# Formation of Discontinuities in Classical Nonlinear Electrodynamics*

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It is shown that discontinuities can develop in the propagation of initially smooth waves governed by a classical nonlinear theory of electrodynamics. The type of theory considered includes as a special case that of Heisenberg and Euler, which describes the modifications that must be made in the Maxwell equations to include the classical limit of the nonlinear vacuum effects of quantum electrodynamics. A particular solution of the equations is constructed by the method of characteristics; this example illustrates how, with the appropriate well-behaved initial conditions, the characteristics can be made to intersect at later times, thus forming discontinuities. The classical approximation fails when the gradient of the field strength becomes large, so that no definite conclusion can be drawn as to the actual physical creation of singularities.

## I. INTRODUCTION

IT is well known[1] that quantum electrodynamics predicts and experiment verifies that the vacuum is polarizable and that two general electromagnetic fields will not superimpose but will interact ("scattering of light by light"). Heisenberg and Euler[2,3] have derived an explicit Lagrangian which describes these nonlinear effects in the classical limit of long wavelengths, low frequencies, and not too excessive field strengths; here the linear Maxwell equations are replaced by *nonlinear* equations. One of the very important differences between linear and nonlinear partial differential equations is that in the latter case solutions may occur which, although the initial values are entirely smooth, develop a surface of discontinuities at a later time.[4] The situation is mathematically analogous to certain hydrodynamical problems, where these discontinuities appear as shocks. In this paper we shall show that any nonlinear classical theory of electrodynamics admits solutions that exhibit this formation of discontinuities.

## II. "SIMPLE WAVE" SOLUTIONS FOR GENERAL LAGRANGIAN

It is easily demonstrated[5] that only two independent scalar invariants can be constructed from the components of the electromagnetic field tensor, $F_{\mu\nu}$; they are, in the usual notation

$$I=F_{\mu\nu}F^{\mu\nu}=2(\mathbf{B}^2-\mathbf{E}^2),$$
$$K=\{\epsilon^{\lambda\rho\mu\nu}F_{\lambda\rho}F_{\mu\nu}\}^2=-(4\mathbf{E}\cdot\mathbf{B})^2.$$

If we impose the conditions that the field equations be linear differential equations, then[6] the only possibility for the Lagrangian (which must be a relativistic scalar) is

$$\mathfrak{L}=aF_{\mu\nu}F^{\mu\nu}=aI, \tag{1}$$

where $a$ is a constant. On the other hand, if the linearity restriction is dropped, we may take as the Lagrangian a general function of the two invariants

$$\mathfrak{L}=\mathfrak{L}(I,K). \tag{2}$$

Since it is our purpose only to show that we can find solutions which involve the formation of discontinuities, we can simplify our problem by considering solutions of a particularly simple form. Thus we assume that, in a particular reference frame, all components of the four-vector potential vanish identically except for $A_2=\phi(x,t)$, where $\phi$ represents a plane wave depending only on $x=x_1$ and on $t$. Then the invariant $K$ is found to vanish identically. It is convenient to transform to the independent variables $\xi=x+t$, $\eta=x-t$; then the remaining invariant $I$ is given by

$$I=4\phi_\xi\phi_\eta, \tag{3}$$

where we introduce a notation in which subscripts on $\phi$ denote partial differentiation. For any such solution and a general Lagrangian of the form of Eq. (2), the variational principle is equivalent to the single partial differential equation:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\mathfrak{L}}{\partial\phi_\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\partial\mathfrak{L}}{\partial\phi_\eta}\right)=0, \tag{4}$$

which may be written

$$\frac{\partial\mathfrak{L}}{\partial I}\phi_{\xi\eta}+2\frac{\partial^2\mathfrak{L}}{\partial I^2}(\phi_\eta{}^2\phi_{\xi\xi}+\phi_\xi{}^2\phi_{\eta\eta}+2\phi_\xi\phi_\eta\phi_{\xi\eta})=0. \tag{5}$$

Since $\mathfrak{L}$ and its derivatives are functions only of $I=4\phi_\xi\phi_\eta$, we find that Eq. (5) is of the general type

$$a\phi_{\xi\xi}+b\phi_{\xi\eta}+c\phi_{\eta\eta}=0, \tag{6}$$

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[1] For example, see any text in quantum electrodynamics, such as J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Cambridge, 1955), p. 298.

[2] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).

[3] V. Weisskopf, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **14**, 1 (1936).

[4] John A. Wheeler and John S. Toll (to be published).

[5] See, for instance, L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addision-Wesley Publishing Company, Cambridge, 1951), p. 63.

[6] L. Landau and E. Lifshitz, reference 5, p. 69.

where $a$, $b$, and $c$ are functions only of $\phi_\xi$ and $\phi_\eta$:

$$a=2\phi_\eta^2\partial^2\mathfrak{L}/\partial I^2, \quad (7)$$

$$c=2\phi_\xi^2\partial^2\mathfrak{L}/\partial I^2, \quad (8)$$

$$b=\partial\mathfrak{L}/\partial I+I\partial^2\mathfrak{L}/\partial I^2. \quad (9)$$

When the solution $\phi(\xi,\eta)$ is such that $b^2>4ac$ throughout a region $R(\xi,\eta)$ of space-time, Eq. (6) is hyperbolic in $R$ and can be solved by the method of characteristics.[7] At each point two characteristic $C_+$ and $C_-$ are defined which have the tangent directions given by:

$$d\eta/d\xi=\rho_\pm/a, \quad (10)$$

where $\rho_\pm$ are the two real solutions of

$$\rho^2-b\rho+ac=0. \quad (11)$$

Functions $F_+(\phi_\xi,\phi_\eta)$ and $F_-(\phi_\xi,\phi_\eta)$ can be constructed which maintain constant values along each of the $C_+$ and $C_-$ characteristics, respectively. We state without proof the known[7] results that an especially simple type of solution of the Eq. (6) can be constructed if the initial values along any curve that is *not* a characteristic are chosen to satisfy:

$$F_-(\phi_\xi,\phi_\eta)=F_0, \quad (12)$$

where $F_0$ is any fixed constant. Then this relation will be satisfied at later points obtained by following $C_+$ characteristics from the initial curve $B$; these $C_+$ characteristics are all straight lines; and the values of $\phi_\xi$ and $\phi_\eta$ are constant along each $C_+$ characteristic. The propagation of solutions of this type, which are called "simple wave" solutions, is easily determined geometrically from the initial values. It is now clear how singularities in the solution can arise. If the initial values are chosen such that the slopes $\rho_+/a$ of Eq. (10) give for some interval on $B$ a converging set of characteristics, these characteristics, which carry different values of $\phi_\xi$ and $\phi_\eta$, will intersect at later times.

At a point where two $C_+$ characteristics intersect, the assignment of values of $\phi_\xi$ and $\phi_\eta$ is no longer uniquely determined. The general situation involves the formation of an envelope by the family of straight lines; this curve separates the $\xi$, $\eta$ plane into two regions, in only one of which the solution is unambiguously determined by the initial conditions. The envelope thus forms a limiting curve beyond which the solution cannot be continued in a unique and unambiguous manner.

Due to the fact that $a$, $b$, and $c$ have the forms given by Eqs. (7), (8), and (9), it is possible to find $F_+$ and $F_-$ as follows: From the theory of characteristics,[7] it follows that along a $C_-$ characteristic $\phi_\xi$ and $\phi_\eta$ are related by

$$(\rho_-)d\phi_\xi+cd\phi_\eta=0. \quad (13)$$

[7] See, for instance, R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948), especially Chap. II.

From Eqs. (6)–(9), we see that $c/\rho_-$ has the form $\phi_\xi^2/P(\phi_\xi\phi_\eta)$, where the function $P$ of the product $\phi_\xi\phi_\eta$ depends on the particular Lagrangian $\mathfrak{L}$ of Eq. (2). Hence, Eq. (13) can be written

$$P(\phi_\xi\phi_\eta)d\phi_\xi+\phi_\xi^2d\phi_\eta=0. \quad (14)$$

The function $F_-(\phi_\xi,\phi_\eta)$, which is constant along the $C_-$ characteristic, is obtained by integrating Eq. (14) with its integrating factor $\{\phi_\xi(\phi_\xi\phi_\eta-P)\}^{-1}$. Once $F_-(\phi_\xi,\phi_\eta)$ is determined, continuous initial values of $\phi_\xi$ and $\phi_\eta$ along any contour $B$ can be chosen which satisfy Eq. (12) and thus yield a simple wave solution. The slopes of the straight $C_+$ characteristics emanating from $B$ are then found by Eq. (10). If these characteristics should all happen to diverge, a new solution with converging characteristics will result from the mapping $x\rightarrow -x$. Consequently we have shown that, except for the case when the characteristics are all parallel (e.g., if the field equations are linear), it is always possible to generate a simple wave solution in which some neighboring characteristics must intersect, thus leading to infinite gradients in the field strengths.

## III. SPECIAL EXAMPLE

The particular Lagrangian of Heisenberg and Euler[2] can be treated by the above methods. The field strengths along the initial curve $B$ can be chosen to be sufficiently small,[8] so that one need only retain the first two terms in the expansion of the Lagrangian in powers of the invariants, which yield

$$\mathfrak{L}=-\tfrac{1}{4}I+\tfrac{1}{32}\beta\{I^2-(7/4)K\}, \quad (15)$$

where $\beta=\hbar e^4/45\pi^2m^4$. Equation (5) or Eq. (6) then becomes

$$\psi_{\xi\eta}=\psi_{\xi\xi}\psi_\eta^2+4\psi_\xi\psi_\eta\psi_{\xi\eta}+\psi_\xi^2\psi_{\eta\eta}, \quad (16)$$

where the explicit appearance of $\beta$ has been removed by introducing:

$$\psi=\beta^{\frac{1}{2}}\phi. \quad (17)$$

Equation (11) can then be solved to give one root as:

$$\rho=\tfrac{1}{2}(4\lambda-1)+\tfrac{1}{2}(12\lambda^2-8\lambda+1)^{\frac{1}{2}}, \quad (18)$$

where $\lambda=\psi_\xi\psi_\eta$. We identify this root with $\rho_-$ and Eq. (13) then becomes

$$\left(\frac{4\psi_\xi\psi_\eta-1}{2}+\frac{[12(\psi_\xi\psi_\eta)^2-8\psi_\xi\psi_\eta+1]^{\frac{1}{2}}}{2}\right)\times d\psi_\xi+\psi_\xi^2d\psi_\eta=0. \quad (19)$$

This is already of the form (14) and its solution may be represented parametrically by

$$\psi_\xi=[\lambda f(\lambda)/J]^{\frac{1}{2}},\quad \psi_\eta=[J\lambda/f(\lambda)]^{\frac{1}{2}}, \quad (20)$$

[8] Since, in the simple wave solution, the field strengths remain constant along $C_+$ characteristics, it is simple to check at the end of our investigation that the contribution of the omitted higher order terms is indeed numerically negligible.

where $\lambda$ is a parameter which must be chosen in the range $0 \leqslant \lambda \leqslant \frac{1}{6}$, in order that the equation remain hyperbolic, and where

$$\frac{f(\lambda)}{\lambda}=\frac{\{(3-12\lambda)[3(1-2\lambda)]^{\frac{1}{2}}+(12\lambda-5)(1-6\lambda)^{\frac{1}{2}}\}^{-2/\sqrt{3}}}{(12\lambda^2-8\lambda+1)^{\frac{1}{2}}+1-4\lambda}. \tag{21}$$

$J$ is a positive, arbitrary constant of integration. (Note that $\psi_\xi\psi_\eta=\lambda$.) Then Eq. (20) gives the values of $\psi_\xi$ and $\psi_\eta$ along the initial curve $B$. A logical choice for this initial curve is the line $\xi=\eta$, or the $x$ axis $(t=0)$. Any continuous assignments of values of $\lambda$ to this initial curve will lead to a particular simple wave solution. For convenience we parametrize the initial curve by

$$\xi=\eta=\frac{1}{k\lambda}-\frac{1}{1-k\lambda}, \tag{22}$$

where $k$ is a real constant $(k>6)$ and the range of $\lambda$ is $0 \leqslant \lambda \leqslant 1/k$. Thus the complete $x$ axis is parametrized by (22). From Eqs. (10) and (18) we then find for the slope of the line passing through that point of $B$ which corresponds to a given $\lambda$:

$$\frac{d\eta}{d\xi}=-\frac{1}{2J}\{(3-12\lambda)[3(1-2\lambda)]^{\frac{1}{2}} +(12\lambda-5)(1-6\lambda)^{\frac{1}{2}}\}^{-2/\sqrt{3}}, \tag{23}$$

with $0 \leqslant \lambda \leqslant 1/k$. Thus the complete solution may be traced out as follows.

Each value of $\lambda$, $0 \leqslant \lambda \leqslant 1/k$, determines (1) a point $(\xi,\eta)$ on $B$, given by Eq. (22), and (2) a straight $C_+$ characteristic passing through this point and with the slope given by Eq. (23); and along which $\psi_\xi$ and $\psi_\eta$ are constant at the values given by Eq. (20). By running over the complete range of $\lambda$, we cover a region of the $\xi,\eta$ plane with straight lines and therefore with values of $\psi_\xi$ and $\psi_\eta$. These values are the simple wave solution.

The computations lead to a family of converging straight lines for the $C_+$ characteristics. The envelope of this family is a curve of two branches, each of which is a time-like curve, and the two branches meet at a cusp. This cusp is the earliest point at which the characteristics intersect and represents the first appearance of a discontinuity in the gradients of the field strengths.

By choosing $k$ to be very large and $J=1/k$, the values of $\psi_\xi$ and $\psi_\eta$ are everywhere smaller than $k^{-\frac{1}{2}}$, according to Eqs. (20) and (21). Thus, for sufficiently large $k$, the higher order terms in the Heisenberg-Euler Lagrangian can be shown to give only a negligible correction to the above discussion. (For $k=10^{10}$, the cusp occurs at about the time $t=5.0\times10^{11}$ in the units used above.) The critical field strength for the Heisenberg-Euler Lagrangian is $4.0\times10^{13}$ gauss, so that all normal laboratory field strengths are indeed small compared to this value. For examples with small fields, the characteristics are all nearly parallel to the light cone. In order to give an easily illustrated example, we show in Fig. 1 a case with $k=6$ and $J=\frac{1}{2}$. (Of course, the field strengths are enormous and the corrections would be appreciable; in fact, the Lagrangian is quite inapplicable anyway at such field strengths, so that this example is purely pedagogical.)

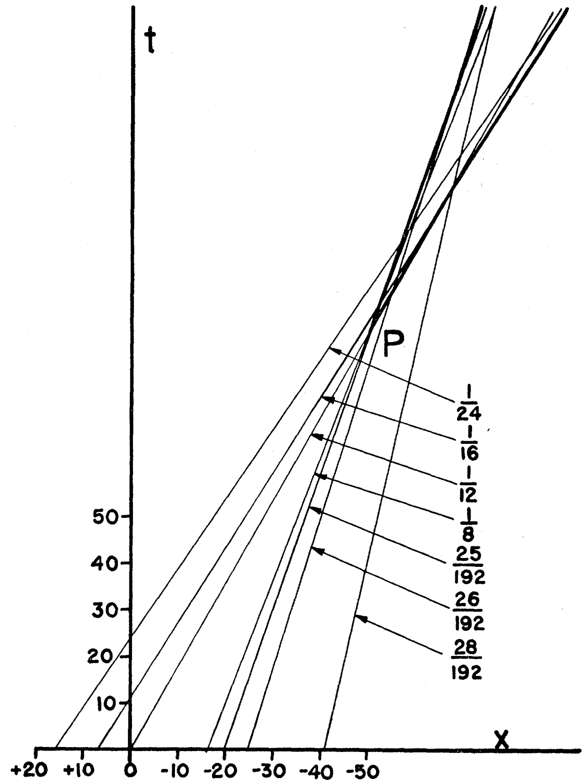


FIG. 1. Construction of a simple wave solution with formation of a singularity. The particular example of Eq. (22) of the text with $k=6$ and $J=\frac{1}{2}$ has been illustrated. The characteristics are straight lines and the small numbers indicate corresponding values of the parameter $\lambda$. The values of the field strengths, given originally along the $x$ axis, remain constant when propagated along each characteristic and the singularity is produced when these characteristics converge to produce the first intersection at the point $P$, where an infinity in the field gradients must appear.

## IV. GENERAL REMARKS

It is clear that we cannot interpret literally the formation of discontinuities in the field gradients; for, the Heisenberg-Euler theory is invalid for fields which change too rapidly,[2] and therefore must break down before a discontinuity is actually achieved. The physical reason for the restriction on the Heisenberg-Euler Lagrangian is that we wish to prevent the formation of real pairs; we see, then, from the character of our solution, that the field must create real pairs in the space-time neighborhood of the singular curve. In this region we no longer have a vacuum situation, the Heisenberg-Euler theory no longer applies, and the more powerful machinery of quantum electrodynamics must be applied to discover just what happens in this case, and how the extension is to be made into the unphysical region.

Because of the small numerical value of the constant coefficients of higher order terms in the Heisenberg-Euler Lagrangian density, these effects are too small to be susceptible to experimental test.[4] The primary importance of our investigation is to point out theoretical methods for the investigation of these nonlinear effects, and to give an example of the kind of phenomenon which may occur. It is hoped that such examples will provide some insight into more realistic situations, and be of some suggestive value in the consideration of the nonlinear aspects of quantum field theories.

### V. ACKNOWLEDGMENTS

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*Note added in proof.*—The question has been raised as to whether the singularities which occur in the solution of the equation we have treated are due to special properties of the equation, or whether any nonlinear hyperbolic equation possesses singular solutions. We have been unable to determine whether there exists a nonlinear hyperbolic equation having the property that for *no* initial conditions does the solution exhibit singularities. Therefore we regard this investigation as merely demonstrating how, for a particular physical situation, singularities may occur, and how they are formed in this case. In this connection, we note that an investigation of the singularities occurring in the solution of another type of nonlinear wave equation has been made by J. B. Keller, Comm. Pure Appl. Math. **10**, 523 (1957).

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# Radiative Corrections to Fermi Interactions*

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The radiative correction to the decay spectrum of polarized muons is recalculated taking into account a mistake in our previous work which was recently pointed out by Berman. The revised values for the radiative correction to $\delta$, $\xi$, and integrated asymmetries for the high- as well as low-energy decay electrons have turned out to be practically identical with the old values. The $\rho$ value determined from experiments, on the other hand, has to be increased by about 1% because of the new correction. Thus the over-all effect of the radiative correction to the $\rho$ value is now an increase of the order of 5.6% when the experimental and theoretical spectral distributions are compared in the region $0\leq p/p_{\max}\leq 0.95$. The radiative corrections to the spectrum and lifetime of the nuclear $\beta$ decay arising from the charge interactions of the electron and proton are also studied. Use of this expression gives a correction of $-1.7\%$ for the lifetime of $O^{14}$. The corrected Feynman–Gell-Mann coupling constant is $G=(1.40\pm0.01)\times10^{-49}$ erg/cm$^3$. In the universal $V-A$ theory of weak interactions, the calculated muon mean life becomes $\tau_\mu=(2.31\pm0.05)\times10^{-6}$ sec. (These three values depend logarithmically on the ultraviolet cutoff $\lambda$ and the corrections to $\tau_\mu$ increase for increasing values of $\lambda$.) It is found that the corrections to the spectral shape of $\beta$ decay are rather large in the case in which the end-point energy $E_m\gg m_ec^2$. The radiative corrections to the lifetime and the total asymmetry for muon decay are found to be well defined and finite for $m_e\rightarrow 0$ in spite of the fact that the differential spectrum itself diverges logarithmically in the same limit. The same situation is encountered in the case of radiative corrections to the nuclear $\beta$ decay. A physical explanation for such behavior of the radiative corrections is attempted. In Appendix A, a simplified expression is given for the determination of the Michel parameter.

## 1. INTRODUCTION

THE lowest order radiative correction to the decay of polarized muons have been studied in previous papers.[1–3] In the present work, the functions which determine the various corrections are reconsidered, taking into account a mistake in the treatment of the low-energy quanta, recently pointed out by Berman.[4] In Sec. 2, the results of this calculation and their effects on the parameters $\delta$, $\rho$, $\xi$, the lifetime, and the integrated asymmetries for high- as well as low-energy electrons are discussed.

As a consequence of this recalculation, the $\rho$ value is increased by an additional amount of about 1%, the

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[1] Behrends, Finkelstein, and Sirlin, Phys. Rev. **101**, 866 (1956). This paper will be quoted as I.

[2] T. Kinoshita and A. Sirlin, Phys. Rev. **107**, 593 (1957). This paper will be quoted as II.

[3] T. Kinoshita and A. Sirlin, Phys. Rev. **107**, 638 (1957). This paper will be quoted as III.

[4] S. M. Berman, Phys. Rev. **112**, 267 (1958).