boldsymbol{\alpha}^b). \quad (22)$$

Superscripts on the Dirac matrices indicate that they operate only on the components of ψ corresponding to a

or b, respectively. ψ is a 16-component spinor whose variable is the relative momentum of the two particles.

$G'(k_\mu)$ is a slightly modified, complete interaction function representing a sum over all possible irreducible diagrams.[5] A G_Δ', whose effect is small, produces a perturbation in the energy given by

$$\Delta E' = \int d^4p d^4k\, \tilde{\psi}(p_\mu) G_\Delta'(k_\mu) \psi(p_\mu + k_\mu). \tag{23}$$

N.S. consider the expression

$$G_1'(k_\nu) = \gamma_4{}^a \gamma_4{}^b \frac{e^2}{2\pi^2} \frac{\gamma_i{}^a \Delta_i{}^b(\boldsymbol{k})}{k^2}$$

$$= \frac{e^2}{2\pi^2}\left[-\frac{1}{K^2} - \frac{A_4{}^b}{K^2} - \sum_{1,2} \frac{\alpha_i{}^a \alpha_i{}^b}{k^2} + \frac{\alpha_i{}^a A_i{}^b}{k^2}\right], \tag{24}$$

where

$$A_i{}^b = \frac{\mu}{4M} \gamma_4{}^b (\boldsymbol{k}\gamma_i - \gamma_i \boldsymbol{k})$$

as the fundamental interaction between the electron and proton. $\psi(p_\mu)$ is found by iteration of the Schrödinger, nonrelativistic wave functions, as explained by Salpeter.[8]

We modify the interaction by placing an F_4 in front of (24) and then proceed to calculate the effects of the small perturbation, $(F_4-1)G_1' = G_\Delta'(k_\mu)$, on the hfs. The first two terms of $(F_4-1)G_1'(k_\mu)$ will be called C' and Q' interactions and the last two T' and P'. In N.S. D is used in place of T. Thus P' is

$$G_{P'}(k_\mu) = [F_4 - 1] \frac{e^2}{2\pi^2} \sum_{1,2} \frac{\alpha_i{}^a A_i{}^b(k_\mu)}{k^2}. \tag{25}$$

As in N.S., all corrections of the mass that arise from single-photon exchanges contribute at high values of the momentum of one of the wave functions in (23). Thus all single-photon terms such as T' must be iterated, and give rise to the so-called uncrossed diagrams, which, of course, correspond exactly to the uncrossed diagrams discussed in the second section.

It is convenient to express ψ in terms of positive and negative energy components:

$$\psi_{\pm\pm}(k_\mu) = \Lambda_\pm{}^a(K) \Lambda_\pm{}^b(K) \psi(k_\mu), \tag{26}$$

where

$$\Lambda_\pm{}^a(p) = \left[\frac{E_a(p) \pm H^a(p)}{2E_a(p)}\right].$$

Then it may be shown that to the required order of accuracy (first order in α) for all values of the momentum p_μ up to $p_\mu \approx \alpha m \exp(137)$,

$$\psi(p_\mu) \approx -[2\pi i F(p_\mu)]^{-1} \times \int d^4k [G_1'(k_\mu) + G_\Delta'(k_\mu)] \psi_{++}(p_\mu + k_\mu). \tag{27}$$

[8] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

To the same order, when (27) is used in (23), $\tilde{\psi}$ may be taken as $\tilde{\psi}_{++}$, that is, only $\tilde{\psi}$ or ψ need be iterated, not both. For every diagram considered by N.S. we must consider three separate ones. For example, the diagram labeled CT in N.S. means that we should iterate with the Coulomb kernel and treat the single exchange of a transverse photon as a perturbation. In addition to this term, we must consider the effect of (1) iterating with the modified Coulomb potential and treating T as a perturbation ($\Delta E_{C'T}$), (2) iterating with the unmodified Coulomb potential and treating the modified transverse interaction as a perturbation ($\Delta E_{CT'}$), and (3) using the modified Coulomb and modified transverse parts ($\Delta E_{C'T'}$).

For example, we give the results for $\Delta E_{C'T}$:

$$\Delta E_{C'T} = -\frac{e^2}{2\pi^2} \int \psi_{++}{}^*(p_\mu) \sum_{1,2} \frac{\alpha_i{}^a \alpha_i{}^b}{k^2} \psi_I(p_\mu') d^4p d^4p', \tag{28}$$

where

$$p_\mu' = p_\mu + k_\mu',$$

and to the required order

$$\psi_I(p_\mu') = -\frac{1}{2\pi i F(p_\mu')} \left(\frac{e^2}{2\pi^2}\right) \times \int \frac{d^4k'}{k'^2} [F_4(k'^2) - 1] \psi_{++}(p_\mu''), \tag{29}$$

where

$$p_\mu'' = p_\mu' + k_\mu'$$

and

$$[F_4(k^2) - 1] = \left[1 + \Lambda^2 \frac{\partial}{\partial \Lambda^2}\right] \frac{k^2}{\Lambda^2 - k^2}. \tag{30}$$

The integral for $\Delta E_{C'T}$ contributes to the required order of magnitude only when $|p'| \approx M$ and $|p|$ and $|p''| \approx \alpha m$ (Bohr momentum). It may also be shown that the integral has its major contributions when $p_4 \approx |p|$, $p_4' \approx |p'|$, $p_4'' \approx |p''|$. Similarly, it may be shown that no corrections of the required order arise unless the intermediate momentum $k_\mu'^2$ is the order of the cutoff (as in the second section). Therefore, in all integrals in which p_μ, p_μ', and p_μ'' appear together we set

$$\begin{aligned} p_\mu &= 0 = p_\mu'', \\ k_\mu' &= -p_\mu' = -k_\mu. \end{aligned} \tag{31}$$

With these approximations our integrals may be considerably simplified. We write

$$\Delta E_{C'T} = \left(\frac{e^2}{2\pi^2}\right)^2 \left(-\frac{1}{2\pi i}\right)^3 \left(1 + \Lambda^2 \frac{\partial}{\partial \Lambda^2}\right) \times \int \psi_{++}{}^*(p_\mu) d^4p \int \frac{d^4p'}{p_\mu'^2 (\Lambda^2 - p_\mu'^2)} \times \sum_{1,2,} \alpha_i{}^a \alpha_i{}^b \frac{1}{F(p')} \int d^4p''\, \psi_{++}(p_\mu''). \tag{32}$$

After interposing projection operators to remove the Dirac matrices $(\Lambda_+{}^a+\Lambda_-{}^a)(\Lambda_+{}^b+\Lambda_-{}^b)=1$ and after using the approximations (31), we may integrate over p'' and p and obtain[9]

$$\Delta E_{C'T}{}^{++}=|\phi(0)|^2(2\pi)^3\left(\frac{e^2}{2\pi^2}\right)^2\left(1+\Lambda^2\frac{\partial}{\partial\Lambda^2}\right)\times\left(\frac{-1}{2\pi i}\int\frac{d^4p'\,\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b}{(\Lambda^2-p_\mu{}'^2)F_{++}(p_\mu')6E_aE_b}\right),\quad(33)$$

where $(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b p_\mu{}'^2/6E_aE_b)$ is the expectation value of a product of Dirac matrices and projection operators. The expectation value is taken between completely N.R. spinors. After summing up the contributions from all diagrams of a given type, that is to say, the contribution from the four energy states of the system, the contribution from the three diagrams for the uncrossed terms, and that for the crossed terms, we may write our results in the notation of N.S. as

$$\Delta E_{\alpha\beta}=\alpha\frac{m}{M}\left[\frac{1}{\pi\mu_p}\right](\text{hfs})T_{\alpha\beta},$$

where

$$T_{\alpha\beta}=\text{Op}\,T_{\alpha\beta}',$$

and Op is given by (13). The values of $T_{\alpha\beta}'$ are tabulated below.

$$T_{CT}'=+4a^2\int\frac{\tilde{K}d\tilde{K}}{(\tilde{K}^2+a^2)^{\frac{1}{2}}[\tilde{K}+(\tilde{K}^2+a^2)^{\frac{1}{2}}](\tilde{K}^2+b^2)}-2\int\frac{(a^2-2)d\tilde{K}}{(\tilde{K}^2+1)^{\frac{1}{2}}(\tilde{K}^2+b^2)}+4\int\frac{d\tilde{K}}{[(\tilde{K}^2+1)^{\frac{1}{2}}+\tilde{K}](\tilde{K}^2+b^2)},$$

$$T_{QT}'=-\mu\left[\int\frac{(4-a^2)d\tilde{K}}{[\tilde{K}+(1+\tilde{K}^2)^{\frac{1}{2}}](\tilde{K}^2+b^2)}-2a^2\int\frac{\tilde{K}d\tilde{K}}{(\tilde{K}^2+a^2)^{\frac{1}{2}}[\tilde{K}+(\tilde{K}^2+a^2)^{\frac{1}{2}}](\tilde{K}^2+b^2)}+2\int\frac{\tilde{K}d\tilde{K}}{\tilde{K}^2+b^2}\left(1-\frac{2\tilde{K}^2}{(\tilde{K}^2+a^2)^{\frac{1}{2}}[\tilde{K}+(\tilde{K}^2+a^2)^{\frac{1}{2}}]}\right)\right]+\frac{\mu}{2}T_{CT}',\quad(34)$$

$$T_{CP}'=\mu T_{CT}',$$

$$T_{QP}'=\mu T_{QT}'-4\mu^2\left[\int_0^{\Lambda_0/M}\frac{\tilde{K}d\tilde{K}}{[(\tilde{K}^2+a^2)^{\frac{1}{2}}+\tilde{K}](\tilde{K}^2+a^2)^{\frac{1}{2}}}\right],$$

$$T_{TT}'=-2\int\frac{\tilde{K}^2d\tilde{K}}{[(\tilde{K}^2+a^2)^{\frac{1}{2}}+\tilde{K}](\tilde{K}^2+b^2)}+\int\frac{\tilde{K}d\tilde{K}}{(\tilde{K}^2+b^2)}+\tfrac{1}{2}\int\frac{(2-a^2)\tilde{K}d\tilde{K}}{[(\tilde{K}^2+1)^{\frac{1}{2}}+\tilde{K}](\tilde{K}^2+b^2)(\tilde{K}^2+1)^{\frac{1}{2}}},$$

$$T_{TP}'=2\mu T_{TT}',$$

$$T_{PP}'=\mu^2T_{TT}'-\frac{2\mu^2}{a^2}\int_0^{\Lambda_0/M}d\tilde{K}\left[\frac{\tilde{K}}{2}-\frac{\tilde{K}^2}{\tilde{K}+(\tilde{K}^2+a^2)^{\frac{1}{2}}}\right],$$

where

$$a=\Lambda/M,$$

$$b=a^2-\tfrac{1}{4}a^4.$$

Now if (10) and (12) are substituted in formula (11), it separates naturally into three types of terms corresponding to a term without μ in front, one with μ, and one with μ^2. If we now proceed to integrate out the fourth component of the momentum for the term without μ, we obtain

$$\Delta E'(\mu^0)=\left[\frac{\alpha m}{\pi\mu_pM}\right](\text{hfs})B,\quad(35)$$

$$B=\frac{6\,\text{Op}}{i\pi^2}\int\frac{d^4\tilde{k}(3\tilde{\omega}^2-2\tilde{K}^2)}{\tilde{k}^4(2\tilde{\omega}+\tilde{k}^2)(\tilde{k}^2-a^2)}=-16\,\text{Op}\int\frac{\tilde{K}^2d\tilde{K}}{a^2}\left\{\frac{1}{8\tilde{K}}+\frac{-3a-\tilde{K}^2}{8(\tilde{K}^2+b^2)(\tilde{K}^2+a^2)^{\frac{1}{2}}}+\frac{\tilde{K}^2(\frac{1}{4}a^4+\frac{3}{2}a^{-2})+4b^2}{8\tilde{K}^2(\tilde{K}^2+b^2)(\tilde{K}^2+1)^{\frac{1}{2}}}\right\},\quad(36)$$

and Op is the same operator which appears above. If, in addition, we add up T_{CT} and T_{TT} and find their related energy, with the aid of algebra, we get the integral (36). It is easily shown that the other terms also agree.

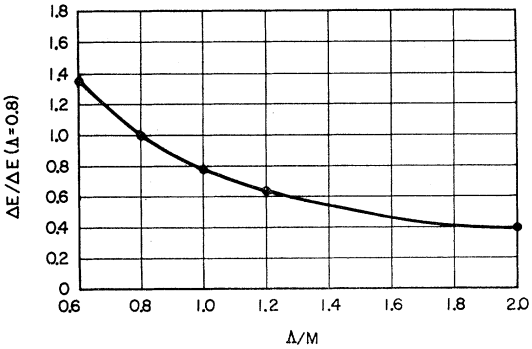

FIG. 2. Correction due to the finite size of the nucleus as a function of the form factor parameter Λ.

[9] For terms of the $C'T'$ type, factors of the form $(F_4-1)^2$ appear in the integrals. These may be written as $D[k^2/(k^2-\Lambda^2)]$, where D is a differential operator, operating on Λ^2.

DISCUSSION

When the value $0.91M$ is given to Λ, the calculated correction becomes $\{(\alpha m/\pi\mu_p M)(\text{hfs})[-73+(9\mu^2/4)\times\ln(2\tilde{K}_0/M)]\}$. Combining this result with the results of N.S., we find[10]

$$\Delta E' = \text{hfs}\frac{\alpha m}{\pi\mu_p M}[-76], \tag{37}$$

or a total shift of -35 parts per million. The exact expression for the hfs then becomes[11]

$$\Delta\nu_{\text{H}} = \frac{16\alpha^2}{3}cR_\infty\left(\frac{\mu_p}{\mu_e}\right)\left(\frac{\mu_e}{\mu_0}\right)^2\left[1+\frac{m}{M}\right]^{-3} \times[1+\tfrac{3}{2}\alpha^2][1-(\tfrac{5}{2}-\ln 2)\alpha^2]\times P, \tag{38}$$

where P is equal to $(1-35\times10^{-6})$, the correction calculated here plus that of N.S. The precision with which the terms of (39) are known is discussed in reference 10. We use the values and probable errors quoted there for c, R_∞, and m/M. The term (μ_e/μ_0) has recently been recalculated[12] and is (1.0011596). The term μ_e/μ_p is subject to some uncertainty because of the polarizability of the oil used for the proton sample. We take the value 658.2087 quoted by Koening, Prodell, and Kusch[13] and assume a rather arbitrary uncertainty of ±0.001. The results may then be written

$$\alpha^{-1} = [137.0391\pm0.0001]P^{\frac{1}{2}}, \tag{39}$$

or

$$\alpha^{-1} = [137.0367\pm0.0001], \tag{40}$$

where the quoted limits are the roots of the sums of the squares of the uncertainties of c, R_∞, m/M, and μ_p/μ_e, and no allowance has been made for uncertainties in the theoretical corrections. Figure 2 shows the variation in the corrections with changes in Λ. Fine-structure measurements give the value

$$\alpha^{-1} = (137.0390)\pm0.0006, \tag{41}$$

which can be used to give the "experimental" value of P:

$$P = [1-1.4\times10^{-6}]\pm18\times10^{-6}. \tag{42}$$

As was pointed out in the Introduction, this disagreement might be due to the virtual absorption and emission of mesons as shown in Fig. 3. The two-photon form factor, $M_{\mu\nu}$, is expected to differ from the product of single-photon form factors assumed in (5). The single-photon form factor, measured by high-energy electron-proton scattering, will contain only diagrams like those shown in Fig. 3(b). If the factor $M_{\mu\nu}$ were accurately known, our result would be much more definite. The intermediate state may be thought of as any one of a complete set, and then the calculations in this paper give the contribution from $\gamma^*+N\rightarrow N$ to the forward-scattering amplitude of one virtual photon ($M_{\mu\nu}$). We are undertaking a crude estimate of the one-meson contribution; see Fig. 3(c). However, because of the strength of the $\pi-N$ interaction, we do not expect that $M_{\mu\nu}$ would accurately be given by the one- and zero-meson terms.

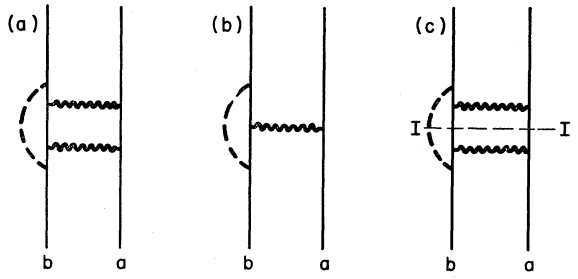

FIG. 3. Single-virtual meson corrections to the two-photon form factor.

ACKNOWLEDGMENTS

We would like to thank Professor Feynman for suggesting this problem and for his continued encouragement and help during the course of the computation.‡

[10] The form of the "Hofstadler Form Factor" used in this article is open to a good deal of theoretical questioning. If one assumes that the form factors have a mass spectral representation, that is to say

$$F_4(q^2) = \int\frac{\rho(m^2)}{q^2-m^2}dm^2.$$

Then our choice corresponds to the use of a $\rho(m^2)\sim\delta'(\Lambda^2-m^2)$. A more natural choice for $\rho(m^2)$ might be $\rho(m^2)=\Sigma\, a_i\delta(\Lambda_i^2-m^2)$. We have repeated our calculation choosing a simple $\rho(m^2)$ with $a_1=-a_2=\Lambda'^2\Lambda''^2/\Lambda'^2-\Lambda''^2$. With $\Lambda'^2=.52$, $\Lambda''^2=2.06$. The root mean square radius in coordinate space for this distribution agrees with the original "Hofstadler Form Factor." The result of the calculation is $\Delta E'=\text{hfs}\,\alpha m/\pi\mu_p M(-74)$ in extremely close agreement with the number quoted above.

[11] Cohen, Crowe, and DuMond, *The Fundamental Constants of Physics* (Interscience Publishers, Inc., New York, 1957).

[12] A. Petermann, Helv. Phys. Acta **30**, 407 (1957). C. M. Sommerfield, Phys. Rev. **107**, 328 (1957).

[13] Koening, Prodell, and Kusch, Phys. Rev. **88**, 191 (1952).

‡ *Note added in proof.*—While this article was in press, it was brought to our attention that a nonrelativistic calculation of the nuclear structure effect has appeared [A. C. Zemach, Phys. Rev. **104**, 1721 (1956)]. Our result of approximately $-8.7\alpha m/M$ is somewhat smaller than the nonrelativistic correction, $-11\alpha m/M$, using the equivalent nonrelativistic form factor and neglecting entirely the effect of the cutoff-dependent term. While we agree with Zemach's conclusion that the major contribution to the hfs does indeed come from low momentum photons, the nuclear structure corrections to the hfs come from the exchange of two rather high momentum virtual photons, as is indicated in our paper. The results depend, in principle, upon the specific form of the proton interactions, especially those with the electromagnetic and π fields, and no simple description in terms of rms radius (obtained from electron-proton scattering data) is *a priori* correct.