Relativistic Hydrodynamics for a Charged Nonviscous Fluid

CHAU-CHIN WEI
Committee on Nuclear Power Development, Taiwan Power Company, Taipei, Taiwan, China
(Received October 8, 1958)

The equations of relativistic hydrodynamics are derived from an alternative variational method and a generalized vorticity equation is obtained.

THE equations of relativistic hydrodynamics can be derived by the application of a variational principle.[1] In this note it will be shown that an alternative variational method, which closely resembles that of field dynamics, can describe the behavior of an ideal compressible fluid.

Let us consider relativistic hydrodynamics in the Minkowski space with coordinates x_μ and metric $g_{\mu\nu}$.

The variational principle can be expressed

$$\delta\int L d^4x=0, \tag{1}$$

where we define L as follows with a slight modification of the nonrelativistic Lagrangian density.[2]

$$L=\rho\left\{\tfrac{1}{2}Ju_\mu u^\nu-u_\mu\left(\frac{\partial\psi}{\partial x_\mu}+\alpha\frac{\partial\beta}{\partial x_\mu}\right)+p-\frac{e}{m}\varphi\right\}, \tag{2}$$

where ρ, u_μ, m, and e are the density and four-velocity of the fluid, and the mass and charge of a particle, respectively. φ is a scalar potential, p is the compressional energy per unit mass, and J is the heat content per unit mass (rest).

First, variation of α and β give, respectively,

$$\frac{d}{d\tau}(\beta)=0, \tag{3}$$

and

$$\frac{d}{d\tau}(\alpha)=0, \tag{4}$$

where

$$d/d\tau\equiv u_\mu\partial/\partial x_\mu \tag{5}$$

denotes the substantial derivative. Equations (3) and (4) indicate the persistence of vorticity and imply that $\alpha=$ const, $\beta=$ const, represent a generalized vortex line.

Varying ψ, we have the equation of continuity

$$\frac{\partial}{\partial x_\mu}(\rho u_\mu)=0. \tag{6}$$

Next we obtain, varying u_μ, the Clebsch transformation[3]

$$Ju_\mu+\frac{e}{m}A_\mu=\frac{\partial\psi}{\partial x^\mu}+\alpha\frac{\partial\beta}{\partial x^\mu}, \tag{7}$$

where A_μ means the vector potential.

Finally, the variation of ρ gives the equation of motion which can be written by means of a certain rearrangement in the form

$$\frac{\partial}{\partial x_\mu}(\tfrac{1}{2}\rho Ju_\mu u_\nu+pg_{\mu\nu})=\frac{e}{m}\rho u^\gamma F_{\mu\gamma}, \tag{8}$$

where we assume $\alpha=\beta=0$, and $F_{\mu\gamma}$ stands for the generalized field tensor.

Equations (3), (4), (6), (7), and (8) describe the relativistic motion of a charged fluid with the equation of thermodynamics and equation of state. For example, we adopt

$$TdS=dJ-\rho^{-1}dp, \tag{9}$$

as the equation of thermodynamics, where T is the temperature, S is the entropy, and we assume the state to be barotropic. Then, from (8) we have

$$u_\mu\frac{\partial}{\partial x_\mu}(Ju_\nu)=-\frac{1}{\rho}\frac{\partial p}{\partial x^\nu}+\frac{e}{m}u^\gamma F_{\nu\gamma}, \tag{10}$$

where we have utilized Eq. (6).

By means of (9), Eq. (10) becomes

$$u^\mu\frac{\partial}{\partial x^\mu}(Ju_\nu)=\left(T\frac{\partial S}{\partial x^\nu}-\frac{\partial J}{\partial x^\nu}\right)+\frac{e}{m}u^\mu F_{\nu\mu}. \tag{11}$$

By differentiation of $u^\mu u_\mu=-1$, we see that

$$u^\mu\frac{\partial}{\partial x^\nu}(Ju_\mu)=-\frac{\partial J}{\partial x^\nu}. \tag{12}$$

Hence, combining (11) and (12), we have

$$u^\mu\left\{\frac{\partial}{\partial x^\mu}(Ju_\nu)-\frac{\partial}{\partial x^\nu}(Ju_\mu)\right\}=T\frac{\partial S}{\partial x_\mu}+\frac{e}{m}u^\mu F^{\nu\mu}, \tag{13}$$

or

$$u^\mu\left\{\frac{\partial}{\partial x^\mu}\left(Ju_\nu+\frac{e}{m}A_\nu\right)-\frac{\partial}{\partial x^\nu}\left(Ju_\mu+\frac{e}{m}A_\mu\right)\right\}=J\frac{\partial S}{\partial x_\nu}. \tag{14}$$

This is the generalization of the vorticity equation.

[1] I. M. Halatnikov, Zhur. Eksptl.' i Teoret. Fiz. **27**, 529 (1954).
[2] N. Mikoshiba, Progr. Theoret. Phys. (Kyoto) **13**, 627 (1955).
[3] For example, H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), sixth edition, p. 248.