where we have made repeated use of the Fourier transform

$$e^{-Ax}=-\frac{1}{2\pi^2}\int \exp(-i\mathbf{l}\cdot\mathbf{x})\left\{\frac{\partial}{\partial A}(l^2+A^2)^{-1}\right\}d\mathbf{l}.$$

[Note that $K'(\beta,\gamma,\delta)=K'(\delta,\gamma,\beta)$, despite the unsymmetric appearance of β and δ in the last integral.] This last integral was integrated numerically with the electronic computer of the Cornell Computing Center to give the values quoted in the text.

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Direct-Interaction Model Calculation of High-Energy Proton-Carbon Scattering*

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An optical potential has been derived which includes, to first order in q^2, the effect of the nucleon-nucleon angular dependence. Nucleon-nucleon scattered amplitudes calculated from the 310-Mev nucleon-nucleon phase shifts have been put into the potential and the proton-carbon scattered amplitudes calculated from it by WKB approximation. Good agreement was obtained with the scattered amplitudes as derived directly from the 313-Mev proton-carbon data using an extension of the analysis employed previously by Bethe. The inclusion of the nucleon-nucleon angular dependence in the potential was found to be important in order to obtain the correct value of the imaginary part of the forward scattered amplitude and the correct proton-carbon angular dependence at moderately small angles. Phase shift solutions 1 and 6 of Stapp *et al.* were investigated and found to give essentially the same agreement with the differential cross section at small angles. Solution 6 was found to give a better fit to the polarization data than solution 1, but the significance of this is not clear.

I. INTRODUCTION

In the direct-interaction model[1] of nuclear reactions one tries to understand the interaction of an incident particle with the nucleus in terms of the interaction of the particle with the individual nucleons composing the nucleus. At high energies the effect of binding is small and the particle-nucleus interaction can be related directly to the particle-nucleon interaction. In particular the optical potential for the elastic scattering of protons from nuclei at high energies can be simply related to the nucleon-nucleon scattering matrix.[2] Since the nucleon-nucleon scattering matrix is determined entirely by the nucleon-nucleon scattering phase shifts, a detailed analysis of the elastic scattering of protons from complex nuclei provides a good test for the direct-interaction model. Furthermore, such an analysis should provide a method for distinguishing between sets of possible nucleon-nucleon phase shifts.

Recently several authors have made such an analysis.[3,4] Bethe, in his analysis of the small-angle scattering and polarization of 313-Mev protons by carbon, showed that the proton-proton scattering phase shifts of Stapp *et al.*[5] and the neutron-proton phase shifts of Gammel and Thaler[6] were in quantitative agreement with the proton-carbon data. He was unable to distinguish between the five different phase shift solutions of Stapp, however; all solutions gave essentially the same agreement.

In his analysis of the experimental data Bethe found the value 8.6 f (1 fermi$=10^{-13}$ cm) for $g_{NI}(0)$, the imaginary part of the spin-independent scattered amplitude at 0°. This agreed with the value calculated from the optical potential derived from the nucleon-nucleon phase shifts. A more reliable value of $g_{NI}(0)$ can be obtained independently from the total neutron cross section at this energy. Since the cross section is nearly constant over a wide energy interval about 313 Mev, its value is known quite accurately. The cross-section data[7] give $g_{NI}(0)=9.45$ f.[8] This is in disagreement both with the proton-carbon data and with the direct-interaction model as calculated in B.

However, because of the over-all success of the

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[1] K. M. Watson, Revs. Modern Phys. **30**, 565 (1958).
[2] B. W. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 1157 (1956).
[3] H. A. Bethe, Ann. Phys. N. Y. **3**, 190 (1958). This paper will be referred to as B.
[4] S. Ohnuma, Phys. Rev. **111**, 1173 (1958).
[5] Stapp, Ypsilantis, and Metropolis, Phys. Rev. **105**, 302 (1957).
[6] The $T=0$ phase shifts calculated by Gammel and Thaler are given in reference 3. Also see J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957).
[7] J. DeJuren, Phys. Rev. **70**, 27 (1950); R. Fox *et al.*, Phys. Rev. **80**, 23 (1950); A. Ashmore *et al.*, Proc. Phys. Soc. (London) **70**, 745 (1957); V. A. Nedzel, Phys. Rev. **94**, 180 (1954).
[8] R. Wilson (private communication).

direct-interaction model demonstrated in B, it seems worthwhile to extend the analysis given there in order to see if, by suitable modification, the model can in fact account for this discrepancy. In this paper the discrepancy is removed by considering two effects neglected in the earlier work. First, departures of the angular dependence of the *nuclear* scattered amplitude from that given by Born approximation are considered. These departures are found to be important even in the region of small angles considered. They introduce a modification in the analytic form of the differential cross section and of the polarization which leads to a different determination of the scattered amplitudes. Secondly, the angular dependence of the nucleon-*nucleon* scattered amplitudes is taken into account. This leads to a modification of the optical potential which, to the order considered here, amounts to an increase in the effective radius of the potential over that determined by electron scattering. It also causes the real and imaginary parts of the modified potential to no longer have the same radial dependence and this is found to strongly affect the angular dependence of the proton-carbon scattered amplitudes. The amplitudes calculated from this potential are found to be in excellent agreement with those deduced directly from the experiments.

It is found that both Stapp phase-shift solutions 1 and 6 give essentially the same agreement with the differential cross-section data. However, due to the larger value of the real part of the spin-dependent scattered amplitude given by solution 6 compared to solution 1, it is found that solution 6 gives better agreement with the polarization data than solution 1.

II. DETERMINATION OF THE PROTON-CARBON SCATTERED AMPLITUDES FROM THE NUCLEON-NUCLEON SCATTERING PHASE SHIFTS

The Born approximation to the elastic scattered amplitude from a spin-zero nucleus, given by the direct-interaction model, is[3]

$$f_N^{\text{Born}}(q)=G(q)F(q)+H(q)F(q)\sin\theta\,\sigma_n. \qquad (1)$$

Here $q=2k\sin\frac{1}{2}\theta$, where θ is the scattering angle and k is the wave number of the incident proton in the laboratory system; σ_n is the component of $\bar{\sigma}$ normal to the scattering plane. $F(q)$ is the nuclear form factor,

$$F(q)=(4\pi/q)\int_0^\infty \rho(r)r\sin(qr)dr, \qquad (2)$$

where $\rho(r)$ is the nuclear density normalized so that $4\pi\int_0^\infty\rho(r)r^2dr=1$. $G(q)$ and $H(q)$ are given by

$$\begin{aligned}G(q)&=\tfrac{1}{2}(k/k_0)[(N-Z)A_0(q)+(N+Z)A_1(q)],\\ H(q)&=\tfrac{1}{2}(k/k_0)\csc\theta[(N-Z)C_0(q)+(N+Z)C_1(q)].\end{aligned} \qquad (3)$$

Here N is the mass number of the nucleus, Z the atomic number, and k_0 the wave number of the incident proton in the proton-nucleon center-of-mass system. $A(q)$ and $C(q)$ are the nucleon-nucleon scattered amplitudes defined in B and are given directly by the nucleon-nucleon scattering phase shifts. The subscripts 0 and 1 refer to the total isotopic spin.

This result assumes that the amplitude for scattering from a bound nucleon differs from that for the scattering from a free nucleon only by the kinematical factor (k/k_0). There is, in reality, also a difference arising from the fact that the final momenta are not the same in the two cases; in the case of scattering from a free nucleon the final momentum is reduced by a factor of $[1-q^2/k^2]^{\frac{1}{2}}$. It is necessary to assume that this has only a small effect upon the scattered amplitude if one wants to relate nucleon-nucleus scattering to the observed nucleon-nucleon scattering. This is certainly a reasonable assumption in the region we are considering ($\theta\leq 9°$), since here the momentum is degraded by at most 1%; even at 25° it is degraded by only 10%.

If an optical potential is assumed, it can be written in the form

$$\begin{aligned}V(r)&=2\pi\frac{\hbar^2c^2}{E}\left[-G(0)v(r)+\frac{H(0)}{k^2}u(r)\boldsymbol{\sigma}\cdot\mathbf{L}\right]\\ &=V_C(r)+V_{\text{s.o.}}(r)\boldsymbol{\sigma}\cdot\mathbf{L}.\end{aligned} \qquad (4)$$

The Born approximation scattered amplitude then is

$$\begin{aligned}f_N^{\text{Born}}(q)&=-\frac{E}{2\pi\hbar^2c^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(r)e^{i\mathbf{k}\cdot\mathbf{r}}d^3r\\ &=G(0)\int v(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3r\\ &\qquad-\frac{H(0)}{k^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}u(r)\boldsymbol{\sigma}\cdot\mathbf{L}e^{i\mathbf{k}\cdot\mathbf{r}}d^3r.\end{aligned} \qquad (5)$$

Comparing Eqs. (1) and (5), we find

$$v(r)=[2\pi^2G(0)]^{-1}\int_0^\infty G(q)F(q)j_0(qr)q^2dq, \qquad (6a)$$

$$u(r)=i[2\pi^2H(0)r]^{-1}\int_0^\infty H(q)F(q)j_1(qr)q^3dq. \qquad (6b)$$

Here j_0 and j_1 are the ordinary spherical Bessel functions of order 0 and 1, respectively. If $G(q)$ is taken to be just the constant $G(0)$, then Eq. (6a) reduces to the usual expression

$$v(r)=\rho(r).$$

Likewise, with $H(q)=H(0)$, Eq. (6b) becomes

$$u(r)=-i(1/r)[d\rho(r)/dr],$$

so that

$$V(r)=\frac{2\pi\hbar^2c^2}{E}\left[-G(0)\rho(r)-iH(0)\frac{1}{k^2r}\frac{d\rho(r)}{dr}\boldsymbol{\sigma}\cdot\mathbf{L}\right]. \quad (7)$$

This is the form of the optical potential usually assumed. We have written it in the same notation as used in B except for the factor $-i$ appearing in the second term, which results from a trival change in the definition of $H(0)$. In our notation a positive $H_R(0)$ gives rise to a negative imaginary spin-orbit potential and a positive real spin-dependent scattered amplitude, whereas a positive $H_I(0)$ results in a positive real potential and a positive imaginary amplitude.

The optical potential defined by Eqs. (4) and (6) describes the nucleon-nucleus scattering as the multiple scattering of the incident nucleon from the individual target nucleons in the nucleus. The amplitude for each scattering is taken to be essentially that for scattering from a free nucleon (when properly averaged over spin and isotopic-spin[3]). Since this is known only in the "far zone", the use of this potential assumes that two successive scatterings do not occur in a distance large compared with the wavelength of the incident particle. This assumption is generally valid at large energies.

Splitting $G(q)$ and $H(q)$ into their real and imaginary parts and then expanding to first order in q^2, we have

$$\begin{aligned}G_R(q)&=G_R(0)(1-\tfrac{1}{4}\lambda_Ra^2q^2+\cdots),\\G_I(q)&=G_I(0)(1-\tfrac{1}{4}\lambda_Ia^2q^2+\cdots),\\H_R(q)&=H_R(0)(1-\tfrac{1}{4}\alpha_Ra^2q^2+\cdots),\\H_I(q)&=H_I(0)(1-\tfrac{1}{4}\alpha_Ia^2q^2+\cdots).\end{aligned} \quad (8)$$

The constant a^2 is introduced here for convenience; it makes λ and α dimensionless. We set it equal to $\frac{2}{3}\langle r^2\rangle$ which, for carbon, is 3.86 f^2. Table I lists the values of the eight parameters in Eq. (8), derived from the various nucleon-nucleon phase shift solutions considered in B.[9] They are obtained by using Eq. (3) together with the values of A and C at 0° and 12° (6° in the laboratory system) given in B. λ_I, α_R, and α_I are all reasonably small, as expected; λ_R, however, is quite large, especially for solution 1. [This is because the small value of $G_R(0)$ is due to the cancellation of nearly equal, but opposite, contributions from different nucleon-nucleon states (see B, Table II), resulting in an exceptionally large slope-to-value ratio at $q^2=0$.] McManus and Thaler[10] have computed the corresponding parameters resulting from the Gammel-Thaler nucleon-nucleon potential.[6] In terms of our notation they found $\lambda_R=1.5$, $\lambda_I=0.31$, $\alpha_R=0.23$, and $\alpha_I=0.19$, which agree in general with the values given in Table I for solution 1, especially as regards the large value of λ_R.

[9] Gammel and Thaler (reference 6) found three different $T=0$ phase shift solutions which, together with solution 1 of Stapp, fit the neutron-proton data. These solutions are referred to as 1*A*, 1*B*, and 1*C*. Likewise, for Stapp solution 6, three different $T=0$ solutions were also found, referred to as 6*A*, 6*B*, and 6*C*. $T=0$ phase shifts have not been calculated for the other Stapp solutions. See reference 3 for details.

[10] H. McManus and R. M. Thaler, Phys. Rev. **110**, 590 (1958).

TABLE I. The 310-Mev nucleon-nucleon scattering parameters as determined from phase shift solutions 1*A* to 1*C* and 6*A* to 6*C*. *G* and *H* are in fermis; λ and α are dimensionless.

Nucleon parameter	1*A*	1*B*	1*C*	6*A*	6*B*	6*C*
$G_R(0)$	5.05	4.48	5.42	3.05	4.77	3.73
$G_I(0)$	12.39	11.92	11.79	11.79	11.79	11.84
$H_R(0)$	6.44	5.99	5.93	15.62	14.39	15.34
$H_I(0)$	34.43	32.64	35.94	26.70	24.63	26.48
λ_R	1.24	1.34	1.16	0.59	0.56	0.41
λ_I	0.24	0.23	0.24	0.31	0.29	0.31
α_R	0.19	0.15	0.15	0.17	0.15	0.18
α_I	0.13	0.14	0.18	0.14	0.12	0.12

For $F(q)$ the best Guassian fit to the form factor as determined by electron scattering[11] is used:

$$F(q)=\exp(-\tfrac{1}{4}a^2q^2). \quad (a=1.96\text{ f}). \quad (9)$$

This approximation agrees with the best Fregeau form factor up to $q^2=1.5$ ($\theta=17°$), well beyond the region being considered. For convenience, Eqs. (8) are replaced by

$$\begin{aligned}G_R(q)&=G_R(0)\exp(-\tfrac{1}{4}\lambda_Ra^2q^2),\\G_I(q)&=G_I(0)\exp(-\tfrac{1}{4}\lambda_Ia^2q^2),\\H_R(q)&=H_R(0)\exp(-\tfrac{1}{4}\alpha_Ra^2q^2),\\H_I(q)&=H_I(0)\exp(-\tfrac{1}{4}\alpha_Ia^2q^2).\end{aligned} \quad (10)$$

As long as q^2 remains small, Eqs. (10) will not differ from Eqs. (8) by much.

Upon combining Eqs. (4), (6), (9), and (10), the optical potential, including now the finite range of the nucleon-nucleon interaction, is found to be

$$\begin{aligned}V(r)=-\frac{2\pi\hbar^2c^2}{\pi^{\frac{3}{2}}a^3E}\Bigg\{&\Bigg[G_R(0)(1+\lambda_R)^{-\frac{3}{2}}\exp\left(-\frac{r^2}{a^2(1+\lambda_R)}\right)\\&+iG_I(0)(1+\lambda_I)^{-\frac{3}{2}}\exp\left(-\frac{r^2}{a^2(1+\lambda_I)}\right)\Bigg]\\&+\frac{2}{k^2a^2}\Bigg[-H_I(0)(1+\alpha_I)^{-\frac{5}{2}}\exp\left(-\frac{r^2}{a^2(1+\alpha_I)}\right)\\&+iH_R(0)(1+\alpha_R)^{-\frac{5}{2}}\\&\times\exp\left(-\frac{r^2}{a^2(1+\alpha_R)}\right)\Bigg]\boldsymbol{\sigma}\cdot\mathbf{L}\Bigg\}.\end{aligned} \quad (11)$$

The effect of the angular dependence of G and H is just to increase a by various amounts, with the result that the real and imaginary parts of V_C and $V_{\text{s.o.}}$ no longer have the same angular dependence and $V_{\text{s.o.}}$ is no longer proportional to $(1/r)(dV_C/dr)$. The dominant

[11] J. H. Fregeau, Phys. Rev. **104**, 225 (1956).

TABLE II. The nuclear scattering parameters for the elastic scattering of 313-Mev protons by carbon. The parameters determined by the least-squares fit to the data are given together with those determined directly from the nucleon-nucleon phase shift solutions 1A to 1C and 6A to 6C. (1A)′ and (6B)′ are the parameters calculated from solutions 1A and 6B neglecting the angular dependence of the nucleon-nucleon scattering. B is the parameter obtained by Bethe.[a]

Nuclear parameter	Least squares	1A	1B	1C	6A	6B	6C	(1A)′	(6B)′	B
$g_{NR}(0)$	3.4	2.5	3.1	3.7	1.9	3.0	2.3	2.4	2.5	1.7
$g_{NI}(0)$	9.45[b]	9.6	9.3	9.3	9.2	9.4	9.4	8.9	8.9	8.6
$h_{NR}(0)$	8.06	2.1	2.2	1.8	10.5	7.2	9.8	1.8	8.4	5.3
$h_{NI}(0)$	23	26.0	24.9	27.7	20.8	19.5	20.7	23.9	18.9	20.3
η_R	2.7	1.8	1.9	1.7	1.0	0.97	0.80	0.38	0.38	
η_I	0.38	0.40	0.39	0.39	0.46	0.43	0.45	0.15	0.15	
μ_R		1.4	1.1	1.5	0.40	0.30	0.45	1.5	0.49	
μ_I		0.28	0.29	0.36	0.19	0.18	0.17	0.26	0.22	

[a] See reference 3.
[b] Derived from the total neutron-carbon cross section.

term in $V(r)$ is the imaginary part of the central potential. Since all the phase shift solutions give about the same value for λ_I, we see that the rms radius of the potential is effectively increased from $(\frac{3}{2})^{\frac{1}{2}}a=2.4$ f to $[\frac{3}{2}(1+\lambda_I)]^{\frac{1}{2}}a=2.7$ f.

Let $f_N(q)=g_N(0)+h_N(q)\sigma_n$ be the scattered amplitude resulting from Eq. (11). $g_N(q)$ and $h_N(q)$ can be written in the form

$$\begin{aligned} g_{NR}(q)&=g_{NR}(0)F(q)[1-\eta_R q^2+\cdots],\\ g_{NI}(q)&=g_{NI}(0)F(q)[1-\eta_I q^2+\cdots],\\ h_{NR}(q)&=\sin\theta h_{NR}(0)F(q)[1-\mu_R q^2+\cdots],\\ h_{NI}(q)&=\sin\theta h_{NI}(0)F(q)[1-\mu_I q^2+\cdots]. \end{aligned} \tag{12}$$

In B the parameters η and μ were assumed to be small enough to be neglected at small angles. But, since $F(q)$ is not even the correct Born approximation to the modified potential [Eq. (11)], it will be seen that they are not necessarily small and their inclusion in the analysis helps to obtain a detailed fit to the data.

For small angles and large ka, the scattered amplitudes from a potential of range a is given in good approximation by

$$g_N(q)=ik\int_0^\infty y[1-\exp(2i\Delta_l)]J_0(qy)dy, \tag{13a}$$

$$h_N(q)=ik\int_0^\infty y\epsilon_l \exp(2i\Delta_l)J_1(qy)dy, \tag{13b}$$

where J_0 and J_1 are the ordinary Bessel functions of order zero and one, $y=(l+\frac{1}{2})/k$, and Δ_l and ϵ_l are given in terms of the scattering phase shifts by

$$\begin{aligned} \Delta_l&=\frac{l+1}{2l+1}\delta_{l,\,l+\frac{1}{2}}+\frac{l}{2l+1}\delta_{l,\,l-\frac{1}{2}},\\ \epsilon_l&=\delta_{l,\,l+\frac{1}{2}}-\delta_{l,\,l-\frac{1}{2}}. \end{aligned} \tag{14}$$

The phase shifts resulting from the optical potential were calculated by WKB approximation and the eight parameters in Eqs. (12) computed from Eqs. (13) and (14). The results are listed in Table II for the various phase shift solutions considered. Since in the presten case $V_{\text{s.o.}}$ is not proportional to $(1/r)(dV_C/dr)$, the relation

$$h_N(q)=(H(0)/G(0))\sin\theta g_N(q) \tag{15}$$

is not strictly valid, even in Born approximation. Therefore $h_N(q)$ was computed directly from Eq. (13b) rather than from Eq. (15). The error caused by using Eq. (15) is not very large for $h_{NI}(q)$, but can be as much as a factor of two for $h_{NR}(q)$ even at $q=0$.

All the solutions give essentially the same value for $g_{NI}(0)$, which is in good agreement with the experimental value of 9.45 f. This experimental value is quite reliably determined from the total neutron-carbon cross section. On the other hand, the theoretical value depends almost entirely upon $G_I(0)$, which, in turn, is determined accurately by the total proton-proton and proton-neutron cross sections. This agreement, obtained on the basis of the two best known experimental quantities, is one of the strongest indications of the reliability of the method used here.

$h_{NR}(0)$ is roughly proportional to $H_R(0)-[2H_I(0)\times G_R(0)/ka^2]$. Thus, while the values of $H_R(0)$ differ by a factor of 2.5 between solutions 1 and 6, the values of $h_{NR}(0)$ differ by as much as a factor of 5. The significance of this large difference in the value of $h_{NR}(0)$ will be discussed further in Sec. IV.

For the Coulomb scattering we use the approximation developed in B. The Coulomb scattered amplitude is

$$f_C(q)=g_C(q)+h_C(q)\sigma_N, \tag{16}$$

with

$$\begin{aligned} g_C(q)&=-\frac{2nk}{q^2}[1+2in\ln(0.54/q)]F(q),\\ h_C(q)&=-i\sin\theta\frac{E-mc^2}{mc^2}(\mu-\tfrac{1}{2})g_C(q). \end{aligned} \tag{17}$$

Here $n=(Ze^2/v)=0.0662$ and μ is the total magnetic moment of the proton. The imaginary part of $g_C(q)$ is assumed to be zero for $q\geq 0.54$.

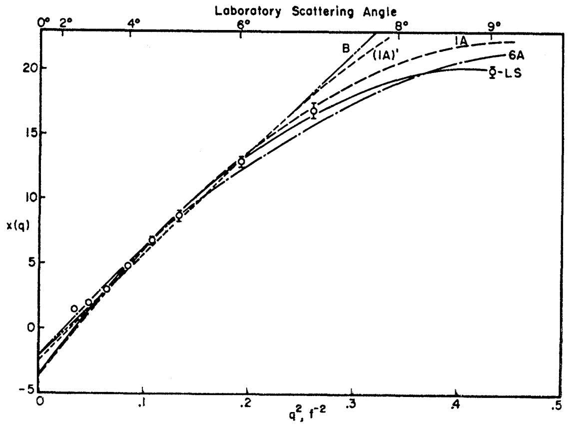

FIG. 1. $x(q)$ calculated from Eq. (24) for several sets of parameters listed in Table III. The curve *LS* is the least-squares fit to the data. The experimental points are derived from the measurements of Chamberlain *et al.* of the differential cross section for the scattering of 313-Mev protons by carbon.

The total scattered amplitude is

$$f(q)=g(q)+h(q)\sigma_n, \tag{18}$$

with

$$\begin{aligned} g(q)&=g_N(q)+g_C(q),\\ h(q)&=h_N(q)+h_C(q). \end{aligned} \tag{19}$$

Then, from Eqs. (12), (17), and (19), the differential and polarized cross sections at small angles are given by

$$\begin{aligned} \frac{d\sigma}{d\Omega}=|g|^2+|h|^2=F^2(q)\bigg\{&\frac{4n^2k^2}{q^4}-\frac{8n^2k}{q^2}g_{NI}(0)\\ &\times(1-\eta_I q^2)\ln\frac{0.54}{q}+\frac{|h|^2}{F^2(q)}+g_{NR}{}^2(0)(1-2\eta_R q^2)\\ &+g_{NI}{}^2(0)(1-2\eta_I q^2)-\frac{4nk}{q^2}g_{NR}(0)(1-\eta_R q^2)\bigg\}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \frac{d\sigma}{d\Omega}P=2\,\mathrm{Re}(g^*h)=\frac{2}{qk}F^2(q)\{&2ankg_{NI}(0)(1-\eta_I q^2)\\ &-4an^2kg_{NR}(0)(1-\eta_R q^2)\ln(0.54/q)-4n^2kh_{NI}(0)\\ &\times(1-\mu_I q^2)\ln(0.54/q)+g_{NR}(0)h_{NR}(0)\\ &\times[1-(\eta_R+\mu_R)q^2]q^2+g_{NI}(0)h_{NI}(0)\\ &\times[1-(\eta_I+\mu_I)q^2]q^2-2nkh_{NR}(0)(1-\mu_R q^2)\}. \end{aligned} \tag{21}$$

Here

$$a\equiv\frac{E-mc^2}{mc^2}(\mu-\tfrac{1}{2})=0.764.$$

III. ANALYSIS OF THE DATA

Chamberlain *et al.*[12] have made accurate measurements of the small-angle scattering and polarization of 313-Mev protons by carbon. In order to deduce the nuclear scattered amplitudes from these data, we proceed with an analysis similar to that done in B. Quantities $x(q)$ and $y(q)$ are defined as follows:

$$\begin{aligned} x(q)\equiv q^2\bigg[&\frac{d\sigma}{d\Omega}\frac{1}{F^2(q)}-\frac{4n^2k^2}{q^4}-\frac{|h|^2}{F^2(q)}\\ &+\frac{8n^2k}{q^2}g_{NI}(0)(1-\eta_I q^2)\ln(0.54/q)\bigg]\\ =&-4nkg_{NR}(0)+[g_{NR}{}^2(0)+g_{NI}{}^2(0)\\ &+4nk\eta_R g_{NR}(0)]q^2\\ &-2[\eta_R g_{NR}{}^2(0)+\eta_I g_{NI}{}^2(0)]q^4, \end{aligned} \tag{22}$$

$$\begin{aligned} y(q)\equiv&\frac{qk}{2F^2(q)}\frac{d\sigma}{d\Omega}P+2ank[g_{NR}(0)(1-\eta_R q^2)2n\\ &\times\ln(0.54/q)-g_{NI}(0)(1-\eta_I q^2)]\\ &+4n^2kh_{NI}(0)(1-\mu_I q^2)\ln(0.54/q)\\ =&-2nkh_{NR}(0)+[g_{NR}(0)h_{NR}(0)+g_{NI}(0)h_{NI}(0)\\ &+2n\mu_R kh_{NR}(0)]q^2-[(\eta_R+\mu_R)g_{NR}(0)h_{NR}(0)\\ &+(\eta_I+\mu_I)g_{NI}(0)h_{NI}(0)]q^4. \end{aligned} \tag{23}$$

In Eq. (22) all the quantities in the brackets are known experimentally except $|h|^2$.[13] However, since for $q^2\ll 1$, $q^2|h|^2$ is small compared to $x(q)$, only a rough estimate of it is needed. We calculate $|h|^2$ from Eqs. (12), (17), and (19) using $h_{NR}(0)=8$ f and $h_{NI}(0)=23$ f. These values are chosen to agree approximately with the values finally deduced from the polarization.[14] Upon using the value 9.45 for $g_{NI}(0)$, Eq. (9) for $F(q)$, and the experimental measurements[12] of $d\sigma/d\Omega$, $x(q)$ can be calculated from Eq. (22). The results are plotted in Fig. 1 as a function of q^2. The errors indicated are only the quoted statistical errors.

Since the first term in Eq. (22) dominates all the others except at the smallest angles where the Coulomb term is large, $x(q)$ is expected to be well determined except at these angles where the Coulomb subtraction makes it uncertain. Using only the values of $x(q)$ from

[12] Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, Phys. Rev. **102**, 1659 (1956).

[13] The factor $(1-\eta_I q^2)$ can be taken equal to unity since the term in which it is a factor is zero for $q\geq 0.54$.

[14] μ_R and μ_I are taken equal to zero in calculating $|h|^2q^2$, so this quantity is over-estimated in the analysis. This effect is unimportant for $q^2\ll 1$, where $|h|^2q^2$ itself is small, but for the largest angle considered (9°) probably results in a value of $x(q)$ 5 to 10% too small.

TABLE III. The parameters a_0, a_1, a_2 and b_0, b_1, b_2 determined by the least-squares fit to the data and calculated from the nucleon-nucleon phase shift solutions 1*A* to 1*C* and 6*A* to 6*C*. (1*A*)′ and (6*B*)′ are the parameters calculated from solutions 1*A* and 6*B* neglecting the angular dependence of the nucleon-nucleon scattered amplitudes. B is the equivalent parameter obtained by Bethe.[a]

Parameter	Least squares	1*A*	1*B*	1*C*	6*A*	6*B*	6*C*	(1*A*)′	(6*B*)′	B
a_0	−3.7	−3.8	−3.5	−4.1	−2.2	−3.3	−2.5	−2.6	−2.8	−1.9
a_1	111	111	102	107	91.0	99.8	95.3	86	86	77.4
a_2	−130	−117	−105	−114	−85.3	−92.8	−88.2	−28	−28	
b_0	−4.5	−1.15	−1.2	−1.0	−5.8	−4.0	−5.5	−1.0	−4.7	−2.9
b_1	247	258	239	265	215	205	219	218	191	184
b_2	−279	−194	−179	−215	−152	−139	−148	−96	−80	

[a] See reference 3.

3.5° to 9°,[15] a least-squares fit of the form

$$x(q)=a_0+a_1q^2+a_2q^4 \tag{24}$$

gives the results

$$\begin{aligned} a_0&=-4nkg_{NR}(0)=-3.7,\\ a_1&=g_{NR}^2(0)+g_{NI}^2(0)+4nk\eta_R g_{NR}(0)=111,\\ a_2&=-2[\eta_R g_{NR}^2(0)+\eta_I g_{NI}^2(0)]=-130.\end{aligned} \tag{25}$$

From the first equation we find $g_{NR}(0)=3.4$ f. This, together with the value of $g_{NI}(0)=9.45$ f taken from the total neutron-carbon cross-section data, gives $\eta_R=2.7$ from the second equation. The determination of η_R in this manner is, of course, very uncertain, but does indicate at least that it is relatively large, i.e., greater than unity. With this value of η_R the last equation gives $\eta_I=0.38$. This value of η_I is more reliable than that for η_R since $g_{NI}^2(0)\gg g_{NR}^2(0)$. These results are listed in Table II for comparison with the same quantities as calculated directly from the nucleon-nucleon phase shifts. In Table III we have listed the parameters a_0, a_1, and a_2 as calculated from the various phase shift solutions.

$x(q)$ calculated from Eqs. (24) and (25) is labeled *LS* in Fig. 1. It fits the experimental points quite well except at the smallest angles (where it was not fitted). Also in Fig. 1, $x(q)$ is shown as calculated from Eq. (24) using the parameters in Table III for phase shift solutions 1*A* and 6*A*. The other solutions give results in between these. Thus, all the solutions give essentially the same fit to the data, in spite of moderate differences in the values which they give for a_0, a_1, and a_2. We shall return to this point in the next section.

Using $x_{LS}(q)$ in Eq. (22), a smoothed value of $d\sigma/d\Omega$ is calculated; it is labeled S and plotted in Fig. 2 along with the experimental points. Also in Fig. 2 we have plotted $d\sigma/d\Omega$ as calculated from Eq. (20) and the phase shift parameters for solutions 1*A* and 6*A*. The divergence of the calculated curves away from the experimental point at 13° is due to the neglect of q^4 terms in Eq. (20).

With the experimental values of g_N and η now determined, Eq. (23) can be used to calculate $y(q)$, using the experimental values of the polarization, P, and the smoothed cross section, $(d\sigma/d\Omega)_S$. For $h_{NI}(0)$ in the first part of Eq. (23) we again use the value of 23 f which is consistent with the value finally found. The result is plotted in Fig. 3. Using only the data from 3.5° to 9°, a least-squares fit of the form

$$y(q)=b_0+b_1q^2+b_2q^4 \tag{26}$$

yields

$$\begin{aligned} b_0&=-2nkh_{NR}(0)=-4.5,\\ b_1&=g_{NR}(0)h_{NR}(0)+g_{NI}(0)h_{NI}(0)+2nk\mu_R h_{NR}(0)\\ &=247,\\ b_2&=-(\eta_R+\mu_R)g_{NR}(0)h_{NR}(0)-(\eta_I+\mu_I)g_{NI}(0)h_{NI}(0)\\ &=-279.\end{aligned} \tag{27}$$

This gives $h_{NR}(0)=8.06$ f. $h_{NI}(0)$ can be calculated

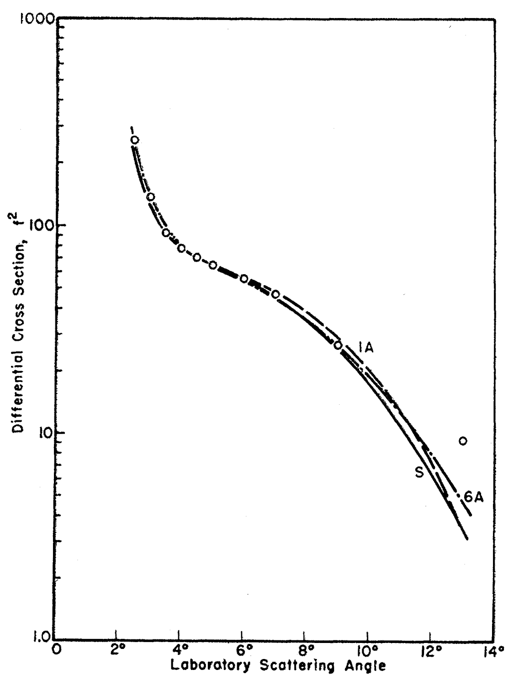

FIG. 2. The differential cross section for the scattering of 313-Mev protons by carbon, calculated from phase shift solutions 1*A* and 6*A*. Curve *S* is calculated from the least-squares parameters given in Table II.

[15] The experimental points from 2.5° to 7° were obtained using an angular resolution of 0.35°. For larger angles the resolution was only 0.84°. However, since for these larger angles the cross section is not as rapidly varying as it is at the smaller angles, the former data are probably as reliable as the latter.

fairly accurately from the second equation in (27) since the third term is only $4.5\mu_R \lesssim 7$. A reasonable value is $h_{NI}(0)=23$ f. These results are listed in Table II. μ_R and μ_I cannot be determined separately from Eqs. (27). In Table III we have listed the values of b_0, b_1, and b_2 as calculated from the various phase shift solutions.

$y(q)$ calculated from Eqs. (26) and (27), and from the phase shift parameters for solutions $1B$ and $6B$, is shown in Fig. 3. The preference for solution $6B$ indicated in Fig. 3 is a result of the larger value it gives for $h_{NR}(0)$. A large value of $h_{NR}(0)$ is needed to give the low polarization at small angles by means of the negative Coulomb interference. This is the main evidence for an imaginary part to the spin-orbit potential, which was noted earlier by Heckrotte.[16] Notice that this result does not depend on the extremely small-angle measurements (2.5°, 3.0°), which are unreliable. Their inclusion in the analysis would make b_0 still more negative and so increase the experimental value of $h_{NR}(0)$ further.

Finally, Fig. 4 shows the experimentally measured polarization along with the phase shift calculations for solutions $1B$ and $6B$. Also shown is the smoothed-out polarization, i.e., P as calculated using the least-squares values of $y(q)$ and $x(q)$. Again, the disagreement at 13° can be interrupted as being due to the neglect of q^6 terms in Eq. (21).

IV. DISCUSSION AND CONCLUSION

The agreement between the experimental and theoretical values of $d\sigma/d\Omega$ shown in Fig. 2 demonstrates rather conclusively the validity of the direct-interaction model. Hence the least-squares values listed in Table II, with the exceptions perhaps of $h_{NR}(0)$ and η_R, may be considered reliable within the variations of the results of phase shift solutions 1 and 6.[17] The corresponding quantities as calculated in B are shown for comparison; $x(q)$ and $y(q)$ calculated from these parameters are shown in Figs. 1 and 3. The difference between these results and those given by the present analysis is due primarily to the inclusion of the q^4 terms in Eqs. (24) and (26), and to the neglect of the data at the smallest angles in making the least-squares fit to $x(q)$. In particular, the larger value of $g_{NR}(0)$ found here is due to the latter. Since solution $6A$, which is in excellent agreement with the differential cross section (Fig. 2), gives a value of $g_{NR}(0)$ similar to that found in B, the smaller value cannot be ruled out. An improvement in the measurement of the differential cross section in the neighborhood of 2° would probably clarify this ambiguity.

To demonstrate the effect of the nucleon-nucleon angular dependence on the proton-carbon scattered amplitude, the scattering parameters have been calculated for solutions $1A$ and $6B$ using the values of $G(0)$ and $H(0)$ given in Table I but taking $\lambda=\alpha=0$. The results, given in columns $(1B)'$ and $(6B)'$ in Tables II and III, show that the main effect of including the angular dependence is to increase $g_N(0)$ and η. The

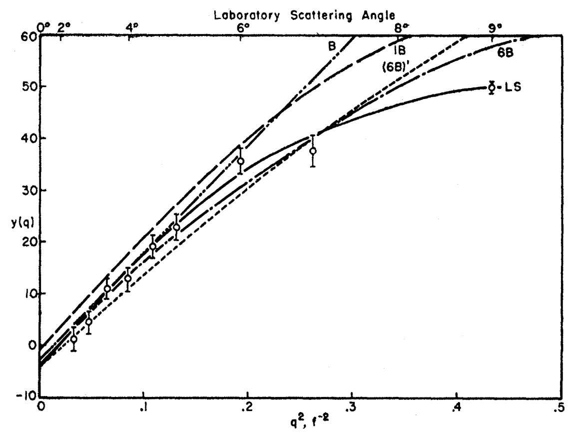

FIG. 3. $y(q)$ calculated from Eq. (26) for several sets of parameters listed in Table II. The curve LS is the least-squares fit to the data. The experimental points are derived from the measurements of Chamberlain *et al.* of the polarization of 313-Mev protons scattered by carbon.

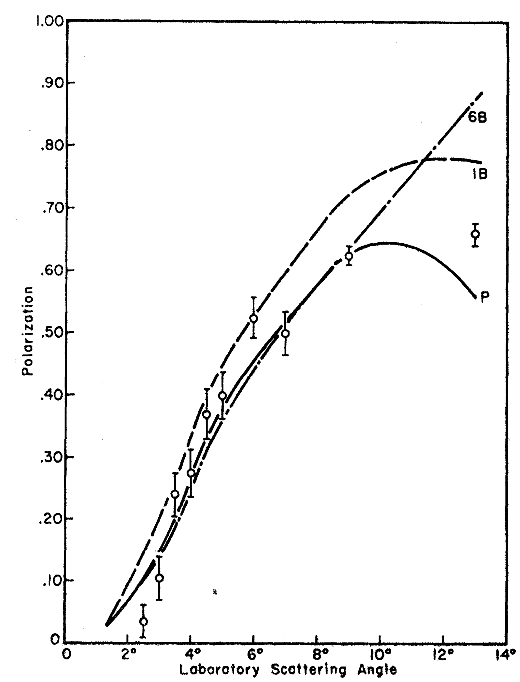

FIG. 4. The polarization of 313-Mev protons scattered by carbon, calculated from phase shift solutions $1B$ and $6B$. Curve P is calculated from the least-squares parameters given in Table II.

[16] W. Heckrotte, Phys. Rev. **101**, 1406 (1956).

[17] The least-squares values of the parameters are based on the absolute differential cross section, which is known to only 20%. This introduces a corresponding uncertainty into these parameters. However, since the fit obtained with the nucleon-nucleon phase shift solutions (Fig. 2) is better than 20%, it is hoped that the absolute calibration is in fact better than 20%.

increase in $g_{NI}(0)$ is just what is needed to obtain agreement with the value of $g_{NI}(0)$ deduced from the total neutron-carbon cross section. Moreover, the large increase in η increases a_2 by nearly a factor of 4, bringing this quantity into agreement with the least-squares value; b_2 is increased by a factor of 2, bringing it closer to the experimental value.

While these results are obtained using a particular shape for the nuclear density, *viz.*, Gaussian, it does not seem likely that these conclusions could be greatly modified by any reasonable change in the distribution. The analysis is limited to small angles where the second moment of the distribution should give the main contribution, and this may be supposed reasonably well known from the electron scattering esperiments.[11] Therefore it appears that in order to fit nucleon-nucleus scattering data by a potential based on the nucleon-nucleon interaction, account must be taken of the nucleon-nucleon angular distribution. Phenomenologically this means the use of optical potentials with different ranges for the real and imaginary parts and with $V_{\mathrm{s.o.}}$ not necessarily proportional to $(1/r)(dV_C/dr)$.

The polarization curves (Fig. 4) indicate the possibility of making a distinction between different sets of nucleon-nucleon phase shifts on the basis of proton-nucleus scattering. However, the slight preference shown here for solution $6B$ is probably not significant. It comes from the larger value of $h_{NR}(0)$ given by solution 6, and any conclusions drawn from the value of $h_{NR}(0)$ are subject to various uncertainties.[18] It is worth emphasizing, however, that the experimental data at the smallest angles were omitted in determining $h_{NR}(0)$; the inclusion of these data in the analysis would increase $h_{NR}(0)$ still further.

The triple-scattering parameter β may also be used to obtain information about the nucleon-nucleus interaction. From the expression

$$\sin\beta = 2\left[\frac{d\sigma}{d\Omega}(1-P^2)^{\frac{1}{2}}\right]^{-1}[g_I(q)h_R(q)-g_R(q)h_I(q)], \quad (28)$$

β can be computed for small angles using Eqs. (12), (17), and (19) and the parameters in Table II. Unfortunately no small-angle measurements have yet been made. The smallest angle for which data are available is 10.4°; here $\beta=-9°\pm18°$ or $30°\pm18°$.[12] [There is an ambiguity in β because only the quantity $\cos(\theta-\beta)$ is measured experimentally.] At 9° phase shift solutions $1A$, $6A$ and $6B$ give β equal to 6.5°, 26°, and 13°, respectively; the least-squares parameters (together with $\mu_R=1$ and $\mu_I=0.25$) give $\beta=25°$. Thus at 9° all solutions give $\beta>0$. While the formulas developed in this paper do not apply for $\theta>9°$, it is certain that at larger angles β will become still more positive.

This conclusion differs from that of other authors,[4,12] who find $\beta<0$ in the neighborhood of 10°. At small angles ($\theta\lesssim 20°$) $g_I(q)$, $h_R(q)$, and $h_I(q)$ are all positive, so that from Eq. (28) we see that β can be negative in this region only if $g_R(q)$ is also positive. Now $g_R(q)\cong F(q)\times[g_{NR}(0)(1-\eta_R q^2)-2nk/q^2]$, which is negative at very small angles. It can become positive only if the Coulomb term becomes smaller (in absolute value) than the nuclear term before the latter goes through zero. In Born approximation $\eta_R=0$, and so this may happen at about 5°. Even in more exact calculations η_R is not large if the angular dependence of the nucleon-nucleon scattering is not included in the potential (see Table II), and so $g_R(q)$ may still become positive. However, because of the large value of η_R found here, $g_{NR}(q)$ decreases more rapidly than the Coulomb term and so $g_R(q)$ is never positive in this region; consequently β is positive. Solution 6 gives a larger value of β than solution 1 because of the very large value it gives for $h_{NR}(0)$. While both solutions are consistent with the present triple-scattering data, an improvement in these data may eventually be able to rule out one of the solutions.†

V. ACKNOWLEDGMENTS

It is a pleasure to thank Professor Bethe for suggesting this problem and for several stimulating discussions. The author also wishes to thank Mr. H. Y. Chiu for his help with the digital computer.

[18] The experimental determination depends mainly on the polarization at small angles, which is poorly known; it also depends on the value of $h_{NI}(0)$ used on the first line of Eq. (23). The theoretical determination, on the other hand, is quite sensitive to the nucleon-nucleon parameters $H_R(0)$ and $H_I(0)$ (Sec. II).

† *Note added in proof.*—A modified phase shift analysis of the proton-proton scattering at 310 Mev [Moravcsik, Cziffra, MacGregor, and Stapp, Bull. Am. Phys. Soc. Ser. II, 4, 49 (1959).] strongly favors solution 1 over solution 6. This may indicate that there is a systematic error in the small angle proton-carbon polarization data.