must be applied to discover just what happens in this case, and how the extension is to be made into the unphysical region.

Because of the small numerical value of the constant coefficients of higher order terms in the Heisenberg-Euler Lagrangian density, these effects are too small to be susceptible to experimental test.[4] The primary importance of our investigation is to point out theoretical methods for the investigation of these nonlinear effects, and to give an example of the kind of phenomenon which may occur. It is hoped that such examples will provide some insight into more realistic situations, and be of some suggestive value in the consideration of the nonlinear aspects of quantum field theories.

V. ACKNOWLEDGMENTS

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Note added in proof.—The question has been raised as to whether the singularities which occur in the solution of the equation we have treated are due to special properties of the equation, or whether any nonlinear hyperbolic equation possesses singular solutions. We have been unable to determine whether there exists a nonlinear hyperbolic equation having the property that for *no* initial conditions does the solution exhibit singularities. Therefore we regard this investigation as merely demonstrating how, for a particular physical situation, singularities may occur, and how they are formed in this case. In this connection, we note that an investigation of the singularities occurring in the solution of another type of nonlinear wave equation has been made by J. B. Keller, Comm. Pure Appl. Math. **10**, 523 (1957).

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Radiative Corrections to Fermi Interactions*

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The radiative correction to the decay spectrum of polarized muons is recalculated taking into account a mistake in our previous work which was recently pointed out by Berman. The revised values for the radiative correction to δ, ξ, and integrated asymmetries for the high- as well as low-energy decay electrons have turned out to be practically identical with the old values. The ρ value determined from experiments, on the other hand, has to be increased by about 1% because of the new correction. Thus the over-all effect of the radiative correction to the ρ value is now an increase of the order of 5.6% when the experimental and theoretical spectral distributions are compared in the region $0\leq p/p_{\max}\leq 0.95$. The radiative corrections to the spectrum and lifetime of the nuclear β decay arising from the charge interactions of the electron and proton are also studied. Use of this expression gives a correction of -1.7% for the lifetime of O^{14}. The corrected Feynman–Gell-Mann coupling constant is $G=(1.40\pm0.01)\times10^{-49}$ erg/cm^3. In the universal $V-A$ theory of weak interactions, the calculated muon mean life becomes $\tau_\mu=(2.31\pm0.05)\times10^{-6}$ sec. (These three values depend logarithmically on the ultraviolet cutoff λ and the corrections to τ_μ increase for increasing values of λ.) It is found that the corrections to the spectral shape of β decay are rather large in the case in which the end-point energy $E_m\gg m_ec^2$. The radiative corrections to the lifetime and the total asymmetry for muon decay are found to be well defined and finite for $m_e\to0$ in spite of the fact that the differential spectrum itself diverges logarithmically in the same limit. The same situation is encountered in the case of radiative corrections to the nuclear β decay. A physical explanation for such behavior of the radiative corrections is attempted. In Appendix A, a simplified expression is given for the determination of the Michel parameter.

1. INTRODUCTION

THE lowest order radiative correction to the decay of polarized muons have been studied in previous papers.[1–3] In the present work, the functions which determine the various corrections are reconsidered, taking into account a mistake in the treatment of the low-energy quanta, recently pointed out by Berman.[4] In Sec. 2, the results of this calculation and their effects on the parameters δ, ρ, ξ, the lifetime, and the integrated asymmetries for high- as well as low-energy electrons are discussed.

As a consequence of this recalculation, the ρ value is increased by an additional amount of about 1%, the

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[1] Behrends, Finkelstein, and Sirlin, Phys. Rev. **101**, 866 (1956). This paper will be quoted as I.

[2] T. Kinoshita and A. Sirlin, Phys. Rev. **107**, 593 (1957). This paper will be quoted as II.

[3] T. Kinoshita and A. Sirlin, Phys. Rev. **107**, 638 (1957). This paper will be quoted as III.

[4] S. M. Berman, Phys. Rev. **112**, 267 (1958).

over-all effect of the radiative corrections on the ρ value being now of the order of 5.6% in the spectral region $0\leq p/p_{max}\leq 0.95$. It is found that the effect of this further correction on the parameters ξ, δ, and the momentum dependence of the asymmetry is small. On the other hand, the corrections to the lifetime are affected considerably.

It is observed that the lowest order radiative corrections to the lifetime and the total integrated asymmetry for muon decay are finite and well-defined in the limit of vanishing electron mass m_e in spite of the fact that the differential spectrum itself diverges in the same limit, since it involves terms proportional to $\alpha \ln(m_\mu/m_e)$. The same situation is found in the case of the nuclear β decay as is discussed in Sec. 4. This behavior of the differential and integrated spectra is examined in Sec. 3. In Sec. 4 the corrections to the lifetime and spectrum of nuclear β decay arising from the charge interactions of the electron and proton are studied.

A simplified theoretical expression for the corrected Michel spectrum is given in Appendix A which may serve to facilitate the comparison of theory and experiment. In Appendix B, the contribution of that part of the radiative corrections that can be interpreted as caused by an acceleration of a classical charge-current is discussed.

2. RADIATIVE CORRECTIONS TO THE DECAY SPECTRUM OF POLARIZED MUON

Let us consider the effect of radiative corrections to the decay spectrum of completely polarized muons. We shall restrict ourselves to the case of the two-component neutrino theory,[5] where only the V and A interactions are present in charge retention order [i.e., in the order $(e\mu)(\nu\nu)$]. According to (II.3.3), the decay spectrum of completely polarized muons is given, to order α, by

$$dN(x,\theta)=\tfrac{1}{2}A\left\{3-2x+\frac{\alpha}{2\pi}f(x)+6\zeta\left(\frac{m_e}{m_\mu}\right)\frac{1-x}{x}+\xi\cos\theta\left[1-2x+\frac{\alpha}{2\pi}g(x)\right]\right\}x^2dxd\Omega, \quad (2.1)$$

where $x=2p_e/m_\mu$ and θ is the angle between the electron and muon momenta.[6] The constants A, ξ, and ζ are defined by (II.2.5), (II.2.33), and (II.3.4), respectively. The term proportional to m_e/m_μ is comparatively small and will be neglected in most of the following considerations (see reference 11). It vanishes exactly in the case of the $V\pm A$ interactions. The effect of the radiative correction is represented by the functions $f(x)$ and $g(x)$.

As was pointed out in reference 4, the previous calculation of $f(x)$ suffered from a mistake in the treatment of the inner bremsstrahlung accompanying the muon decay. The same is true of the function $g(x)$. This may be traced back to the evaluation of the integral

$$I=\frac{1}{4\pi}\sum_i\int_0^{k_{max}}\frac{d^3k}{\epsilon}\left[\frac{(p_2\cdot e_i)}{(p_2\cdot\kappa)}-\frac{(p_1\cdot e_i)}{(p_1\cdot\kappa)}\right]^2, \quad (2.2)$$

where p_1, p_2, and κ are the four-momenta of muon, electron, and photon, respectively, and the e_i's are the polarization vectors of the photon.

This integral was evaluated before in a manner that was inconsistent with the treatment of the virtual quanta. A consistent procedure is to regard κ as the four-momentum of a vector meson of mass λ_{min} and energy ϵ, and sum over all directions of polarization of the vector mesons.[7,8] In Appendix C, the integral I is evaluated in the rest system of particle 1 and the result is shown to be equal to the previous one plus a correction term which is independent of k_{max}. From this one readily finds that the corrected result for any of the five Fermi interactions is obtained if one replaces V of (I.25b) by $V+C$, where

$$C=C(x)=1-\tfrac{1}{6}\pi^2-(\omega+\ln x-1)(\omega+\ln x-2\ln 2), \quad (2.3)$$

with $\omega=\ln(m_\mu/m_e)=5.332$. Adding the virtual photon term to the inner bremsstrahlung term, one thus arrives at the results[9]

$$f(x)=(6-4x)R(x)+(6-6x)\ln x+\frac{1-x}{3x^2}\times[(5+17x-34x^2)(\omega+\ln x)-22x+34x^2], \quad (2.4)$$

and

$$g(x)=(2-4x)R(x)+(2-6x)\ln x-\frac{1-x}{3x^2}\left[(1+x+34x^2)(\omega+\ln x)+3-7x-32x^2+\frac{4(1-x)^2}{x}\ln(1-x)\right], \quad (2.5)$$

in the case of the two-component theory where

$$R(x)=2\sum_{n=1}^{\infty}\frac{x^n}{n^2}-\tfrac{1}{3}\pi^2-2+\omega\left[\tfrac{3}{2}+2\ln\left(\frac{1-x}{x}\right)\right]-\ln x(2\ln x-1)+\left(3\ln x-1-\frac{1}{x}\right)\ln(1-x). \quad (2.6)$$

[5] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957); A. Salam, Nuovo cimento **5**, 299 (1957); L. Landau, Nuclear Phys. **3**, 127 (1957).

[6] With this definition of θ, (2.1) is valid for both the π^+-μ^+-e^+ and π^--μ^--e^- sequences.

[7] An alternative way is to treat the real and virtual photons as exactly massless. However, extreme care must then be taken in order to avoid improper cancellation of the infrared divergences.

[8] See, for example, F. Coester, Phys. Rev. **83**, 798 (1951), and J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1956) (especially Secs. 6–5 and 15–2).

[9] We have heard recently that Berman's result for the spectrum agrees exactly with our formula (2.4) [S. M. Berman (private communication)]. This fact was temporarily obscured by the presence of a spurious term in Eq. (4) of the initial version of reference 4. We are grateful to Mr. Berman for his private communication.

TABLE I. Radiative corrections to the isotropic and cosθ terms of the muon decay spectrum and related functions.

x	$10^2(\alpha/2\pi)f(x)$	$10^2(\alpha/2\pi)F(x)$	$10^2(\alpha/2\pi)g(x)$	$10^2(\alpha/2\pi)G(x)$	$10^2 h_{V,A}(x)$	$O_r(x)/O(x)$	$a_r(x,\ -1,\ 0)$
0	...	−0.417	...	−0.681	...	0.9974	0
0.1	69.177	−0.505	−7.727	−0.709	24.706	0.9980	0.170
0.2	25.219	−0.678	−3.677	−0.773	9.700	0.9992	0.201
0.3	13.301	−0.900	−3.116	−0.897	5.542	1.0007	0.181
0.4	7.564	−1.145	−2.965	−1.121	3.438	1.0021	0.142
0.5	4.027	−1.370	−2.746	−1.470	2.014	1.0031	0.092
0.6	1.540	−1.531	−2.310	−1.931	0.856	1.0035	0.033
0.7	−0.368	−1.573	−1.555	−2.426	−0.230	1.0034	−0.038
0.8	−1.981	−1.436	−0.327	−2.751	−1.415	1.0027	
0.9	−3.681	−1.027	1.799	−2.463	−3.067	1.0016	
0.95	−5.009	−0.659	3.748	−1.769	−4.554	1.0009	
0.99	−8.012	−0.198	7.623	−0.580	−7.855	1.0003	
1.0	...	0	...	0	...	...	

The new function $R(x)$ differs from the old function $u(x)$ defined in II by $C(x)$ of (2.3). It is to be noted that the function $R(x)$ depends on ω only linearly whereas $u(x)$ depends on it quadratically.

For later use, we shall define the functions

$$F(x)=2\int_x^1 f(x)x^2dx,$$
$$G(x)=-6\int_x^1 g(x)x^2dx. \tag{2.7}$$

The functions $f(x)$, $g(x)$, $F(x)$, and $G(x)$ are tabulated in Table I for several values of x.

We shall now proceed to discuss some aspects of the revised spectrum (2.1).

(a) Michel Parameter

It was shown in II that the isotropic part of the spectrum (II.3.6) can be approximated very well by an uncorrected Michel formula in the range $0.3\lesssim x\lesssim 0.95$. This still holds for our new spectrum (2.1). Because of this, the normalized isotropic spectrum

$$S(x)=\frac{2}{1+(\alpha/2\pi)F(0)}\left(3-2x+\frac{\alpha}{2\pi}f(x)\right) \tag{2.8}$$

may be approximated by

$$S_1(x)=12(1-x)+8\rho_{\text{eff}}(\tfrac{4}{3}x-1), \tag{2.9}$$

which defines the effective Michel parameter ρ_{eff}. These functions are normalized by[10]

$$\int_0^1 S(x)x^2dx=\int_0^1 S_1(x)x^2dx=1. \tag{2.10}$$

Our problem is to find such a value for ρ_{eff} that $S(x)$ is most closely approximated by the linear function $S_1(x)$. If one applies the method described in II, one obtains $\rho_{\text{eff}}=0.71_0$ as the best fit for the range $0.3\lesssim x\lesssim 0.95$. As an alternative, however, let us determine ρ_{eff} here by fitting $x^2S(x)$ with $x^2S_1(x)$ by means of the method of least squares. In the range $0\leq x\leq 0.95$, the best least-squares fit is then obtained for

$$\rho_{\text{eff}}=0.70_8, \tag{2.11}$$

which is very close to the value mentioned above.[11] This is consistent with the observation of II that $S(x)$ is almost linear over a large region of electron momentum x. We have excluded the interval $0.95<x$ in the determination of ρ_{eff} in order to give a value which is fairly independent of the details of the experimental resolution. The consideration of the experimental resolution is necessary in that region on account of the fact that the corrections to the differential spectrum diverge logarithmically as $x\rightarrow 1$.

If the local two-component theory of the neutrino is exactly correct, one would therefore observe the Michel spectrum (2.9) characterized by (2.11) in the range $x\lesssim 0.95$ instead of the uncorrected value $\rho=0.75$. Thus the radiative correction to the ρ value is of the order of 5.6% if the observation is limited to that spectral region.[12]

If one applied the above method to the old spectrum (I.23), one would have obtained $\rho_{\text{eff}}=0.71_6$ [i.e., an effect of 4.5% instead of (2.11)].[10] Thus the new value of the radiative correction to the Michel parameter is larger than the old value by about 1%. In spite of the rather large numerical difference between $f(x)$ of (2.2) and (II.A.1), the difference in ρ_{eff} is therefore not very

[10] In our previous paper (reference 2), the value of ρ_{eff} was chosen as 0.70_6. This was caused by a failure to normalize the spectrum in a proper manner. The same remark applies to our previous determination of δ_{eff} (which was 0.74_6).

[11] In obtaining Eq. (2.11), the term of order m_e/m_μ of Eq. (2.1) has been neglected. It is estimated that if one uses for ζ the maximum value allowed by the theoretical relation $\zeta\leq(1-\xi^2)^{\frac{1}{2}}$ and the present experimental lower limit $\xi>0.75$, this term could increase the correction to the ρ value by about 0.7%. However, such a value of ζ would require a value of $|g_A|^2$ considerably larger than $|g_V|^2$ contrary to what is now believed.

[12] The latest experimental values of the Michel parameter ρ, with the Berman correction included, are 0.68±0.05 [L. Rosenson, Phys. Rev. **109**, 958 (1958)], 0.69±0.02 [K. M. Crowe (private communication)], and 0.741±0.022 [Dudziak, Sagane, and Vedder, University of California Radiation Laboratory Report UCRL-8202, and W. F. Dudziak, UCRL-8202 Supplement (unpublished)]. These values should be compared with the uncorrected value $\rho=0.75$.

large. This is because the *main* contribution to the difference of old and new spectra is approximately equal to

$$(\alpha/2\pi)(6-4x)\{1-\tfrac{1}{6}\pi^2-(\omega-1)(\omega-2\ln 2)\}, \quad (2.12)$$

except for $x\approx 0$. Since this has the same momentum dependence as the uncorrected spectrum, it modifies the Michel parameter only slightly. On the other hand, the contribution of (2.12) has an appreciable effect on the radiative correction to the lifetime.

(b) Radiative Correction to the Lifetime

The mean life τ of the muon decay is obtained by integrating (2.1) over all possible values of x and Ω. If one denotes the zeroth order mean life by τ_0, the fractional change of the lifetime due to the radiative correction is given by[13]

$$\frac{\tau-\tau_0}{\tau_0}=-\frac{\alpha}{2\pi}F(0)=-\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\approx 4.2\times 10^{-3}. \quad (2.13)$$

Thus the radiative correction suppresses the rate of muon decay by about 0.4%. This should be compared with the 3% increase of the decay rate in the previous calculation. It is not understood why the radiative correction suppresses rather than enhances the decay rate. A physical explanation of this fact would be desirable.

It is interesting to point out that the lifetime is independent of the quantity ω and hence approaches a finite value in the limit $m_e\to 0$. On the other hand, the differential spectrum (2.1) diverges logarithmically for $m_e\to 0$. This is a consequence of the breakdown of perturbation theory in this limit. These points are discussed in more detail in Sec. 3.

(c) Correction to the Parameter δ

The parameter δ may be determined in principle by measuring the momentum dependence of $dN(x,\theta)-dN(x,\theta+\pi)$. The magnitude of the radiative correction to δ can be estimated in analogy with (a), by approximating the function

$$T(x)=-\frac{6}{1+(\alpha/2\pi)G(0)}\left(1-2x+\frac{\alpha}{2\pi}g(x)\right), \quad (2.14)$$

by means of

$$T_1(x)=12(1-x)+24\delta_{\mathrm{eff}}(\tfrac{4}{3}x-1). \quad (2.15)$$

Note that these functions are normalized by[10]

$$\int_0^1 T(x)x^2dx=\int_0^1 T_1(x)x^2dx=1. \quad (2.16)$$

The best least-squares fit of $x^2T(x)$ by $x^2T_1(x)$ is obtained by the choice

$$\delta_{\mathrm{eff}}=0.72_6, \quad (2.17)$$

for the range $x\leq 0.95$. The uncorrected two-component neutrino theory gives $\delta=0.75$. Thus the radiative correction decreases δ by about 3% if the observation is restricted to the range $x\leq 0.95$. If one applied this method to the old spectrum (II.3.9), one should have found $\delta_{\mathrm{eff}}=0.73_0$.[10] Thus there is no significant change in the corrected value δ_{eff}.

(d) Momentum Dependence of the Integrated Asymmetry

The integrated asymmetry corresponding to electrons of energy larger than a certain value $p_e=xm_\mu/2$ has been discussed in II. The new values for the ratio $Q_r(x)/Q(x)$ (see II) is given in Table I. These corrections are very small and may be neglected in view of the accuracy of the present experiments.

(e) Correction to the Parameter ξ

The parameter ξ, which is defined theoretically in terms of the weak coupling constants, is determined from the *observed* total integrated asymmetry a by the relation

$$\xi=-3a\left(\frac{1+(\alpha/2\pi)F(0)}{1+(\alpha/2\pi)G(0)}\right). \quad (2.18)$$

From Table I, it is seen that the effect of the radiative correction is to increase ξ by only 0.3% with respect to its uncorrected value. This has the same magnitude as the old value given in II but the direction is reversed. We again notice that $G(0)$ is independent of ω and thus the lowest order radiative correction to ξ remains finite in the limit $m_e\to 0$.

(f) Forward-Backward Asymmetry of Low-Energy Positrons

The lowest order radiative correction to the asymmetry of low-energy positrons in μ^+-decay is very large. This was studied in III and was used in comparing the theory with a recent experiment.[14] Let us express the angular distribution of the decay positrons of energy less than x by $1+a(x)\cos\theta$. The theoretical expression for $a(x)$, including the lowest order radiative correction, is given by the function $a_r(x,\xi,\zeta)$ defined in III. We shall give the new values for the function $a_r(x,-1,0)$ in Table I. Comparing the present table of $a_r(x,-1,0)$ with that of III, it is seen that they coincide within 2%. This difference is much smaller than the over-all effect of the radiative correction and thus the results of III are not affected by the modification of the functions $F(x)$ and $G(x)$.

[13] This numerical result agrees with Berman's (reference 4).

[14] Pless, Brenner, Williams, Bizzarri, Hildebrand, Milburn, Shapiro, Strauch, Street, and Young, Phys. Rev. **108**, 159 (1957).

3. RADIATIVE CORRECTIONS IN THE LIMIT $m_e/m_\mu \to 0$

As is seen from Eqs. (2.4) and (2.5), the differential spectrum (2.1) contains terms depending on $\omega = \ln(m_\mu/m_e)$. As a consequence, if one considers a hypothetical problem in which the ratio m_e/m_μ is arbitrarily close to zero, the spectrum approaches $+\infty$ and $-\infty$ for $x \lesssim 0.7$ and $x \gtrsim 0.7$, respectively.[15] This means that the lowest order perturbation theory is not adequate even for small α when the ratio m_e/m_μ becomes very small, the relevant expansion parameter being of the order $\alpha \ln(m_e/m_\mu)$. In this limit, higher order processes, such as multiple photon emission and creation of electron-positron pairs, may become important.

At first hand, this situation may look similar to that of the infrared catastrophe, since both divergences are associated with vanishing of the mass of a particle. There is the important difference, however, that in the present case there is nothing that compensates this divergence in the same order of perturbation theory. Moreover, the divergence of Eq. (2.1) occurs for all values of the electron momentum whereas the usual infrared divergence is associated with the low-energy quanta only.

The appearance of the ω term itself is obviously related to the nature of the electron propagation function. In fact, the divergence of the lowest order corrections to the differential spectrum in the limit $m_e \to 0$ is a consequence of the vanishing of the denominator of the electron propagator when the electron and photon have no mass and their momenta are parallel. It is therefore not entirely trivial that the lowest order corrections to the lifetime turn out to be independent of m_e and approach a finite value for $m_e \to 0$.

In order to gain insight in the details of the cancellation of the terms involving ω, it is instructive to perform the integrations over the photon and electron momenta in a somewhat different manner from that of Sec. 2. For simplicity, we shall discuss the isotropic part only. Let us denote the virtual and real photon corrections to the muon spectrum by $dN_v(p_e)$ and $dN_r(\mathbf{p}_e,\mathbf{k})$, respectively, where the latter has not been integrated yet with respect to the photon momentum. If one integrates dN_r over all possible values of $\mathbf{k}$ and adds it to dN_v, one obtains the differential spectrum (2.1). Further integration with respect to p_e from 0 to p_e^0 gives the quantity $F(0) - F(x_0)$ (apart from a numerical factor) where $F(x)$ is defined by (2.7) and $x_0 = 2p_e^0/m_\mu$. Instead of this procedure, we shall divide the domain of integration into three parts characterized by

$$\left.\begin{array}{ll}\text{(a)} & p_e+k\leq p_e^0,\\ \text{(b)} & p_e^0<p_e+k\leq m_\mu/2,\\ \text{(c)} & m_\mu/2<p_e+k,\end{array}\right\} 0\leq p_e\leq p_e^0.$$

The resulting integrals will be denoted by $N_a(p_e^0)$, $N_b(p_e^0)$, and $N_c(p_e^0)$, respectively. We shall also write $N_0(p_e^0)$ to represent the integral $\int_0^{p_e^0} dN_0(p_e)$ of the uncorrected spectrum. Similarly $N_v(p_e^0) = \int_0^{p_e^0} dN_v(p_e)$.

In the domains (a) and (b), the photon can be emitted in any direction and thus N_a and N_b contain terms depending on ω. On the other hand, the function N_c should not depend on ω since the photon cannot be emitted in this case in the same direction as the electron. Explicit calculation of $N_a(p_e^0)$ and $N_v(p_e^0)$ shows that not only the ordinary infrared divergences but also the terms involving ω cancel when these two are added together and hence the sum N_a+N_v is finite and well-defined even in the limit $m_e \to 0$. It follows that the ω dependence of $F(0)-F(x_0)$ arises entirely from the term N_b.

To see another feature of the problem, let us assume for a while that the mass of the electron is arbitrarily small and moreover restrict our attention to those configurations in which the photon and the electron are emitted in practically the same direction. Under such circumstances, one may imagine that, as the radiative interaction is switched on, occasionally electrons of momentum p_e+k in the uncorrected spectrum are "shifted" to a state of momentum p_e by the emission of a quantum k. All the electrons in $N_a(p_e^0)$ may then be regarded as having originated from corresponding electrons in $N_0(p_e^0)$ since $p_e+k\leq p_e^0$ holds for domain (a). Taking account of the virtual photon process, the number of electrons "counted" in $N_0(p_e^0)+N_a(p_e^0)+N_v(p_e^0)$ would thus be approximately equal to that in $N_0(p_e^0)$ of the same momentum interval. On the other hand, $N_b(p_e^0)$ consists of those electrons that have "drifted" into the interval $(0,p_e^0)$ from higher energy regions of the uncorrected spectrum, having lost their energies by emission of photons. Such electrons therefore do not correspond to any in $N_0(p_e^0)$.[16]

Since the electron has actually a nonvanishing mass and the photon is not always emitted in the direction of electron, this argument is of course not rigorous. It is not difficult to see, however, that its essential feature is not affected by the removal of our assumptions.

We have thus found two properties that characterize the separation of the integration domain (a) from (b) and (c). It is unlikely that this is a mere coincidence. One would rather be tempted to infer that there is a close relation between them to the extent that the former holds because of the latter. Somewhat inaccurately, one might express this by saying that N_a+N_v

[15] In this section, the behavior of the spectrum and lifetime as a function of m_e is studied in the limit $m_e \to 0$. It is to be noted that these quantities may turn out to be discontinous at $m_e=0$. This is because the photon mass $\lambda_{\min}$ is always considered to satisfy the relation $\lambda_{\min} \ll m_e$ in our problem which is not the case if $m_e=0$.

[16] Under the assumption of this paragraph, $N_c(p_e^0)$ should be disregarded since it would represent those electrons that have "drifted" from the region of uncorrected spectrum which lies beyond the maximum energy.

does not diverge for $m_e \to 0$ because the number of electron is "conserved" in the transition $N_0(p_e^0) \to N_0(p_e^0)+N_a(p_e^0)+N_v(p_e^0)$.

The absence of the $\alpha \ln(m_\mu/m_e)$ terms in the decay lifetime, i.e., the completely integrated spectrum, is expected from our argument since the domain of integration (b) disappears in that case. This result has been verified for all the ten Fermi interactions using the results of I and Eq. (2.4) of this paper. The same behavior has been observed in the case of the integrated asymmetry in muon decay.

The arguments developed in this section suggest strongly that this is not a peculiar property of the radiative correction to muon decay but a property that is also found for the radiative corrections to other decay processes. This is supported in Sec. 4 by the analysis of the neutron beta decay in the $V-A$ theory.

4. RADIATIVE CORRECTIONS TO BETA DECAY

The lowest order radiative corrections to the nuclear beta decay arising from the charge interactions of the proton and electron (assuming that the nucleons have no anomalous magnetic moment) may be readily evaluated using some of the general expressions derived in I as well as Eq. (C.4) of this paper.[17] In the approximation in which the electron momentum p is neglected in comparison with the proton mass m_p, one obtains the following expression for the radiative corrections to the electron spectrum in the $V-A$ theory:

$$\Delta P d^3p = \frac{\alpha}{2\pi} P^0 d^3p \bigg\{ 6\ln\left(\frac{\lambda}{m_p}\right) + 3\ln\left(\frac{m_p}{m_e}\right) + \tfrac{3}{2} + 4\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right) \times \left[\frac{E_m - E}{3E} - \tfrac{3}{2} + \ln\left(\frac{2(E_m-E)}{m_e}\right)\right] + \frac{2}{\beta}\left[L(\beta) - L(-\beta) + L\left(\frac{2\beta}{1+\beta}\right) + \tfrac{1}{2}L\left(\frac{1-\beta}{2}\right) - \tfrac{1}{2}L\left(\frac{1+\beta}{2}\right)\right] + \frac{1}{\beta}\tanh^{-1}\beta \times \left[2(1+\beta^2) + \frac{(E_m-E)^2}{6E^2} - 2\ln\left(\frac{2}{1-\beta}\right)\right]\bigg\}, \quad (4.1)$$

where $\beta = p/E$,

$$P^0 d^3p = \frac{8G^2}{(2\pi)^4}(E_m - E)^2 d^3p \quad (4.2)$$

is the uncorrected spectrum, and G is the Feynman–Gell-Mann coupling constant. The quantity λ is the ultraviolet cutoff, E_m is the maximum energy of the electron (including the rest mass), and $L(x)$ is the Spence function[18]

$$L(x) = \int_0^x \frac{dt}{t} \ln(|1-t|). \quad (4.3)$$

As usual, Eq. (4.1) includes the contributions of the vertex and wave function renormalizations as well as that arising from the inner bremsstrahlung. Contrary to the case of muon decay, the ultraviolet divergences have not cancelled for the V and A interactions. This is because the order $(pn)(e\nu)$ is quite different from $(e\mu)(\nu\nu)$ from the point of view of the electromagnetic interaction.

Our main concern in this section is to investigate the behavior of Eq. (4.1) in the limit $m_e \ll E$ (or $\beta \to 1$). In this case (4.1) is reduced to

$$\Delta P d^3p = \frac{\alpha}{\pi^4} G^2 E_m^5 (1-x)^2 x^2 dx \times \bigg\{ 6\ln\left(\frac{\lambda}{m_p}\right) + 3\ln\left(\frac{m_p}{2E_m}\right) + \tfrac{3}{2} - \frac{2\pi^2}{3} + 4(\ln x - 1)\left[\frac{1-x}{3x} - \tfrac{3}{2} + \ln\left(\frac{1-x}{x}\right)\right] + \frac{(1-x)^2}{6x^2}\ln x + \Omega\left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4\ln\left(\frac{1-x}{x}\right)\right]\bigg\}, \quad (4.4)$$

where $x = E/E_m$ and $\Omega = \ln(2E_m/m_e)$. The spectrum (4.4) contains a term proportional to Ω and thus diverges logarithmically in the limit $m_e \to 0$. If one integrates it with respect to x in the range $0 \leq x \leq 1$, the fractional change of the lifetime is found to be

$$\frac{\Delta\tau}{\tau_0} = -\frac{\alpha}{2\pi}\left[6\ln\left(\frac{\lambda}{m_p}\right) + 3\ln\left(\frac{m_p}{2E_m}\right) - 2.85\right]. \quad (4.5)$$

The term containing Ω has dropped out in this expression. Thus the relevant expansion parameter for the lifetime correction is apparently $\alpha \ln(m_p/2E_m)$ in contrast to that of the differential spectrum which is $\alpha \ln(m_p/m_e)$. Consequently, the radiative correction to the lifetime is finite in the limit $m_e \to 0$ in exact coincidence with the case of muon decay. Incidentally, one may expect that the radiative correction to the muon lifetime is also of order $\alpha \ln(m_\mu/2E_m)$. This happens to vanish however, because $E_m = m_\mu/2$ in this case. Thus,

[17] Throughout this section, we mean by the radiative correction that part of the electromagnetic corrections of order α which is not included in the usual Coulomb correction. This contribution is the same for both electron and positron emissions and should be added to the conventional Coulomb F function correction.

[18] Tables and useful properties of $L(x)$ are given by K. Mitchell, Phil. Mag. 40, 351 (1949).

terms of order α become the only corrections as was discussed in previous sections. This will explain why the lifetime correction to β-decay can be much larger than that to muon decay.

When the observed electron mass is used, the most important contribution to the spectrum comes from the term $\ln(m_p/m_e)$. If one neglects the smaller terms of (4.1) which are momentum dependent, the resultant lifetime is close to the value obtained from (4.1) for the observed electron mass. This approximate lifetime, however, depends on the term $\ln(m_p/m_e)$ and hence diverges logarithmically in the limit $m_e \rightarrow 0$. Comparing this with (4.5), it is seen that the neglected terms become important in this artificial limit and exactly cancel the $\ln m_e$ term of the major part.

Unfortunately, formulas (4.1), (4.4), and (4.5) are not unambiguous because of the presence of ultraviolet divergences. To estimate the magnitude of the radiative corrections, however, we shall put $\lambda = m_p$ in the following, hoping that future theory will justify the approximate correctness of such a choice. In order to avoid possible misunderstanding, any numerical values quoted below that depend on the value of λ will be labelled by an asterisk as a warning that such numerical values are not completely defined. It should be noted however, that the corrections to the lifetime and coupling constant of β decay become even larger if one chooses $\lambda > m_p$.

Under the assumption $\lambda = m_p$, Eq. (4.5) gives a decrease in the lifetime of 1.9*% in the case of neutron decay.[19] Of course, the use of (4.5) is not fully justified in this case because the approximation $E \gg m_e$ is used in deriving it. The above result is expected to be approximately correct, however, since most of the contribution to it comes from the energy independent term $3\ln(m_p/2E_m)$. As a check, one may calculate $\Delta\tau/\tau_0$ in the opposite limit in which $\beta \approx 0$. From (4.1), one then gets a decrease of 2.3*% for the neutron decay. One may expect that the actual correction due to the interaction of the radiation field with the charges will lie between these two estimates.

The case of O^{14}, a positron emitter with $E_m \sim 2.3$ Mev, is of particular importance because it has been used in the evaluation of the vector coupling constant and in the prediction of the muon lifetime.[20] Use of formula (4.5) gives $\Delta\tau/\tau_0 = -1.7$*%. This is somewhat smaller than that given by Berman who has estimated $\Delta\tau/\tau_0 = -(2.6 \pm 0.5)$*%.

In this section we have not considered in detail the magnetic moment contributions and other possible structure effects arising from the pion clouds. Berman[4] has estimated that the magnetic moment contributions are of the order of 0.2*%. Here we shall simply make use of his result. The corrected Feynman–Gell-Mann coupling constant then becomes $G = (1.40 \pm 0.01) \times 10^{-49}$ erg/cm^3. From the radiative corrections for the decay of O^{14} and for the muon decay, the muon decay mean life is now calculated to be $(2.31 \pm 0.05) \times 10^{-6}$* sec if the universal Fermi interaction is assumed, while the experimental value is $(2.22 \pm 0.02) \times 10^{-6}$ sec. Thus if the numerical values are compared, there appears to be a small discrepancy between theory and experiment of the order of 5%.

The situation is somewhat paradoxical because a discrepancy of 5% seems to be rather small to be explained by a renormalization of g_V due to the pion cloud but not sufficiently small as to be compatible with the estimated errors. Concerning the possible ambiguities due to the logarithmic dependence on the cutoff in the case of neutron decay, it should be noticed that for $\lambda > m_p$, the disagreement is even larger. In fact, to get even a 1% decrease in the calculated value of τ_μ, it would be necessary to choose $\lambda \ll m_p$, which on the one hand is unreasonable from the physical point of view and, on the other hand, is inconsistent with our present knowledge of the domain of validity of the local quantum electrodynamics of the electron proton system.[21] Structure effects of the pion cloud on the electromagnetic correction have been at least partially investigated by Berman who has analyzed in detail the magnetic moment interactions, which appear to be very small. Further study of these structure effects seems to be of great interest. It would be also interesting, perhaps, to investigate in detail possible effects of the nuclear structure of O^{14} on the electromagnetic corrections.

Nonetheless, it seems premature for us to draw any definite conclusion as to the significance of this discrepancy. Several experimental and theoretical factors such as the determination of the nuclear matrix element in $O^{14} \rightarrow N^{14} + e^+ + \nu$, the end point of e^+, and the experimental muon lifetime enter into this comparison. Any future modification may change considerably the present situation.[22] Clearly, more experimental and theoretical work is needed to clarify this important point.

When E_m is very large, the momentum dependent terms of Eqs. (4.1) or (4.4) contribute considerably to the modification of the spectral shape. To gain an idea of the order of magnitude involved, we have considered a hypothetical case where $E_m = 30 m_e$. Upon using Eq.

[19] Strictly speaking, it is inconsistent to put $\lambda = m_p$ in (4.1) since it has been derived under the assumption $\lambda^2 \gg m_p^2$. Re-evaluation of the Feynman integrals without this assumption (but assuming still $\lambda^2 \gg m_e^2$) shows, however, that the value of (4.5) is decreased only by an amount 0.002 when the cutoff λ is chosen to be m_p. This value is included in our estimate of the radiative corrections.

[20] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[21] S. D. Drell, Ann. Phys. N. Y. 4, 75 (1958).

[22] A very interesting possibility is to postulate that the universal weak interactions are carried through an intermediate charged vector meson of mass $\sim 1000 m_e$. This would give a theoretical value of $\tau_\mu = (2.25 \pm 0.05)$ μsec and would predict an effective ρ value of 0.763 which is compatible with the latest Berkeley and Columbia determinations. At the same time it would give a dynamical reason for the tetrahedron scheme of the universal interactions. Unfortunately, this hypothesis seems to predict a rate for the $\mu \rightarrow e + \gamma$ decay which is faster than is allowed by the present experimental upper limit.

(4.4), it is found that the ratio of the heights of the spectrum at $x=0.2$ and $x=0.9$ is increased by 6.7% when the terms of order α are included. The increase will be 21% if one considers the ratio at $E=m_e$ and $E=0.9E_m$.

For a value of E_m of the order of a few electron masses, the effects on the spectral shape are considerably smaller but not negligible. For example, for $E_m \approx 5m_e$ the ratio of the heights of the spectrum at $E=m_e$ and $E=0.9E_m$ is increased by 2.5%.

It is important to notice that the correction function to the spectral shape is actually independent of the ultraviolet cutoff λ. This is because one has to normalize the theoretical expression by

$$\int_{m_e}^{E_m} (P^0+\Delta P)(d^3p/dE)dE=1/\tau,$$

where τ is the experimental mean life, in order to compare it with experiments. The normalized expression is independent of λ up to terms of order α.

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APPENDIX A. SPECTRUM OF UNPOLARIZED MUONS (FOUR-COMPONENT THEORY)

In comparing theory with the experiments on the decay of unpolarized muons, it has become customary to determine first the Michel parameter ρ_M by fitting the experimental curves with the general expression for the electron spectrum in the four-component neutrino theory, and then the value of ρ_M thus determined is compared with prediction of a particular theory. In the case of the two-component neutrino theory, this value is to be compared with $(\rho_M)_{\text{theoretical}}=\frac{3}{4}$. The inclusion of the lowest order radiative corrections into the general four-component theoretical spectrum was treated in reference 1. [See formulas (I.28a) and (I.28c).]

We would like to call attention here to an alternative expression[23]

$$\tau P(x)dx=\frac{4x^2dx[1+h(x)]}{1+\Lambda_1+\frac{2}{3}\rho_M(\Lambda_2-\Lambda_1)} \times[3(1-x)+2\rho_M(\tfrac{4}{3}x-1)], \quad \text{(A.1)}$$

which is equivalent to Eqs. (I.28a) and (I.28c) up to terms of order α but has a simpler structure than the latter. In this formula, terms of order m_e/p_e are neglected. An advantage of this formula lies in the fact that it depends directly on the original expression for the Michel parameter ρ_M defined for instance by (II.2.6, 7). (The effect of the radiative correction appears as a corrective factor multiplying the uncorrected Michel spectrum.) The old expression (I.28c), although completely equivalent to (A.1) up to terms of order α, is somewhat more complicated than (A.1) because it is expressed in terms of a parameter ρ which differs somewhat from the original Michel parameter ρ_M. Making use of the definition of ρ given in (I.28a) and neglecting terms of order m_e/p_e, one finds the relation

$$\rho=\frac{\rho_M}{1+\Lambda_1+\frac{2}{3}\rho_M(\Lambda_2-\Lambda_1)}. \quad \text{(A.2)}$$

The equivalence up to the lowest order in α of (I.28c) and (A.1) can be easily established with help of (A.2).

In reference 1, the function $h(x)$ was chosen as a suitable average of the corrections to the different interactions ($S=P$, $V=A$, and T), taking advantage of the fact that they all have a very similar behavior. Since it is now believed that the dominant interactions in the muon decay are V and A, however, it is more reasonable to choose as $h(x)$ the corrections calculated for the V and A interactions. We shall therefore choose

$$h(x)=\frac{\alpha}{2\pi}\frac{f(x)}{3-2x}, \quad \text{(A.3)}$$

where $f(x)$ is defined by (2.4). The quantities Λ_1 and Λ_2 are given by

$$\Lambda_{1,2}=4\int_0^1 h(x)[3(1-x)\,;\,x]x^2dx. \quad \text{(A.4)}$$

Analytical evaluation of (A.4) gives

$$\Lambda_1=0.0132, \quad \Lambda_2=-0.0216. \quad \text{(A.5)}$$

The function $h(x)$ diverges logarithmically for $x\rightarrow 1$. Because of this reason, this quantity was expressed in reference 1 as a function of x and the experimental energy resolution ΔE. In practice, however, the experimental resolution function is folded into the theoretical expression and then comparison is made with the experimental curve. In the theoretical expression, it is therefore sufficient to use the differential spectrum with infinite resolution. We have accordingly set $\Delta E=0$ in the function $h(x)$ of Eq. (A.1).

APPENDIX B. CLASSICAL CONTRIBUTION TO INNER BREMSSTRAHLUNG

It would be of some interest to see how much of the radiative correction to the muon decay could be interpreted as an effect of classical radiation due to acceleration of the charge-current. For this purpose, let us note that the energy radiated per decay is given by

$$\Delta U=\frac{\epsilon\Delta P_\gamma d^3p_e d^3\kappa}{Pd^3p_e}, \quad \text{(B.1)}$$

[23] An expression very similar to (A.1) was first suggested by R. Behrends (private communication).

where ΔP_γ is given by (I.17), ϵ and κ are the energy and momentum of the photon, and Pd^3p_e is the probability of the nonradiative decay [see (I.2)]. Now, if one expresses ΔU as a function of the circular frequency ν of the emitted radiation, using the relations $\epsilon=\hbar\nu$ and $\kappa=\hbar\nu/c$, and then goes to the limit $\hbar\to 0$, one finds the following relation:

$$\lim_{\hbar\to 0}\Delta U=\frac{e^2}{c}\frac{\nu^2}{(2\pi)^2}\sum_{i=1,2}\left[\frac{(p_2\cdot e_i)}{(p_2\cdot\nu)}-\frac{(p_1\cdot e_i)}{(p_1\cdot\nu)}\right]^2 d\nu d\Omega, \quad \text{(B.2)}$$

where ν in the scalar products stands for the 4-vector $[\nu,(\nu/c)\mathbf{n}]$ and $\mathbf{n}$ is the direction of propagation of radiation.

As is well known, (B.2) can also be derived from purely classical considerations if one assumes that the radiation energy is emitted by an instantaneous acceleration of the charge-current. In the second case, the classical contribution of inner bremsstrahlung to the muon decay spectrum may be defined as the number of "equivalent quanta" which is obtained by dividing the energy formula (B.2) by $\hbar\nu$. Actually such a definition is not unambiguous since the classical term will diverge for the long-wavelength limit when it is integrated over the photon energy. It is not impossible, however, to estimate the magnitude of the classical contribution in a more or less qualitative manner.

For this purpose, it is convenient to separate the part of the spectrum due to real photons of energy larger than a certain quantity, say the rest energy of the electron, from that due to lower energy real photons and the virtual photons. As is easily seen, the first part is given by $(\alpha/2\pi)b_i(x)$, where $b_i(x)$ are the functions defined by (I.25) if one sets $\omega_<=0$ in that expression. The second part is then given as the difference between the entire radiative correction and the first part. Each part consists of "classical" and "nonclassical" terms. The classical term of the first part is given by $(\alpha/2\pi)2V(x)$ (with $\omega_<=0$) where $V(x)$ is defined by (I.25d).

In the case of V and A interactions, the two parts of the spectrum mentioned above are found to contribute roughly equal amounts to the radiative correction to the ρ value. Furthermore, in the region $0.3\lesssim x\lesssim 0.9$, the first part is dominated by the contribution of the classical term arising from (B.2). It may thus be said that the classical radiation of frequency $\nu\geq m_ec^2/\hbar$ may account for about half of the radiative correction of the ρ value. The classical contribution is also characterized by the fact that it is fairly independent of the electron momentum throughout the range of the spectrum.

APPENDIX C. EVALUATION OF THE INTEGRAL I

The integral I, defined in (2.2), can be easily evaluated in the rest system of particle 1. It is convenient to work in the Coester representation of the vector meson field in which the scalar quanta have been eliminated and the field operators obey the same commutation relations as those of the massless photon.[8] Summing over the three directions of polarization of the vector meson, one obtains

$$I=\tfrac{1}{2}\beta^2\int_{-1}^{1}dx\int_0^{k_{\max}}\frac{k^2dk}{\epsilon}\frac{(1-x^2k^2/\epsilon^2)}{(\epsilon-\beta xk)^2}, \quad \text{(C.1)}$$

where $\epsilon=(k^2+\lambda_{\min}^2)^{\frac{1}{2}}$ and $\beta=p_e/E_e$. Integration over k can be easily performed by choosing $v=k/\epsilon$ as the new variable. One thus finds

$$I=I_0+C, \quad \text{(C.2)}$$

where

$$I_0=\tfrac{1}{2}\beta^2\int_{-1}^{1}\ln\left(\frac{k_{\max}}{\lambda_{\min}}\right)\frac{1-x^2}{(1-\beta x)^2}dx, \quad \text{(C.3)}$$

$$C=2\ln2\left(\frac{1}{\beta}\tanh^{-1}\beta-1\right)+1+\frac{1}{2\beta}\tanh^{-1}\beta\left[2+\ln\left(\frac{1-\beta^2}{4}\right)\right]+\frac{1}{\beta}[L(\beta)-L(-\beta)]+\frac{1}{2\beta}\left[L\left(\frac{1-\beta}{2}\right)-L\left(\frac{1+\beta}{2}\right)\right]. \quad \text{(C.4)}$$

The function $L(x)$ is defined by (4.3).

The integral I_0 was previously used as the result of Eq. (2.2). The correction function C is independent of $k_{\max}$. Thus it may be used in any problem involving the radiative corrections of two charged fermions. In the case $E_2\gg m_2$, (C.4) reduces to

$$C=\left[1-\ln\left(\frac{2E_2}{m_2}\right)\right]\ln\left(\frac{E_2}{2m_2}\right)+1-\tfrac{1}{6}\pi^2. \quad \text{(C.5)}$$

This expression can be used in the case of the muon decay and it may be expressed in the form given in Eq. (2.3).