Solution of Maxwell's Equations in Terms of a Spinor Notation: the Direct and Inverse Problem*

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Maxwell's equations for fields with sources in media in which the dielectric constant and permeability are unity are written in terms of a spinor notation which resembles the one used for Dirac's equation for the electron. One can introduce Green's functions and expansions in terms of complete sets of orthogonal functions, analogous to those used in the quantum theory of the electron, to solve Maxwell's equations in more compact form than in terms of the conventional vector notation.

In addition, the new notation enables us to solve in a simple way an "inverse radiation problem" which we describe as follows:

Consider at time $t<0$ the electromagnetic field to be zero. At time $t=0$ sources are turned on and then later turned off. The electromagnetic field, which results after this process has been completed, will be a radiation field. We can solve the problem of finding the nature of the sources which will lead to a *prescribed* final radiation field. It is shown that, in general, the sources are not unique but additional conditions can be given which will make them so.

1. INTRODUCTION. THE SPINOR FORM OF MAXWELL'S EQUATIONS

MAXWELL'S equations in free space with sources are:

$$\begin{aligned} \mathrm{curl}\mathbf{E}+\partial\mathbf{H}/\partial t&=0, & \mathrm{curl}\mathbf{H}-\partial\mathbf{E}/\partial t&=4\pi\mathbf{j}, \\ \mathrm{div}\mathbf{H}&=0, & \mathrm{div}\mathbf{E}&=4\pi\rho. \end{aligned} \tag{1.1}$$

[In (1.1) we have used Gaussian units with $c=1$.]

As in a previous paper,[1] we introduce two 4-component column vectors ψ and Φ:

$$\psi=\begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{bmatrix}, \qquad \Phi=\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Phi_3\end{bmatrix}, \tag{1.2}$$

$$\begin{aligned} &\psi_0\equiv 0, && \Phi_0=\rho, \\ &\psi_1=H_1-iE_1, && \Phi_1=j_1, \\ &\psi_2=H_2-iE_2, && \Phi_2=j_2, \\ &\psi_3=H_3-iE_3, && \Phi_3=j_3, \end{aligned} \tag{1.2a}$$

$$(H_1=H_x,\ H_2=H_y,\ H_3=H_z,\ \text{etc.}),$$

and 4×4 matrices α^i $(i=0, 1, 2, 3)$:

$$\alpha^0\equiv I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \quad \alpha^1=\begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix},$$

$$\alpha^2=\begin{bmatrix}0&0&-1&0\\0&0&0&i\\-1&0&0&0\\0&-i&0&0\end{bmatrix}, \quad \alpha^3=\begin{bmatrix}0&0&0&-1\\0&0&-i&0\\0&i&0&0\\-1&0&0&0\end{bmatrix}. \tag{1.3}$$

Maxwell's equations have the form

$$-\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x^j}\psi=-4\pi\Phi, \tag{1.4}$$

where, in (1.4), and later

$$x^0=x_0=t,\ x^1=-x_1=x,\ x^2=-x_2=y,\ x^3=-x_3=z. \tag{1.5}$$

Equations essentially identical with (1.4) have also been given in by Ohmura[2] and an analogous but more cumbersome set has been given by Oppenheimer.[3] Molière[4] and Good[5] give sets of equations similar to (1.4) for the truncated set of Maxwell's equations in which the divergence equations have been omitted.

In reference 1, it was shown that the usual transformation properties for the fields and sources are obtained. In particular, it was shown that the field function ψ transformed like the wave function for a spin 1 particle.

The Hermitian operators α^i $(i=1, 2, 3)$ satisfy the following multiplication laws:

$$\begin{aligned} &(\alpha^1)^2=(\alpha^2)^2=(\alpha^2)^2=I, \\ &\alpha^1\alpha^2=i\alpha^3=-\alpha^2\alpha^1, \\ &\alpha^2\alpha^3=i\alpha^1=-\alpha^3\alpha^2, \\ &\alpha^3\alpha^1=i\alpha^2=-\alpha^1\alpha^3. \end{aligned} \tag{1.6}$$

As a consequence of these multiplication rules one obtains the following important identity:

$$\begin{aligned} &\left(-\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\right)\left(-\frac{1}{i}\sum_{k=0}^{3}\alpha^k\frac{\partial}{\partial x^k}\right) \\ &\qquad=\left(-\frac{1}{i}\sum_{k=0}^{3}\alpha^k\frac{\partial}{\partial x^k}\right)\left(-\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\right) \\ &\qquad=\Box^2\psi=\left(\nabla^2-\frac{\partial^2}{\partial t^2}\right)\psi. \end{aligned} \tag{1.7}$$

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[1] H. E. Moses, Nuovo cimento Suppl. **7**, 1 (1958). Also New York University Report IMM-NYU 238, January, 1957 (unpublished).

[2] T. Ohmura, Progr. Theoret. Phys. (Kyoto) **16**, 684 (1956).
[3] J. R. Oppenheimer, Phys. Rev. **38**, 725 (1931).
[4] G. Molière, Ann. Physik **6**, 146 (1949).
[5] R. H. Good, Jr., Phys. Rev. **105**, 1914 (1957).

On applying $-(1/i)\sum_{j=0}^{3}\alpha^j(\partial/\partial x_j)$ to both sides of (1.4), we obtain

$$\Box^2\psi=\frac{4\pi}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\Phi. \tag{1.8}$$

We see that when there are no source terms, ψ and hence the components of the electromagnetic field satisfy the wave equation. If there are sources, on taking the top component of the vectors on both sides of the equation (1.8) and using $\psi_0\equiv 0$, we obtain as a necessary condition for the solution of (1.4), the equation of continuity:

$$\sum_{j=0}^{3}\frac{\partial\Phi_j}{\partial x^j}=\frac{\partial\rho}{\partial t}+\mathrm{div}\mathbf{j}=0. \tag{1.9}$$

We can also write

$$\left(-\frac{1}{i}\frac{\partial}{\partial t}-H_0\right)\psi=-4\pi\Phi, \tag{1.10}$$

where H_0 is given by

$$H_0=\frac{1}{i}\sum_{j=1}^{3}\alpha^j\frac{\partial}{\partial x^j}. \tag{1.11}$$

The operator H_0 is analogous to Dirac's Hamiltonian. We can now state our primary objective: we solve (1.10) by working in the spectral representation of the operator H_0 instead of using the usual Fourier transformations in terms of wave numbers. This approach is used in references 4 and 5, but the treatment there is incomplete and also somewhat cumbersome because the equations analogous to (1.10) are not the complete set of Maxwell's equations.

In the Appendix we shall show how the energy conservation laws may be derived simply in terms of these definitions.

2. THE "EIGENFUNCTIONS" OF H_0; THE x REPRESENTATION; THE p REPRESENTATION

We shall work essentially in the spectral representation of the operator H_0. We shall introduce four column vectors (which we term basic vectors) which form a complete, orthonormal set which give the spectral representation of H_0. Rather than give a detailed motivation for this set of vectors, we shall instead give them explicitly. We shall designate the four basic vectors by $\chi(\mathbf{x}|\mathbf{p},\epsilon)$ where $\mathbf{p}$ is a three-dimensional vector and where ϵ has the values ± 1, 0, τ. That is, the four values of ϵ label the four basic vectors. Explicitly, the set of basic vectors which we use is

$$\chi(\mathbf{x}|\mathbf{p},\epsilon)=\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2}(\eta_x^2+\eta_y^2)^{\frac{1}{2}}}\begin{bmatrix}0\\-(\eta_x\eta_z-i\epsilon\eta_y)\\-(\eta_y\eta_z+i\epsilon\eta_x)\\\eta_x^2+\eta_y^2\end{bmatrix}\quad\text{for}\quad\epsilon=\pm1,$$

$$\chi(\mathbf{x}|\mathbf{p},0)=\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}\begin{bmatrix}0\\\eta_x\\\eta_y\\\eta_z\end{bmatrix}, \tag{2.1}$$

$$\chi(\mathbf{x}|\mathbf{p},\tau)=\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

In (2.1), $\boldsymbol{\eta}$ is the unit vector in the direction of $\mathbf{p}$, i.e.,

$$\boldsymbol{\eta}=\mathbf{p}/p, \tag{2.1a}$$

where $p=|\mathbf{p}|$. Also $\mathbf{p}\cdot\mathbf{x}$ is the usual three-dimensional scalar product of $\mathbf{p}$ and $\mathbf{x}$.

It is easily verified that the basic vectors are orthonormal to each other:

$$\sum_{i=0}^{3}\int\chi_i^*(\mathbf{x}|\mathbf{p},\epsilon)\chi_i(\mathbf{x}|\mathbf{p}',\epsilon')d\mathbf{x}=\delta(\mathbf{p}-\mathbf{p}')\delta_{\epsilon\epsilon'}, \tag{2.2}$$

where

$$\delta_{\epsilon\epsilon'}=0,\quad \epsilon'\neq\epsilon$$
$$\delta_{\epsilon\epsilon}=1, \tag{2.2a}$$

and the asterisk means complex conjugate. This set also satisfies the completeness relation:

$$\sum_{\epsilon}\int\chi_i^*(\mathbf{x}|\mathbf{p},\epsilon)\chi_j(\mathbf{x}'|\mathbf{p},\epsilon)d\mathbf{p}=\delta(\mathbf{x}-\mathbf{x}')\delta_{ij}. \tag{2.3}$$

As a consequence we can expand any four-component column vector $A(\mathbf{x})$

$$A(\mathbf{x})=\begin{bmatrix}A_0(\mathbf{x})\\A_1(\mathbf{x})\\A_2(\mathbf{x})\\A_3(\mathbf{x})\end{bmatrix}$$

in the following way:

$$A_i(\mathbf{x})=\sum_{\epsilon}\int\chi_i(\mathbf{x}|\mathbf{p},\epsilon)A(\mathbf{p},\epsilon)d\mathbf{p}, \tag{2.4}$$

where

$$A(\mathbf{p},\epsilon)=\sum_{i=0}^{3}\int\chi_i^*(\mathbf{x}|\mathbf{p},\epsilon)A_i(\mathbf{x})d\mathbf{x}. \tag{2.5}$$

If we choose, we may regard $A_i(\mathbf{x})$ and $A(\mathbf{p},\epsilon)$ as representing an abstract vector A in Hilbert space in two different representations, which we may call the $\mathbf{x}$-representation and the $\mathbf{p}$-representation, respectively, where the norm of A is

$$\sum_{i=0}^{3}\int A_i^*(\mathbf{x})A_i(\mathbf{x})dx=\sum_{\epsilon}\int A^*(\mathbf{p},\epsilon)A(\mathbf{p},\epsilon)d\mathbf{p}.$$

The basic vectors $\chi(\mathbf{x}|\mathbf{p},\pm1)$ are true eigenfunctions of H_0 which have their top components zero. They satisfy

$$H_0\chi(\mathbf{x}|\mathbf{p},\epsilon)=\epsilon p\chi(\mathbf{x}|\mathbf{p},\epsilon),\ (\epsilon=\pm1),\ \text{where } p=|\mathbf{p}|. \quad (2.6)$$

Also

$$H_0\chi(\mathbf{x}|\mathbf{p},0)=-p\chi(\mathbf{x}|\mathbf{p},\tau),\quad (p=|\mathbf{p}|) \quad (2.7)$$

and

$$H_0\chi(\mathbf{x}|\mathbf{p},\tau)=-p\chi(\mathbf{x}|\mathbf{p},0). \quad (2.8)$$

Another relation that will prove very useful is the divergence property

$$\begin{aligned}&\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\chi_i(\mathbf{x}|\mathbf{p},\pm1)=0,\\ &\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\chi_i(\mathbf{x}|\mathbf{p},0)=\frac{ip}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{p}\cdot\mathbf{x}}.\end{aligned} \quad (2.9)$$

The relations (2.9) enable us to separate the longitudinal from the transverse field in a simple fashion.

We are now able to expand our time-dependent electromagnetic field vector $\psi(\mathbf{x};t)$ sources $\Phi(\mathbf{x};t)$ in terms of the basic vectors $\chi(\mathbf{x}|\mathbf{p},\epsilon)$:

$$\begin{aligned}\psi(\mathbf{x};t)&=\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\psi(\mathbf{p},\epsilon;t)d\mathbf{p},\\ \Phi(\mathbf{x};t)&=\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi(\mathbf{p},\epsilon;t)d\mathbf{p},\end{aligned} \quad (2.10)$$

Since we require $\psi_0(\mathbf{x};t)\equiv0$, we see from (2.1) that in the $\mathbf{p}$-representation we must have

$$\psi(\mathbf{p},\tau;t)=0. \quad (2.11)$$

Hence we may write

$$\psi(\mathbf{x};t)=\psi^T(\mathbf{x};t)+\psi^L(\mathbf{x};t), \quad (2.12)$$

where

$$\begin{aligned}\psi^T(\mathbf{x};t)&=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\psi(\mathbf{p},\epsilon;t)d\mathbf{p},\\ \psi^L(\mathbf{x};t)&=\int\chi(\mathbf{x}|\mathbf{p},0)\psi(\mathbf{p},0;t)d\mathbf{p}.\end{aligned}$$

Now ψ^T is the transverse field, since from (2.9) we have $\sum_{i=1}^{3}(\partial/\partial x^i)\psi_i^T=0$, which leads to $\mathrm{div}\mathbf{E}^T=\mathrm{div}\mathbf{H}^T=0$. Likewise ψ^L is the longitudinal part of the field.

We now consider the sources. The equation of continuity which is a necessary condition for the solution of Maxwell's equations,

$$\sum_{i=0}^{3}\frac{\partial\Phi_i}{\partial x^i}=0, \quad (2.13)$$

lead, on using (2.9) and (2.10), to a restriction on $\Phi(\mathbf{p},\epsilon;t)$, namely

$$\Phi(\mathbf{p},0;t)=-\frac{i}{p}\frac{\partial\Phi(\mathbf{p},\tau;t)}{\partial t}. \quad (2.14)$$

The condition that $\Phi_i(\mathbf{x};t)$ is real leads to necessary and sufficient symmetry conditions on $\Phi(\mathbf{p},\epsilon;t)$, namely

$$\begin{aligned}\Phi^*(\mathbf{p},\pm1;t)&=\Phi(-\mathbf{p},\pm1;t),\\ \Phi^*(\mathbf{p},\tau;t)&=\Phi(-\mathbf{p},\tau;t),\\ \Phi^*(\mathbf{p},0;t)&=-\Phi(-\mathbf{p},0;t).\end{aligned} \quad (2.15)$$

3. THE SOLUTION OF THE INITIAL VALUE PROBLEM. SEPARATION OF THE LONGITUDINAL AND TRANSVERSE FIELDS

We shall now use the expansion of ψ and Φ in terms of the complete set of basic vectors to solve Maxwell's equations with sources. As in Sec. 2, let us write

$$\begin{aligned}\psi(\mathbf{x};t)&=\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\psi(\mathbf{p},\epsilon;t)d\mathbf{p},\\ \Phi(\mathbf{x};t)&=\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi(\mathbf{p},\epsilon;t)d\mathbf{p}.\end{aligned} \quad (3.1)$$

On substituting $\psi(\mathbf{x};t)$ and $\Phi(\mathbf{x};t)$ as given by (3.1) into Maxwell's equations (1.10) and using (2.6)–(2.8), we obtain

$$\begin{aligned}-\frac{1}{i}\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)&\frac{\partial\psi(\mathbf{p},\epsilon;t)}{\partial t}\\ &+\int\chi(\mathbf{x}|\mathbf{p},0)p\psi(\mathbf{p},\tau;t)d\mathbf{p}\\ &+\int\chi(\mathbf{x}|\mathbf{p},\tau)p\psi(\mathbf{p},0;t)d\mathbf{p}\\ &-\sum_{\epsilon=\pm1}\int\epsilon p\chi(\mathbf{x}|\mathbf{p},\epsilon)\psi(\mathbf{p},\epsilon;t)d\mathbf{p}\\ &=-4\pi\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi(\mathbf{p},\epsilon;t)d\mathbf{p}.\end{aligned} \quad (3.2)$$

By identifying coefficients of $\chi(\mathbf{x}|\mathbf{p},\epsilon)$, we obtain equations for $\psi(\mathbf{p},\epsilon;t)$ in terms of $\Phi(\mathbf{p},\epsilon;t)$. The equations for the transverse part of the field are obtained by identifying the coefficients of $\chi(\mathbf{x}|\mathbf{p},\pm1)$, and we find

$$-\frac{1}{i}\frac{\partial}{\partial t}\psi(\mathbf{p},\epsilon;t)-\epsilon p(\mathbf{p},\epsilon;t)=-4\pi\Phi(\mathbf{p},\epsilon;t),\ (\epsilon=\pm1). \quad (3.3)$$

As before, we require

$$\psi(\mathbf{p},\tau;t)\equiv0. \quad (3.4)$$

Hence, by identifying the coefficients of $\chi(\mathbf{x}|\mathbf{p},\tau)$,

$$\psi(\mathbf{p},0;t)=-\frac{4\pi}{p}\Phi(\mathbf{p},\tau;t), \quad (3.5)$$

and by identifying the coefficients of $\chi(\mathbf{x}|\mathbf{p},0)$

$$-\frac{1}{i}\frac{\partial}{\partial t}\psi(\mathbf{p},0;t)=-4\pi\Phi(\mathbf{p},0;t). \quad (3.6)$$

In order that $\psi(\mathbf{p},0;t)$ be given both by (3.5) and (3.6), we obtain a relation that Φ must satisfy:

$$-\frac{1}{i}\frac{1}{p}\frac{\partial\Phi(\mathbf{p},\tau;t)}{\partial t}=-\Phi(\mathbf{p},0;t), \quad (3.7)$$

which is just Eq. (2.14); this was seen to be equivalent to the equation of continuity.

In our notation (as in the usual treatment) the transverse and longitudinal fields uncouple. The transverse field in the $\mathbf{p}$-representation is given by (3.3). Because of the first derivatives in time, one can consider an initial value problem which this poses.

On the other hand, the longitudinal field itself, rather than its derivative, depends on the sources and is therefore simpler to obtain than the transverse field. No initial value problem is involved.

On using the expressions for the eigenfunction (2.1),

$$\begin{aligned}\psi_i^L(\mathbf{x};t)&=-4\pi\int\frac{\chi_i(\mathbf{x}|\mathbf{p},0)}{p}\Phi(\mathbf{p},\tau;t)d\mathbf{p}\\&=-4\pi\sum_i\int\int\frac{\chi_i(\mathbf{x}|\mathbf{p},0)}{p}\chi_j^*(\mathbf{x}'|\mathbf{p},\tau)\\&\qquad\times\Phi_j(\mathbf{x}';t)d\mathbf{x}'d\mathbf{p}\\&=\frac{-4\pi}{(2\pi)^3}\int\int\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p}d\mathbf{p}\ \eta_i\Phi_0(\mathbf{x}';t)d\mathbf{x}',\\&\qquad(i=1,2,3), \quad (3.8)\end{aligned}$$

where $\eta_i=\eta_x$, etc. Since $\Phi_0(\mathbf{x};t)=\rho(\mathbf{x};t)$, we see that

$$\begin{aligned}\psi_i^L(\mathbf{x};t)&=-\frac{i}{2\pi^2}\frac{\partial}{\partial x^i}\int\int\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p^2}d\mathbf{p}\rho(\mathbf{x}';t)d\mathbf{x}'\\&=i\frac{\partial}{\partial x^i}\int\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'. \quad (3.9)\end{aligned}$$

We thus obtain the familiar result that the longitudinal magnetic field is zero,

$$\mathbf{H}^L\equiv0, \quad (3.10)$$

and

$$\mathbf{E}^L(\mathbf{x};t)=-\mathrm{grad}\int\frac{\rho(\mathbf{x}';t)}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'. \quad (3.11)$$

Having disposed of the longitudinal field, we shall now discuss the initial value problem for the transverse field.

The general solution of the differential equation (3.3) is

$$\begin{aligned}\psi(\mathbf{p},\epsilon;t)&=e^{-i\epsilon p(t-t_0)}\psi(\mathbf{p},\epsilon;t_0)+4\pi ie^{-i\epsilon pt}\\&\qquad\times\int_{t_0}^{t}\Phi(\mathbf{p},\epsilon;t')e^{i\epsilon pt'}dt',\quad(\epsilon=\pm1). \quad (3.12)\end{aligned}$$

The first term on the right represents a solution of Maxwell's equation without sources, while the second term shows the effects of sources. It should be noted that $\Phi_0(\mathbf{x},t)\equiv\rho(\mathbf{x};t)$ has no effect whatsoever on the transverse field. We can also write the solution (3.12) in terms of the $\mathbf{x}$-representation:

$$\begin{aligned}\psi_i^T(\mathbf{x};t)&=\sum_{\epsilon=\pm1}\int\chi_i(\mathbf{x}|\mathbf{p},\epsilon)\psi^T(\mathbf{p},\epsilon;t)d\mathbf{p}\\&=\sum_{\epsilon=\pm1}\int\int\chi_i(\mathbf{x}|\mathbf{p},\epsilon)\\&\qquad\times\chi_j^*(\mathbf{x}'|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t_0)}\psi_j(\mathbf{x}',t_0)d\mathbf{x}'d\mathbf{p}\\&\qquad+4\pi i\int_{t_0}^{t}dt'\int\int\sum_{\epsilon=\pm1}\chi_i(\mathbf{x}|\mathbf{p},\epsilon)\\&\qquad\times\chi_j^*(\mathbf{x}'|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t')}\Phi_j(\mathbf{x}';t')d\mathbf{x}'d\mathbf{p}. \quad (3.13)\end{aligned}$$

Hence

$$\begin{aligned}\psi^T(\mathbf{x};t)&=\int d\mathbf{x}'G(\mathbf{x};t|\mathbf{x}';t_0)\psi^T(\mathbf{x}';t_0)\\&\qquad+4\pi i\int_{t_0}^{t}dt'\int d\mathbf{x}'G(\mathbf{x};t|\mathbf{x}';t')\Phi(\mathbf{x}';t') \quad (3.14)\end{aligned}$$

where $G(\mathbf{x};t|\mathbf{x}';t')$ is the matrix Green's function given by

$$\begin{aligned}G_{ij}(\mathbf{x};t|\mathbf{x}';t')&=\sum_{\epsilon=\pm1}\int\chi_i(\mathbf{x}|\mathbf{p},\epsilon)\\&\qquad\times\chi_j^*(\mathbf{x}'|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t')}d\mathbf{p}. \quad (3.15)\end{aligned}$$

The matrix elements are easily evaluated:

$$G_{0j}(\mathbf{x},t|\mathbf{x}',t')=0,\quad(j=0,1,2,3)$$

$$\left.\begin{aligned}G_{ii}(\mathbf{x},t|\mathbf{x}',t')&=\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^{i2}}\right]B(\mathbf{x},t|\mathbf{x}',t'),\\&\qquad(i=1,2,3)\end{aligned}\right\} \quad (3.16)$$

$$G_{12}(\mathbf{x},t|\mathbf{x}',t')=\left[-i\frac{\partial^2}{\partial t\partial x^3}+\frac{\partial^2}{\partial x^1\partial x^2}\right]B(\mathbf{x},t|\mathbf{x}',t'),$$

and cyclically,

$$G_{ij}(\mathbf{x},t|\mathbf{x}',t')=G_{ji}(\mathbf{x},t|\mathbf{x}',t');$$

where

$$B(\mathbf{x},t|\mathbf{x}',t')=\frac{\eta(|\mathbf{x}-\mathbf{x}'|^2-|t-t'|^2)}{4\pi|\mathbf{x}-\mathbf{x}'|}, \qquad (3.17)$$

where

$$\begin{aligned}\eta(x)&=1 \quad \text{for} \quad x>0,\\ \eta(x)&=0 \quad \text{for} \quad x<0.\end{aligned} \qquad (3.18)$$

If there were no sources, one would obtain only the first term of (3.14) which would represent a radiation field in terms of its value at time $t=t_0$. This radiation field could also be written

$$\sum_{\epsilon=\pm1}\int \chi(\mathbf{x}|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t_0)}\psi(\mathbf{p},\epsilon;t_0)d\mathbf{p}. \qquad (3.19)$$

It is useful to note that the radiation field (3.19) represents a superposition of solutions $\chi(\mathbf{x}|\mathbf{p},\epsilon)e^{-i\epsilon pt}$ of Maxwell's equations without sources. Such solutions represent circularly polarized radiation of frequency $|p|$ travelling in the direction $\mathbf{p}$ for $\epsilon=1$ and $-p$ for $\epsilon=-1$. To show this, one sets $\eta_x=1$, $\eta_y=\eta_z=0$, and obtains from (2.1)

$$\begin{aligned}
E_x&=0,\\
E_y&=\frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\frac{(\epsilon+1)}{2}\cos[p(x-\epsilon t)],\\
E_z&=\frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\frac{(\epsilon-1)}{2}\sin[p(x-\epsilon t)],\\
H_x&=0,\\
H_y&=\frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\frac{(\epsilon-1)}{2}\sin[p(x-\epsilon t)],\\
H_z&=\frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\frac{(\epsilon+1)}{2}\cos[p(x-\epsilon t)],
\end{aligned} \qquad (3.20)$$

which are just the usual forms for circularly polarized electromagnetic waves.

4. THE PROBLEM OF FINDING SOURCES WHICH WILL GIVE A PRESCRIBED RADIATION FIELD

Let us consider a particular situation where $\Phi(x;t)\equiv 0$ for $t<t_0$ and for $t>t_1$. At time $t<t_0$ and $t>t_1$ we shall have radiation fields which we call initial and final fields.

A physically interesting problem is to obtain the final field from the initial field and sources. The problem is analogous to the scattering problem in quantum mechanics. Since we have given the general solution of Maxwell's equations in the previous section, we can easily solve this problem.

We can also solve the "inverse" problem which may be described in the following way: We prescribe the initial field and the final field. We are required to find the sources which lead from the initial field to the final field. A particular case is that which occurs when the initial field is zero. We shall then want to find the sources which give a prescribed radiation pattern.

Let us now assume that there are no sources for $t<t_0$. At $t=t_0$ the sources are switched on, permitted to vary in time in any desired fashion, and then switched off again at $t\geq t_1$. From (3.12) it is clear that for $t<t_0$ we have a solution of Maxwell's equations without sources which is a purely transverse field and which we may write as

$$\begin{aligned}\psi(\mathbf{x};t)&\equiv\psi^T(\mathbf{x};t)\\ &=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t_0)}\psi(\mathbf{p},\epsilon;t_0)d\mathbf{p}\\ &=\int G(\mathbf{x};t|\mathbf{x}',t_0)\psi^T(\mathbf{x}';t_0)d\mathbf{x}', \quad (t\leq t_0).\end{aligned} \qquad (4.1)$$

In the time interval $t_0<t<t_1$, the transverse field is given by (3.12) or (3.14).

The longitudinal field is given by (3.9).

For times $t\geq t_1$ corresponding to the switching off again of the sources, we have another solution of Maxwell's equations without sources which is a pure transverse field:

$$\begin{aligned}\psi(\mathbf{x};t)&\equiv\psi^T(\mathbf{x};t)\\ &=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)e^{-i\epsilon p(t-t_1)}\psi(\mathbf{p},\epsilon;t_1)\\ &=\int G(\mathbf{x};t|\mathbf{x}';t_1)\psi^T(\mathbf{x}';t_1)d\mathbf{x}', \quad t\geq t_1\end{aligned} \qquad (4.2)$$

where

$$\psi(\mathbf{p},\epsilon;t_1)=e^{-i\epsilon p(t_1-t_0)}\psi(\mathbf{p},\epsilon;t_0)+4\pi i e^{-i\epsilon pt_1}\int_{t_0}^{t_1}\Phi(\mathbf{p},\epsilon;t')e^{i\epsilon pt'}dt', \quad (\epsilon=\pm1). \qquad (4.3)$$

It will be useful to introduce the notation

$$\begin{aligned}F(\mathbf{p},\epsilon)&=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{t_0}^{t_1}\Phi(\mathbf{p},\epsilon;t')e^{i\epsilon pt'}dt'\\ &=\frac{1}{(2\pi)^{\frac{3}{2}}}\sum_i\int_{t_0}^{t_1}\int\chi_i^*(\mathbf{x}|\mathbf{p},\epsilon)\Phi_i(\mathbf{x};t')e^{i\epsilon pt'}d\mathbf{x}dt', \quad (\epsilon=\pm1).\end{aligned} \qquad (4.4)$$

Equation (4.3) can be written

$$\psi(\mathbf{p},\epsilon;t_1)=e^{-i\epsilon p(t_1-t_0)}\psi(\mathbf{p},\epsilon;t_0)+2(2\pi)^{\frac{3}{2}}ie^{-i\epsilon pt_1}F(\mathbf{p},\epsilon), \quad (\epsilon=\pm1). \qquad (4.5)$$

In terms of the $\mathbf{p}$-representation, the initial electromagnetic field is $\psi(\mathbf{p},\epsilon;t_0)$, while the final one is $\psi(\mathbf{p},\epsilon;t_1)$, both fields being given. We are required to find the sources $\Phi(\mathbf{x};t)$. We shall show that the solution is not unique, but that we can obtain essentially unique results by imposing additional conditions.

From (4.5) we can find $F(p,\epsilon)$ from the initial and final fields:

$$F(\mathbf{p},\epsilon)=\frac{-i}{2(2\pi)^{\frac{3}{2}}}[e^{i\epsilon pt_1}\psi(\mathbf{p},\epsilon;t_1)-e^{i\epsilon pt_0}\psi(\mathbf{p},\epsilon;t_0)],\quad \epsilon=\pm1. \tag{4.6}$$

If we can find $\Phi(\mathbf{p},\epsilon;t)$ and hence $\Phi(\mathbf{x};t)$ from $F(\mathbf{p},\epsilon)$, our problem is solved.

Let us define the function $F(\mathbf{p},\epsilon;k)$ as being the Fourier transform of $\Phi(\mathbf{p},\epsilon;t)$ with respect to time, for $\epsilon=\pm1$.

$$\begin{aligned}F(\mathbf{p},\epsilon;k)&=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{t_0}^{t_1}\Phi(\mathbf{p},\epsilon;t)e^{ikt}dt,\\ \Phi(\mathbf{p},\epsilon;t)&=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}F(\mathbf{p},\epsilon;k)e^{-ikt}dk.\end{aligned} \tag{4.7}$$

The condition that $\Phi(\mathbf{x};t)$ be real leads to Eqs. (2.15) and hence also

$$F^*(\mathbf{p},\epsilon;k)=F(-\mathbf{p},\epsilon;-k). \tag{4.7a}$$

Also the condition that $\Phi(\mathbf{x};t)$ vanish for $t>t$, and $t<t_0$ leads to the requirement that $F(\mathbf{p},\epsilon;k)$ be an entire function in the complex $\mathbf{k}+$plane and that $e^{-ikt_1}F(\mathbf{p},\epsilon;k)$ and $e^{-ikt_0}F(\mathbf{p},\epsilon;k)\to 0(1/|k|)$ as $|k|\to\infty$ in the upper and lower half planes, respectively.

It is clear that if we are given the initial and final fields and hence $F(\mathbf{p},\epsilon)$, we can take *any* function $F(\mathbf{p},\epsilon;k)$ which satisfies (4.7a) and the analyticity conditions such that

$$F(\mathbf{p},\epsilon;\epsilon p)\equiv F(\mathbf{p},\epsilon), \tag{4.8}$$

and obtain $\Phi(\mathbf{p},\epsilon;t)$ for $\epsilon=\pm1$ using the second of Eqs. (4.7).

We can find a suitable real source $\Phi(\mathbf{x};t)$ from

$$\Phi(\mathbf{x};t)\equiv\sum_{\epsilon}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi(\mathbf{p},\epsilon;t)d\mathbf{p}, \tag{4.9}$$

where $\Phi(\mathbf{p},\pm1;t)$ is given by (4.7), $\Phi(\mathbf{p},\tau;t)$ is any arbitrary function which satisfies (2.15) and which vanishes identically when $t<t_0$ or $t>t_1$, and

$$\Phi(\mathbf{p},0;t)=(i/p)\partial\Phi(\mathbf{p},\tau;t)/\partial t.$$

We note that there are two types of lack of uniqueness in $\Phi(\mathbf{x};t)$. One type consists in the arbitrary choice of function $\Phi(\mathbf{p},\tau;t)$ which satisfies (2.15), but vanishes identically for times $t<t_0$ and for times $t>t_1$. This arbitrariness corresponds to the possibility of having any arbitrary charge distribution $\rho(x;t)$ for $t_0\leq t\leq t_1$ which would give rise to the longitudinal field given by (3.9).

When this charge distribution is switched off, it will not affect the final transverse field in any way whatever. Thus the final radiation field is given by the sources

$$\Phi(\mathbf{x};t)=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi(\mathbf{p},\epsilon;t)d\mathbf{p}$$

only. To this source we may add Φ^{Arb}, where

$$\Phi^{\mathrm{Arb}}(\mathbf{x};t)=\int\chi(\mathbf{x}|\mathbf{p},\tau)\Phi(\mathbf{p},\tau;t)d\mathbf{p}+\int\chi(\mathbf{x}|\mathbf{p},0)\Phi(\mathbf{p},0;t)d\mathbf{p}$$

which, when expressed in terms of the arbitrary charge density ρ is

$$\begin{aligned}\Phi_0^{\mathrm{Arb}}(\mathbf{x};t)&=\rho(\mathbf{x};t),\\ \Phi_i^{\mathrm{Arb}}(\mathbf{x};t)&=\frac{\partial}{\partial x^i}\frac{1}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\rho(\mathbf{x}';t)}{\partial t}d\mathbf{x}',\quad(i=1,2,3).\end{aligned} \tag{4.10}$$

[It is clear that Φ^{Arb} satisfies the equation of continuity

$$\sum_{i=0}^{3}\frac{\partial\Phi_i^{\mathrm{Arb}}}{\partial x^i}=0,$$

since $\nabla^2(1/4\pi|\mathbf{x}|)=-\delta(\mathbf{x})$.]

The choice of $\rho(\mathbf{x};t)$ and hence also Φ^{Arb} is not an essential lack of uniqueness in the inverse problem because this source does not give rise to transverse fields at *any* time.

The second lack of uniqueness in the inverse problem is far more important. It concerns the choice of function $F(\mathbf{p},\epsilon;k)$ for $\epsilon=\pm1$. Except for the requirement that this function satisfy (4.7a) and (4.8) and the analyticity conditions, it is arbitrary. We shall therefore impose additional conditions which will make the problem unique.

5. THE STATEMENT OF THE INVERSE PROBLEM WHICH LEADS TO UNIQUE SOLUTIONS

It will be convenient to choose the origin of time so that $t_0=-T$ and $t_1=T$ where $T>0$. This can always be done by taking $T=\frac{1}{2}(t_1-t_0)$ and introducing a new time coordinate $t'=t-\frac{1}{2}(t_1+t_0)$. We shall assume the new time coordinate is always used in what follows and drop the prime. We shall prescribe the initial and final fields in the $\mathbf{p}$-representation which are now $\psi(\mathbf{p},\epsilon;-T)$ and $\psi(\mathbf{p},\epsilon;T)$, respectively. Henceforth, whenever ϵ appears it will be restricted to the values $\epsilon=\pm1$.

The Ansatz which will lead to unique sources for *any* choice of initial and final fields is the requirement that the source vector $\Phi(\mathbf{x};t)$ be represented by

$$\Phi(\mathbf{x};t)=\Phi^E(\mathbf{x})h^E(t)+\Phi^U(\mathbf{x})h^U(t), \tag{5.1}$$

where $h^E(t)$ is a prescribed real even function of t and $h^U(t)$ is a prescribed real odd function of t (the superscript U stands for "uneven" and is used instead of O to prevent confusion). Neither $h^E(t)$ nor $h^U(t)$ is allowed to be identically zero. Our statement of the inverse problem is that we shall be able to find unique real 4-component column vectors $\Phi^E(\mathbf{x})$ and $\Phi^U(\mathbf{x})$ which are functions of $\mathbf{x}$ only for any initial and final fields.

Let us first introduce the Fourier transforms of $h^E(t)$ and $h^U(t)$:

$$\begin{aligned} g^E(k)&=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-T}^{+T}h^E(t)e^{ikt}dt,\\ g^U(k)&=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-T}^{+T}h^U(t)e^{ikt}dt. \end{aligned} \tag{5.2}$$

It is easily seen that

$$\begin{aligned} g^{E*}(k)&=g^E(k),\\ g^E(-k)&=g^E(k),\\ g^{U*}(k)&=-g^U(k),\\ g^U(-k)&=-g^U(k). \end{aligned} \tag{5.3}$$

From (4.7) and the relation

$$\Phi(\mathbf{p},\epsilon;t)=\Phi^E(\mathbf{p},\epsilon)h^E(t)+\Phi^U(\mathbf{p},\epsilon)h^U(t), \tag{5.4}$$

we see that

$$F(\mathbf{p},\epsilon;k)=\Phi^E(\mathbf{p},\epsilon)g^E(k)+\Phi^U(\mathbf{p},\epsilon)g^U(k), \tag{5.5}$$

where

$$\begin{aligned} \Phi^{E,U}(\mathbf{p},\epsilon)&=\sum_i\int\chi_i^*(\mathbf{x}|\mathbf{p},\epsilon)\Phi_i^{E,U}(\mathbf{x})d\mathbf{x},\\ \Phi^{E,U}(\mathbf{x})&=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi^{E,U}(\mathbf{p},\epsilon)d\mathbf{p}. \end{aligned} \tag{5.6}$$

The reality conditions on $\Phi^E(\mathbf{x})$ and $\Phi^U(\mathbf{x})$ lead to the requirement that

$$\Phi^{E,U}(-\mathbf{p},\epsilon)=\Phi^{E,U*}(\mathbf{p},\epsilon). \tag{5.7}$$

Equation (4.8), together with (5.3) yields, finally,

$$F(\mathbf{p},\epsilon)=\Phi^E(\mathbf{p},\epsilon)g^E(p)+\epsilon\Phi^U(\mathbf{p},\epsilon)g^U(p). \tag{5.8}$$

Now it will be convenient to introduce $F^E(\mathbf{p},\epsilon)$ and $F^U(\mathbf{p},\epsilon)$ defined by

$$\begin{aligned} F^E(\mathbf{p},\epsilon)&=\tfrac{1}{2}[F(\mathbf{p},\epsilon)+F^*(-\mathbf{p},\epsilon)],\\ F^U(\mathbf{p},\epsilon)&=\tfrac{1}{2}[F(\mathbf{p},\epsilon)-F^*(-\mathbf{p},\epsilon)]. \end{aligned} \tag{5.9}$$

It is clear that

$$\begin{aligned} F(\mathbf{p},\epsilon)&=F^E(\mathbf{p},\epsilon)+F^U(\mathbf{p},\epsilon),\\ F^E(-\mathbf{p},\epsilon)&=F^{E*}(\mathbf{p},\epsilon),\\ F^U(-\mathbf{p},\epsilon)&=-F^{U*}(\mathbf{p},\epsilon). \end{aligned} \tag{5.10}$$

Therefore, using (5.3) and (5.7), we have

$$\begin{aligned} F^E(\mathbf{p},\epsilon)&=\Phi^E(\mathbf{p},\epsilon)g^E(p),\\ F^U(\mathbf{p},\epsilon)&=\epsilon\Phi^U(\mathbf{p},\epsilon)g^U(p). \end{aligned} \tag{5.11}$$

Furthermore, on using (4.6), we find the following solutions for $\Phi^{E,U}(\mathbf{p},\epsilon)$ in terms of the initial and final fields:

$$\begin{aligned} \Phi^E(\mathbf{p},\epsilon)=\frac{-i}{4(2\pi)^{\frac{3}{2}}g^E(p)}&[e^{i\epsilon pT}\psi(\mathbf{p},\epsilon;t)\\ &-e^{-i\epsilon pT}\psi^*(-\mathbf{p},\epsilon;T)-e^{-i\epsilon pT}\psi(\mathbf{p},\epsilon;-T)\\ &+e^{i\epsilon pT}\psi^*(-\mathbf{p},\epsilon;T)], \end{aligned} \tag{5.12}$$

$$\begin{aligned} \Phi^U(\mathbf{p},\epsilon)=\frac{-i\epsilon}{4(2\pi)^{\frac{3}{2}}g^U(p)}&[e^{i\epsilon pT}\psi(\mathbf{p},\epsilon;t)\\ &+e^{-i\epsilon pT}\psi^*(\mathbf{p},\epsilon;-T)-e^{-i\epsilon pT}\psi(\mathbf{p},\epsilon;-T)\\ &-e^{i\epsilon pT}\psi^*(-\mathbf{p},\epsilon;T)]. \end{aligned} \tag{5.12a}$$

Finally, we have the sources

$$\begin{aligned} \Phi(\mathbf{x};t)=&\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi^E(\mathbf{p},\epsilon)d\mathbf{p}h^E(t)\\ &+\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\Phi^U(\mathbf{p},\epsilon)d\mathbf{p}h^U(t). \end{aligned} \tag{5.13}$$

We shall now consider two examples of the inverse problem.

Example 1.—We shall take the field before the sources are switched on to be zero. After the sources are switched off the field is to consist of a circularly polarized wave travelling in the positive x direction with frequency ν. Hence

$$\begin{aligned} &\psi(\mathbf{p},\epsilon;-T)\equiv0,\\ &\psi(\mathbf{p},\epsilon;T)\equiv K\delta(p_x-\nu)\delta(p_y)\delta(p_z)\delta_{\epsilon,+1},\quad \nu>0. \end{aligned} \tag{5.14}$$

Furthermore, we shall take

$$\begin{aligned} h^E(t)&=A\delta(t),\\ h^U(t)&=B\delta'(t). \end{aligned} \tag{5.15}$$

All sufficiently short time $h^E(t)$ and $h^U(t)$ functions can be approximated by the functions given in (5.15). Furthermore, we may take T to be arbitrarily small. In fact we shall take T to be zero *after* the various integrations over time have been performed.

Then, in terms of the $\mathbf{x}$-representation, the field is identically zero for $t<0$ and for $t>0$ is given by [see (3.20)]

$$E_x=0,\qquad H_x=0,$$

$$E_y=\frac{K}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\cos[\nu(x-t)],\quad H_y=\frac{-K}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\sin[\nu(x-t)],$$

$$E_z=\frac{-K}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\sin[\nu(x-t)],\quad H_z=\frac{K}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\cos[\nu(x-t)]. \tag{5.16}$$

The calculations for the sources are quite straightforward and one obtains

$$\Phi^E(\mathbf{x})=\frac{K}{A16\pi^{\frac{5}{2}}}\begin{bmatrix}0\\0\\-\cos\nu x\\\sin\nu x\end{bmatrix},\quad \Phi^U(\mathbf{x})=\frac{K}{B16\pi^{\frac{5}{2}}\nu}\begin{bmatrix}0\\0\\\sin\nu x\\\cos\nu x\end{bmatrix}, \tag{5.17}$$

and

$$\Phi(\mathbf{x};t)=\frac{K}{16\pi^{\frac{5}{2}}}\begin{bmatrix}0\\0\\-\cos\nu x\delta(t)+\nu^{-1}\sin\nu x\delta'(t)\\\sin\nu x\delta(t)+\nu^{-1}\cos\nu x\delta'(t)\end{bmatrix}. \tag{5.18}$$

One can add to this source the arbitrary charge distribution $\rho(\mathbf{x};t)$ which we may choose, if we wish, to take the form

$$\rho(\mathbf{x};t)=\rho(\mathbf{x})\delta(t),$$

where $\rho(\mathbf{x})$ is arbitrary. This gives rise to the arbitrary additional sources

$$\Phi^{\mathrm{Arb}}(\mathbf{x};t)=\begin{bmatrix}\rho(\mathbf{x})\delta(t)\\ \frac{1}{4\pi}\frac{\partial}{\partial x}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'\delta'(t)\\ \frac{1}{4\pi}\frac{\partial}{\partial y}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'\delta'(t)\\ \frac{1}{4\pi}\frac{\partial}{\partial z}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'\delta'(t)\end{bmatrix}. \tag{5.19}$$

Having obtained the sources, we can now obtain the fields for all time using techniques given in Sec. 3. The transverse field is given by

$$\psi^T(\mathbf{x};t)=\frac{K}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\times\left\{\eta(t)e^{i\nu(x-t)}\begin{bmatrix}0\\0\\-i\\1\end{bmatrix}+i\nu^{-1}\delta(t)\begin{bmatrix}0\\0\\\sin\nu x\\\cos\nu x\end{bmatrix}\right\}, \tag{5.20}$$

where $\eta(t)$ is the Heaviside function:

$$\begin{aligned}\eta(t)&=0\quad\text{for}\quad t<0\\&=1\quad\text{for}\quad t>0.\end{aligned} \tag{5.20a}$$

The longitudinal field is

$$\psi^L(\mathbf{x};t)=i\begin{bmatrix}0\\\partial/\partial x\\\partial/\partial y\\\partial/\partial z\end{bmatrix}\int\frac{\rho(x')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'\delta(t). \tag{5.21}$$

Example 2.—In the present example we shall study the case in which the time dependence of the sources is the same as before and where we again require the initial field to be identically zero. In contrast to the previous example, however, we shall require the final transverse field to be highly concentrated in space immediately after the sources are switched off. We shall then see that as long as the sources are on, they will also be highly concentrated in space.

It would be nice to consider the field immediately after the switchoff to be

$$\psi^A(\mathbf{x})=R\delta(\mathbf{x}), \tag{5.22}$$

where R is the four-component vector

$$R=\begin{bmatrix}0\\R_1\\R_2\\R_3\end{bmatrix}, \tag{5.22a}$$

R_i being any complex number $R_j=L_j+iM_j$. However, ψ^A is not suitable for a final field because it is not purely transverse. Hence we shall subtract the part whose divergence is not zero.

We may expand $\psi^A(\mathbf{x})$ as

$$\psi^A(\mathbf{x})=\sum_{\epsilon=\pm1}\int\chi(\mathbf{x}|\mathbf{p},\epsilon)\psi^A(\mathbf{p},\epsilon)d\mathbf{p}+\int\chi(\mathbf{x}|\mathbf{p},0)\psi^A(\mathbf{p},0)d\mathbf{p}. \tag{5.23}$$

Now, the part for which the divergence does not vanish is

$$\psi^{AL}(\mathbf{x})=\int\chi(\mathbf{x}|\mathbf{p},0)\psi^A(\mathbf{p},0)d\mathbf{p}. \tag{5.24}$$

Moreover, since

$$\psi^A(\mathbf{p},0)=\sum_i\int\chi_i^*(\mathbf{x}|\mathbf{p},0)\psi_i^A(\mathbf{x})d\mathbf{x} \tag{5.25}$$

we have, on using the explicit form of $\chi_i(\mathbf{x}|\mathbf{p},0)$,

$$\begin{aligned}\psi_i^{AL}(\mathbf{x})&=\sum_j\int d\mathbf{x}'\psi_j^A(\mathbf{x}')\int\chi_i(\mathbf{x}|\mathbf{p},0)\chi_j^*(\mathbf{x}'|\mathbf{p},0)d\mathbf{p}\\&=-\frac{1}{4\pi}\sum_{j=1}^{3}R_j\frac{\partial^2}{\partial x^i\partial x^j}\frac{1}{|\mathbf{x}|},\quad(i=1,2,3).\end{aligned} \tag{5.26}$$

Hence, we shall take as the field immediately after the sources are switched off,

$$\psi_i = R_i\delta(\mathbf{x}) + \frac{1}{4\pi}\sum_{j=1}^{3} R_j \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{|\mathbf{x}|}, \quad (i=1,2,3) \tag{5.27}$$

which is still highly localized near the origin and is now a purely transverse wave.

The sources which give rise to (5.27) are easily calculated:

$$\Phi_0^E(\mathbf{x}) \equiv 0,$$

$$\Phi_i^E(\mathbf{x}) = \frac{1}{4\pi A}\left[M_i\delta(\mathbf{x}) + \frac{1}{4\pi}\sum_{j=1}^{3} M_j \frac{\partial^2}{\partial x^i\partial x^j}\frac{1}{|\mathbf{x}|}\right], \tag{5.28}$$

$$(i=1,2,3)$$

$$\Phi_0^U(\mathbf{x}) = 0,$$

$$\begin{aligned}
\Phi_1^U(\mathbf{x}) &= \frac{1}{(4\pi)^2 B}\left[L_3\frac{\partial}{\partial x^2} - L_2\frac{\partial}{\partial x^3}\right]\frac{1}{|\mathbf{x}|},\\
\Phi_2^U(\mathbf{x}) &= \frac{1}{(4\pi)^2 B}\left[L_1\frac{\partial}{\partial x^3} - L_3\frac{\partial}{\partial x^1}\right]\frac{1}{|\mathbf{x}|},\\
\Phi_3^U(\mathbf{x}) &= \frac{1}{(4\pi)^2 B}\left[L_2\frac{\partial}{\partial x^1} - L_1\frac{\partial}{\partial x^2}\right]\frac{1}{|\mathbf{x}|}.
\end{aligned} \tag{5.28a}$$

In addition to these sources, one can add the arbitrary sources given by (5.19).

For $t>0$, it is easy to show that the field is given by

$$\psi_i(\mathbf{x};t) = \sum_j G_{ij}(\mathbf{x};t|0;0)R_j, \tag{5.29}$$

where G is Green's function given by (3.16).

APPENDIX. DERIVATION OF THE ENERGY CONSERVATION LAW FROM THE SPINOR FORM

One can derive the energy conservation law for Maxwell's equations in a manner similar to the derivation of the equation of continuity in Dirac's equations.

Let us define two different types of inner products of column vectors. Consider two vectors $A(\mathbf{x})$ and $B(\mathbf{x})$ which are given by

$$A(\mathbf{x}) = \begin{bmatrix} A_0(\mathbf{x}) \\ A_1(\mathbf{x}) \\ A_2(\mathbf{x}) \\ A_3(\mathbf{x}) \end{bmatrix}, \quad B(\mathbf{x}) = \begin{bmatrix} B_0(\mathbf{x}) \\ B_1(\mathbf{x}) \\ B_2(\mathbf{x}) \\ B_3(\mathbf{x}) \end{bmatrix}. \tag{1}$$

In (1) and later, $\mathbf{x}$ represents collectively the three-dimensional space coordinates. The first type of inner product is the Hermitian inner product in finite-dimensional vector space.

$$A(\mathbf{x})\cdot B(\mathbf{x}) = \sum_{i=0}^{3} A_i^*(\mathbf{x})B_i(\mathbf{x}), \tag{2}$$

where the asterisk means the complex conjugate. The second inner product is the Hermitian inner product used in Hilbert space:

$$(A,B) = \sum_{i=0}^{3}\int A_i^*(\mathbf{x})B_i(\mathbf{x})d\mathbf{x} = \int A(\mathbf{x})\cdot B(\mathbf{x})d\mathbf{x}. \tag{3}$$

The operators α^i are Hermitian with respect to both inner products.

Let us write

$$\begin{aligned}
A(\mathbf{x}) &= \frac{i}{8\pi}\psi(\mathbf{x}),\\
B(\mathbf{x}) &= \frac{-1}{i}\sum_{i=0}^{3}\alpha^i\frac{\partial}{\partial x^i}\psi.
\end{aligned} \tag{4}$$

Because of Maxwell's equations (1.4), we have

$$\begin{aligned}
(A,B) &= \frac{1}{8\pi}\sum_{j=0}^{3}\int \psi(\mathbf{x})\cdot\alpha^j\frac{\partial}{\partial x^j}\psi(\mathbf{x})d\mathbf{x}\\
&= \tfrac{1}{2}i\int \psi(\mathbf{x})\cdot\Phi(\mathbf{x})d\mathbf{x}.
\end{aligned} \tag{5}$$

Also

$$\begin{aligned}
(B,A) &= \frac{1}{8\pi}\sum_{j=0}^{3}\int \alpha^j\frac{\partial}{\partial x^j}\psi(\mathbf{x})\cdot\psi(\mathbf{x})d\mathbf{x}\\
&= -\tfrac{1}{2}i\int \Phi(\mathbf{x})\cdot\psi(\mathbf{x})d\mathbf{x}.
\end{aligned} \tag{6}$$

Since the operators α^i are Hermitian, Eq. (6) becomes

$$\frac{1}{8\pi}\sum_{j=0}^{3}\int \frac{\partial}{\partial x^j}\psi(\mathbf{x})\cdot\alpha^j\psi(\mathbf{x})d\mathbf{x} = -\tfrac{1}{2}i\int\Phi(\mathbf{x})\cdot\psi(\mathbf{x})d\mathbf{x}. \tag{7}$$

On adding (5) and (7), we obtain

$$\begin{aligned}
&\sum_{i=0}^{3}\int\frac{\partial}{\partial x^i}S^i(\mathbf{x})d\mathbf{x}\\
&= \frac{\partial}{\partial t}\int S^0(\mathbf{x})d\mathbf{x} + \sum_{i=1}^{3}\int\frac{\partial}{\partial x^i}S^i(\mathbf{x})d\mathbf{x} = \int P(\mathbf{x})d\mathbf{x},
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
S^i(\mathbf{x}) &= \frac{1}{8\pi}\psi(\mathbf{x})\cdot\alpha^i\psi(\mathbf{x}), \quad (i=0,1,2,3)\\
P(\mathbf{x}) &= \mathrm{Re}\, i\psi(\mathbf{x})\cdot\Phi(\mathbf{x}) = \sum_{i=1}^{3} E_i j_i
\end{aligned} \tag{9}$$

where Re means "real part." It is easy to see that $P(\mathbf{x})$ is the time rate of change at which the electric field does work per unit volume on the sources. Hence $S^0=(1/8\pi)\sum_i \psi_i^*\psi_i$ is the energy density and the components S^i ($i=1, 2, 3$) are the components of the Poynting vector. Since the field ψ is arbitrary, we can strip off the integrals in (9) and obtain the familiar differential form of the conservation of energy:

$$\partial S^0/\partial t+\mathrm{div}\mathbf{S}=\mathbf{E}\cdot\mathbf{j}. \tag{10}$$

Various other conservation laws can be obtained in an analogous way.

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Symmetry Laws and Strong Interactions*

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An attempt is made to explore the possible connection between symmetry laws in internal space (e.g., isospin space) and symmetry laws in Lorentz space with special attention to the question: Why are the strong interactions parity-conserving? For direct (non-derivative-type) pion-nucleon interactions, CP invariance and charge independence are sufficient to guarantee the separate conservation of P and C, as previously pointed out. For derivative-type pion-nucleon interactions, charge independence and G invariance (rotational and inversion invariance in three-dimensional isospin space) require that parity (and CP) be conserved; in addition we can also show that the charge-triplet pion must be pseudoscalar, provided that the virtual Yukawa process $\pi^0 \rightleftarrows p+\bar{p}$ is allowed or, equivalently, the π^0 can be regarded as a bound state of a proton and an antiproton as far as symmetry laws are concerned. For the K couplings, analogous conditions cannot be obtained from the usual assumption of charge independence alone. However, if the K couplings (rather than the π couplings) exhibit a higher internal symmetry in the sense that the K couplings are universal, the high K symmetry plus charge independence in the usual sense imply parity conservation both in the case of CP-invariant nonderivative-type K interactions and in the case of G-invariant derivative-type K interactions. The high K symmetry also implies that the relative $N\Xi$ parity as well as the relative $\Lambda\Sigma$ parity is even. It is conjectured that, if the K couplings must be of a derivative type, only *ps-pv* coupling is allowed, which means that the K particle is pseudoscalar. The global symmetry model which cannot be reconciled with our assumption of the high K symmetry is re-examined. The high K symmetry is destroyed in a specific and definite manner by the π couplings, and relations among the various coupling constants are inferred from the baryon mass spectrum. Some empirical implications of our model are discussed. Whereas G invariance requires the symmetric appearance of the two chiral spinors $\frac{1}{2}(1+\gamma_5)\psi$ and $\frac{1}{2}(1-\gamma_5)\psi$ for strangeness-conserving processes, for strangeness-nonconserving processes G conjugation carries charge-conserving interactions into inadmissible interactions that do not conserve electric charge. Hence, if we take the point of view that parity-conserving interactions are generated by G conjugation, we have some understanding of the puzzling fact that strangeness conservation and parity conservation have the same domain of validity. Further theoretical speculations are made.

I.

RECENTLY some progress has been made in our understanding of weak interactions. With the empirical observation of a statistically well-established asymmetry in the decay of Λ particles[1] and with the advent of the universal VA theory which accounts for parity nonconservation in weak processes regardless of whether or not neutrinos are involved,[2–4] the original "puzzle" that arose from the curious behavior of the pionic decay modes of K particles has largely disappeared. Yet there remain deeper (and perhaps more difficult) questions unanswered: Why do baryons and mesons interact sometimes strongly and sometimes weakly? "Why are the strong interactions parity-symmetric,"[5] or, more specifically, why can't we insert $1+\gamma_5$ for the strangeness-conserving $[p,\Lambda^0,K^+]$ interaction? Why are the parity-conserving interactions 10^{11} to 10^{14} times stronger than the parity-nonconserving interactions?

It is not at all evident to us now whether the present (unsatisfactory) quantum field theory of elementary particles is capable of coping with these formidable questions. Yet we cannot help but be struck by the empirical facts that strongly interacting particles possess internal degrees of freedom such as isospin and strangeness that leptons do not seem to possess; that symmetry laws concerning these internal degrees of freedom are approximate, just as the "law" of the conservation of parity is approximate; and that the conservation of strangeness (or equivalently the conservation of I_3) seems to have the same domain of validity as the conservation of parity for those interactions that involve only strongly interacting particles. From these empirical

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[1] Crawford, Cresti, Good, Gottstein, Lyman, Solmitz, Stevenson, and Ticho, Phys. Rev. **108**, 1102 (1957); F. Eisler *et al.*, Phys. Rev. **108**, 1353 (1957).
[2] E. C. G. Sudarshan and R. E. Marshak, Suppl. Nuovo cimento (to be published).
[3] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[4] J. J. Sakurai, Nuovo cimento **7**, 649 (1958).
[5] W. Pauli (private communication to V. Weisskopf, January, 1957).