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Low-Temperature Behavior of a Dilute Bose System of Hard Spheres. II. Nonequilibrium Properties*

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The calculation of a previous paper is extended to cover nonequilibrium properties. The phenomena of superfluidity, critical velocity, and "infinite heat conductivity" are given natural explanations. By using classical kinetic theory on the wave packets, hydrodynamical equations for reversible flow are derived and the dependence of the two "sound velocities" on the temperature studied. The relationship between macroscopic sound vibrations and microscopic excitations is analyzed. The work is confined to the model of a dilute hard-sphere Bose system.

1. INTRODUCTION

In the present paper we continue the study of a dilute hard-sphere Bose system by the pseudopotential method. The energy level calculation of the previous paper[1] is first extended to those levels for which there is macroscopic but incomplete occupation of a free-particle state with nonvanishing momentum. Such an extension leads to the concept of quasi-equilibrium states, the thermodynamical behavior of which is discussed. The separation into two components, the superfluid component and the normal fluid component, in this discussion is a natural and explicit *mathematical* notion, and not based on physical or heuristic arguments. Superfluidity and the existence of heat transfer in the absence of a temperature gradient are also natural consequences.

To discuss the hyerodynamics of the system, classical kinetic theory concepts are borrowed for the wave packets formed out of the quantum mechanical energy levels. Reversible flow is discussed in such a picture and hydrodynamical equations of motion obtained. The dependence of the "sound velocities" on temperature is then analyzed. The question of whether the superfluid flow is irrotational is not resolved in this paper. Also, it is to be emphasized that the general problem of transport phenomena in quantum mechanics is not discussed in this paper.

2. GALILEAN TRANSFORMATION AND QUASI-EQUILIBRIUM STATES

In paper I,[1] the energy spectrum of a dilute Bose system of hard spheres is found to be

$$E(\xi,m_{\mathbf{k}})=4\pi a\rho N[1+(1-\xi)^2]+\sum_{\mathbf{k}\neq0} m_{\mathbf{k}}k(k^2+16\pi a\xi\rho)^{\frac{1}{2}}, \quad (1)$$

where

$$N^{-1}\sum_{\mathbf{k}\neq0} m_{\mathbf{k}}=(1-\xi)+O(N^{-1}). \quad (2)$$

In each of these eigenstates only the occupation number for the unperturbed individual (free) particle state with $\mathbf{k}=0$ is of the order of N.

By applying a Galilean transformation to the whole system, we can easily obtain a new eigenstate in which there is a macroscopic occupation of a free-particle state with $\mathbf{k}\neq0$. Such an eigenstate can also be explicitly generated by using a unitary transformation with the unitary operator

$$\exp\left[i\sum_{j=1}^{N}\mathbf{k}_s\cdot\mathbf{r}_j\right]$$

on an eigenstate of paper I. In the new eigenstate the occupation number for the free-particle state $\mathbf{k}=\mathbf{k}_s$ is ξN [neglecting terms proportional to $(\rho a^3)^{\frac{1}{2}}$]. The corresponding energy and momentum for the new eigenstate are

$$E(\xi,m_{\mathbf{k}},\mathbf{k}_s)=N\mathbf{k}_s^2+\sum_{\mathbf{q}\neq0} m_{\mathbf{q}}(\omega_{\mathbf{q}}+2\mathbf{q}\cdot\mathbf{k}_s)+4\pi a\rho N[1+(1-\xi)^2], \quad (3)$$

$$\mathbf{P}=N\mathbf{k}_s+\sum_{\mathbf{q}\neq0} m_{\mathbf{q}}\mathbf{q}, \quad (4)$$

where $\mathbf{q}$ is the momentum of the phonons relative to $\mathbf{k}_s$,

$$\mathbf{q}=\mathbf{k}-\mathbf{k}_s, \quad (5)$$

and

$$\omega_{\mathbf{q}}=q(q^2+16\pi a\xi\rho)^{\frac{1}{2}}. \quad (6)$$

The m_q's are positive integers that satisfy

$$N^{-1}\sum_{\mathbf{q}\neq0} m_{\mathbf{q}}=(1-\xi)+O(N^{-1}), \quad (7)$$

and ξ is a parameter between 0 and 1.

The parameters ξ, $m_{\mathbf{q}}$, and $\mathbf{k}_s$ are, in essence, quantum numbers describing the various energy states of the whole macroscopic system. (The states discussed in paper I correspond to those with $\mathbf{k}_s=0$.) It is important to recognize, however, that they are not absolute quantum numbers. For example, as discussed in paper I, each phonon can decay into two phonons of longer wavelengths with a mean life τ and two phonons can

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958); referred to as I in this paper.

scatter each other, changing into other phonons of different wavelengths. It is only when these effects can be treated as small perturbations that ξ, $m_{\mathbf{q}}$, and $\mathbf{k}_s$ become quantum numbers. In such an approximation one can calculate the thermodynamical equilibrium of the system at a given density ρ, a given temperature T, and a given total momentum $\mathbf{P}$, and obtain the equilibrium values of ξ, $m_{\mathbf{q}}$, and $\mathbf{k}_s$. The calculation of the equilibrium properties made in paper I was exactly such a calculation for the special case of $\mathbf{P}=0$. (One can prove that for $\mathbf{P}=0$, the equilibrium value of $\mathbf{k}_s$ vanishes. Now by the method of steepest descent, one can neglect the nonequilibrium values of such parameters as $\mathbf{k}_s$ in a calculation of the equilibrium properties. Therefore it was legitimate to neglect altogether states with $\mathbf{k}_s\neq 0$ in paper I.) For the case $\mathbf{P}\neq 0$ the equilibrium values of the parameters ξ, $m_{\mathbf{q}}$, and $\mathbf{k}_s$ can be directly obtained from (3) and (4) or can be obtained by a Galilean transformation applied to the case $\mathbf{P}=0$ discussed in paper I. The most probable value of $\mathbf{k}_s$ is thus seen to be

$$\mathbf{k}_s^{\mathrm{m.p.}}=N^{-1}\mathbf{P}. \tag{8}$$

In this paper we turn our attention to states which are not in equilibrium. Because of the existence of a mean life τ for the phonons, it is clear that any deviation of ξ and $m_{\mathbf{q}}$ from their equilibrium values would disappear over a relaxation time which is of the order of τ or maybe smaller. On the other hand, by its very nature the quantity $\mathbf{k}_s$ is a long-range-order parameter. It is much more difficult to have transitions between states with different values of $\mathbf{k}_s$. In order to have such transitions it is necessary to have a *coherent* change involving simultaneously $\sim\xi N$ particles. We may, therefore, expect states with $\mathbf{k}_s$ different from its most probable value to exist as quasi-equilibrium systems, and we shall discuss their thermodynamical behavior in the next section.

It may be emphasized that the application of thermodynamics to a macroscopic system in quasi-equilibrium is a familiar subject in physics. The entire subject of static problems in elasticity deals with systems in quasi-equilibrium. To see this, we notice that for an infinite solid in absolute thermodynamical equilibrium the statistical average value of any shearing strain must be zero, independently of the amount of stress applied on the surface. (We do not consider here the stress produced by an external gravitational field.) This state of absolute equilibrium can usually be achieved by developing slips or cracks in the solid to relieve the system of the applied stress. However, because of the long time-interval required to attain such absolute thermodynamical equilibrium, it is usual practice to extend the application of thermodynamics to the quasi-equilibrium system of an elastic solid under strain.

Similarly, for a system of Bose particles with the total momentum $\mathbf{P}$, we shall consider quasi-equilibrium states with a value of $\mathbf{k}_s$ not necessarily equal to the $\mathbf{k}_s^{\mathrm{m.p.}}$ of (8).

3. THERMODYNAMICAL FUNCTIONS FOR THE QUASI-EQUILIBRIUM STATES

In this section we shall derive the thermodynamical functions for these quasi-equilibrium states. Throughout the present paper we shall be concerned only with the degenerate phase, i.e., $T<T_c$.

The partition function $Q(\mathbf{k}_s,\mathbf{P})$ for the quasi-equilibrium system with $\mathbf{k}_s$ and total momentum $\mathbf{P}$ can be defined as

$$Q(\mathbf{k}_s,\mathbf{P})=\int_0^1 d\xi \sum_{m_{\mathbf{q}}} \exp[-\beta E(\xi,m_{\mathbf{q}},\mathbf{k}_s)], \tag{9}$$

where the sum extends over all $m_{\mathbf{q}}$ that satisfy (4) and (7). Using the method of steepest descent, it is straightforward to find the most probable values of ξ and $m_{\mathbf{q}}$. These values are given by

$$(1-\bar{\xi})\rho=(8\pi^3)^{-1}\int \bar{m}_{\mathbf{q}} d^3\mathbf{q}, \tag{10}$$

$$\bar{m}_{\mathbf{q}}=\frac{\zeta \exp[-\beta(\omega_q-\mathbf{q}\cdot\mathbf{u})]}{1-\zeta \exp[-\beta(\omega_q-\mathbf{q}\cdot\mathbf{u})]}, \quad (\mathbf{q}\neq 0) \tag{11}$$

where the parameters ζ and $\mathbf{u}$ are determined by

$$\kappa T \ln\zeta=-a\pi^{-2}\int d^3q\, \bar{m}_{\mathbf{q}}[1-q(q^2+16\pi a\bar{\xi}\rho)^{-\frac{1}{2}}], \tag{12}$$

$$\Omega^{-1}\mathbf{P}=\rho\mathbf{k}_s+(8\pi^3)^{-1}\int d^3\mathbf{q}\, \bar{m}_{\mathbf{q}}\mathbf{q}. \tag{13}$$

With the aid of these parameters it is convenient to describe the present system as composed of two separate components, the "superfluid" and the "normal" fluid [similar to the two-fluid model first proposed by Tisza[2,3]]. We introduce, *as purely formal terminologies*, the following definitions for the normal fluid density ρ_n, superfluid density ρ_s, the normal fluid velocity $\mathbf{v}_n$, and the superfluid velocity $\mathbf{v}_s$:

$$\rho_n\equiv(1-\bar{\xi})\rho, \tag{14}$$

$$\rho_s\equiv\bar{\xi}\rho, \tag{15}$$

$$\mathbf{v}_n\equiv\mathbf{v}_s+\mathbf{u}, \tag{16}$$

$$\mathbf{v}_s\equiv 2\mathbf{k}_s. \tag{17}$$

The factor 2 in (17) occurs because we had chosen units such that $2m=1$.

By using (10) and (11), we find that ρ_n and $\mathbf{v}_n$ are

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[2] L. Tisza, J. phys. radium **1**, 164 (1940).

[3] See F. London, *Superfluids* (John Wiley and Sons, Inc., New York, 1954).

related to $\bar{m}_{\mathbf{q}}$ by

$$\rho_n=(8\pi^3)^{-1}\int d^3\mathbf{q}\ \bar{m}_{\mathbf{q}},$$
$$\mathbf{v}_n=\mathbf{v}_s+(8\pi^3\rho_n)^{-1}\int d^3\mathbf{q}\ \bar{m}_{\mathbf{q}}\boldsymbol{\nabla}_{\mathbf{q}}\omega_q, \tag{18}$$

where $\boldsymbol{\nabla}_{\mathbf{q}}$ is the partial differential operator with respect to $\mathbf{q}$. To derive the second equation of (18) use has been made of the fact that

$$\int \bar{m}_{\mathbf{q}}[\boldsymbol{\nabla}_{\mathbf{q}}\omega_{\mathbf{q}}-\mathbf{u}]d^3\mathbf{q}=0.$$

Thus, ρ_n is the average number of phonons per unit volume and $\mathbf{v}_n$ the average group velocity of these phonons. Similarly, we define the superfluid momentum per particle as $\mathbf{k}_s$ and the normal fluid momentum per particle as $\mathbf{k}_n$ where

$$\mathbf{k}_n=\mathbf{k}_s+(8\pi^3\rho_n)^{-1}\int m_{\mathbf{q}}\mathbf{q}d^3\mathbf{q}, \tag{19}$$

so that

$$\Omega^{-1}\mathbf{P}=\rho_s\mathbf{k}_s+\rho_n\mathbf{k}_n.$$

The Helmholtz free energy for this quasi-equilibrium system is related to $Q(\mathbf{k}_s,\mathbf{P})$ by

$$F=-\kappa T\ \ln Q(\mathbf{k}_s,\mathbf{P}). \tag{20}$$

In terms of the above-defined parameters, F can be written as

$$\Omega^{-1}F=\rho_s\mathbf{k}_s^2+\rho_n[\mathbf{k}_s^2+(\mathbf{k}_n-\mathbf{k}_s)\cdot\mathbf{v}_n]+\kappa T\rho_n\ln\zeta$$
$$+\kappa T(8\pi^3)^{-1}\int d^3\mathbf{q}\ \ln\{1-\zeta\ \exp[-\beta(\omega_q-\mathbf{q}\cdot\mathbf{u})]\}$$
$$+4\pi a(\rho^2+\rho_n^2). \tag{21}$$

The other thermodynamical functions for this quasi-equilibrium system can be defined in terms of F by the familiar relations

$$S=-\left(\frac{\partial F}{\partial T}\right)_{\Omega,\mathbf{k}_s,\mathbf{P}},\quad E=F+TS,\quad p=-\left(\frac{\partial F}{\partial\Omega}\right)_{T,\mathbf{k}_s,\mathbf{P}}.$$

These functions can be explicitly written as

$$\Omega^{-1}E=\rho k_s^2+(8\pi^3)^{-1}\int \bar{m}_{\mathbf{q}}(\omega_q+\mathbf{q}\cdot\mathbf{v}_s)d^3q$$
$$+4\pi a(\rho^2+\rho_n^2), \tag{22}$$

$$\Omega^{-1}S=-\kappa\rho_n\ \ln\zeta+(8\pi^3T)^{-1}\int \bar{m}_q(\omega_q-\mathbf{q}\cdot\mathbf{u})d^3\mathbf{q}-\kappa(8\pi^3)^{-1}$$
$$\times\int \ln\{1-\zeta\ \exp[-\beta(\omega_q-\mathbf{q}\cdot\mathbf{u})]\}d^3\mathbf{q}, \tag{23}$$

$$p=-\kappa T(8\pi^3)^{-1}\int \ln\{1-\zeta\ \exp[-\beta(\omega_q-\mathbf{q}\cdot\mathbf{u})]\}d^3\mathbf{q}$$
$$+\pi^{-2}a\rho_s\int \bar{m}_{\mathbf{q}}\omega_q d^3\mathbf{q}+4\pi a(\rho^2+\rho_n^2). \tag{24}$$

It is of interest to notice that both $\Omega^{-1}F$ and $\Omega^{-1}E$ are functions depending on the four independent variables T, Ω, $\mathbf{k}_s$, and $\mathbf{P}$. However, because of Galilean invariance it is possible to construct other functions which depend only on three independent variables. If, for example, we define the Gibbs free-energy function per particle, μ, by

$$\rho\mu\equiv\Omega^{-1}F+p-(\mathbf{k}_n-\mathbf{k}_s)\rho_n\cdot\mathbf{v}_n-\rho\mathbf{k}_s^2, \tag{25}$$

then

$$d\mu=-sdT+\rho^{-1}dp-(\mathbf{k}_n-\mathbf{k}_s)\cdot d\mathbf{u}, \tag{26}$$

where s is the average entropy per particle,

$$s=N^{-1}S. \tag{27}$$

For our present system the explicit form of μ is

$$\mu=\kappa T\ \ln\zeta+8\pi a(\rho+\rho_n). \tag{28}$$

4. SUPERFLUIDITY AND CRITICAL VELOCITY

The existence of quasi-equilibrium states is directly related to the phenomena of superfluidity. We shall discuss some general aspects of these relationships:

(i) In these quasi-equilibrium systems, $\mathbf{k}_s$ is not determined by the total momentum of the system. This additional freedom allows for, for example, heat flow (i.e., entropy transfer) in the absence of a temperature gradient. (The detail of these transport problems will be treated in the next section.)

(ii) From (11), we see that there must be an upper limit to the relative velocity $\mathbf{u}$ between the superfluid component and the normal fluid component. Since $\bar{m}_{\mathbf{q}}$ must be positive, we determine this upper limit to be

$$|\mathbf{u}|=|\mathbf{v}_n-\mathbf{v}_s|<(16\pi a\xi\rho)^{\frac{1}{2}}. \tag{29}$$

(iii) The existence of a superfluid flow and its lack of viscosity can be understood from a mechanical point of view by examining the energy spectrum of the system. The following simple argument, due to Landau,[4] is particularly instructive in this connection.

For simplicity, let us suppose the superfluid velocity to vanish ($\mathbf{k}_s=\frac{1}{2}\mathbf{v}_s=0$). Consider now an external object with velocity $\mathbf{v}_e$ moving through this system. We shall show that due to the nature of the energy spectrum (3) and (4), this external object cannot transfer momentum and energy to the system through pure excitation of any finite number of phonons, provided

$$|\mathbf{v}_e|<(16\pi a\xi\rho)^{\frac{1}{2}}. \tag{30}$$

To see this, let us assume the external object can excite a certain number of phonons, say, $m_{\mathbf{k}}$ ($m_{\mathbf{k}}\geqq 0$); the amount of energy exchange and momentum exchange must then be given by

$$\delta E=\sum m_{\mathbf{k}}\omega_{\mathbf{k}},\quad \delta\mathbf{P}=\sum m_{\mathbf{k}}\mathbf{k}.$$

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[4] L. D. Landau, J. Phys. U.S.S.R. **5**, 71 (1940)

From (6), and the fact that $m_{\mathbf{k}} \geqq 0$, we have the inequality

$$\delta E \geqq (16\pi a\bar{\xi}\rho)^{\frac{1}{2}}|\delta\mathbf{P}|. \tag{31}$$

On the other hand, we have the relation

$$\delta E = \mathbf{v}_e \cdot \delta\mathbf{P}.$$

Consequently, for $|\mathbf{v}_e|$ smaller than the critical velocity $(16\pi a\bar{\xi}\rho)^{\frac{1}{2}}$, the only way to have energy and momentum transfer is through scatterings between the external object and the existing phonons. The amount of viscosity experienced by the external object thus depends on the number of phonons present which vanishes at zero temperature.

It may be emphasized that the present system behaves very differently from a free Bose system. In particular, from (29) and (30) it is clear that the phenomena of superfluidity depends critically on the fact that there are interactions ($a \neq 0$) between these Bose particles.

5. APPLICATION OF KINETIC THEORY TO THE QUASI-EQUILIBRIUM SYSTEMS

In a quasi-equilibrium state for which $\mathbf{v}_n - \mathbf{v}_s \neq 0$, there exists a steady flow of mass, entropy, and energy relative to each other. To calculate quantitatively the amount of these fluxes, pure thermodynamical considerations become inadequate. In the following we shall extend the concepts used in ordinary kinetic theory to the present problem.

(i) We first visualize each phonon of momentum $\mathbf{k}$ ($\mathbf{k} = \mathbf{q} + \mathbf{k}_s$) as a wave packet which moves with a group velocity

$$\mathbf{v}_s + \boldsymbol{\nabla}_{\mathbf{q}}\omega_{\mathbf{q}}.$$

Through any small plane surface $\mathbf{A}$ in the system, there is, then, a steady flow of these phonons at the rate of

$$\delta\bar{m}_{\mathbf{q}} = \Omega^{-1}\bar{m}_{\mathbf{q}}(\mathbf{v}_s + \boldsymbol{\nabla}_{\mathbf{q}}\omega_{\mathbf{q}}) \cdot \mathbf{A} \tag{32}$$

per unit time. Similarly, there is also a steady flow of superfluid particles at the rate of

$$\delta\rho_s = \rho_s \mathbf{v}_s \cdot \mathbf{A} \tag{33}$$

per unit time passing through $\mathbf{A}$. Using (18) we find that $\mathbf{J}_\rho$, defined as the average flux of particles per unit area, is given by

$$\mathbf{J}_\rho = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s. \tag{34}$$

(ii) Next, let us define T_{ij} to be the jth component of the average flux of the ith component of momentum per unit area. We should expect T_{ij} to be composed of two parts: one that is due to the average motion of the two fluids

$$\rho_s(k_s)_i(v_s)_j + \rho_n(k_n)_i(v_n)_j, \tag{35}$$

and the other due to the mutual interactions between particles and the relative motions of the phonons which, according to kinetic theory, is simply

$$\delta_{ij}p, \tag{36}$$

where p is the gas pressure given by (24) and δ_{ij} is the Kronecker δ-symbol. Combining these two effects, we find

$$T_{ij} = \delta_{ij}p + \rho_s(k_s)_i(v_s)_j + \rho_n(k_n)_i(v_n)_j. \tag{37}$$

(iii) To calculate the entropy flux $\mathbf{J}_s$, we recall that according to (23) the entropy function S can be written as

$$\Omega^{-1}S = (8\pi^3)^{-1}\int \bar{m}_{\mathbf{q}} s_{\mathbf{q}} d^3\mathbf{q}, \tag{38}$$

where

$$\begin{aligned} s_{\mathbf{q}} = -\kappa \ln\zeta + T^{-1}(\omega_q - \mathbf{q}\cdot\mathbf{u}) \\ -\kappa(\bar{m}_{\mathbf{q}})^{-1}\ln\{1 - \zeta\exp[-\beta(\omega_q - \mathbf{q}\cdot\mathbf{u})]\}. \end{aligned} \tag{39}$$

Again from kinetic considerations we expect that accompanying any change $\delta\bar{m}_{\mathbf{q}}$ there is also a change δS in entropy, with

$$\Omega^{-1}\delta S = (8\pi^3)^{-1}\int \delta\bar{m}_{\mathbf{q}} s_{\mathbf{q}} d^3\mathbf{q}.$$

Combining with (32), we obtain the expression for $\mathbf{J}_s$ as

$$\mathbf{J}_s = (8\pi^3)^{-1}\int \bar{m}_{\mathbf{q}} s_{\mathbf{q}}(\mathbf{v}_s + \boldsymbol{\nabla}_{\mathbf{q}}\omega_{\mathbf{q}}) d^3\mathbf{q}. \tag{40}$$

Using (40) and (11), $\mathbf{J}_s$ can be written in a very simple form,

$$\mathbf{J}_s = \rho s \mathbf{v}_n, \tag{41}$$

where s is defined in (27). This equation shows that in the absence of a temperature gradient and a mass flux (i.e., T = constant, $\mathbf{J}_\rho = 0$), there could be maintained a steady quasi-stationary entropy flux.

(iv) In a similar way we can also try to obtain the expression for the energy flux $\mathbf{J}_E$. Consider a virtual variation of $\delta\bar{m}_{\mathbf{q}}$ and $\delta\rho_s$ in the expression of energy density given by (22). We find

$$\Omega^{-1}\delta E = \epsilon_s\delta\rho_s + (8\pi^3)^{-1}\int \epsilon_{\mathbf{q}}\delta\bar{m}_{\mathbf{q}} d^3\mathbf{q}, \tag{42}$$

where

$$\epsilon_s = 8\pi a\rho + \mathbf{k}_s^2 + a\pi^{-2}\int \mathbf{q}^2\omega_{\mathbf{q}}^{-1}\bar{m}_{\mathbf{q}} d^3\mathbf{q}, \tag{43}$$

$$\epsilon_{\mathbf{q}} = \omega_{\mathbf{q}} + \mathbf{q}\cdot\mathbf{v}_s + \mathbf{k}_s^2 + 8\pi a(\rho + \rho_n). \tag{44}$$

Substituting (32) and (33) into (42), we find the energy flux $\mathbf{J}_E$ to be

$$\mathbf{J}_E = \rho_s\epsilon_s\mathbf{v}_s + (8\pi^3)^{-1}\int \bar{m}_{\mathbf{q}}\epsilon_{\mathbf{q}}(\mathbf{v}_s + \boldsymbol{\nabla}_{\mathbf{q}}\omega_{\mathbf{q}}) d^3\mathbf{q}, \tag{45}$$

which, with the aid of (43), (44), and (11), can also be written as

$$\mathbf{J}_E = \mathbf{v}_n(\rho\epsilon + p) - \mathbf{u}(\rho_s\mu + \rho_s\mathbf{k}_s^2), \tag{46}$$

where μ is given by (25) and

$$\rho\epsilon = \Omega^{-1}E. \tag{47}$$

6. TRANSPORT EQUATIONS

In preceding sections we considered various quantities such as the thermodynamic functions and mass flux, momentum flux, etc., for a quasi-equilibrium system in which ρ_s, ρ_n, $\mathbf{k}_s$, and $\mathbf{k}_n$ are constant parameters characterizing the system. We shall now extend our considerations to a system in which these parameters may be slowly varying functions in space.

We shall assume that there exists a length, say l, which is much longer than the mean free path of the phonon. Yet, over this length l any variation of ρ_s, ρ_n, k_s, and k_n can be neglected. The entire volume Ω can then be subdivided into smaller volumes of linear size $\sim l$. We may apply our previous considerations to each of these small volumes and treat these small systems as in quasi-equilibrium. In the language of ordinary kinetic theory, the neglect of any spatial variation over a length long compare to the mean free path means that only reversible processes are being considered.

The transport equations for this problem are, then, simply the equations for conservation of mass, momentum, entropy, and energy. These equations can be written as

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\mathbf{J}_\rho, \tag{48}$$

$$\frac{\partial}{\partial t}(\rho_n\mathbf{k}_n+\rho_s\mathbf{k}_s) = -\frac{\partial}{\partial x_j}T_{ij}, \tag{49}$$

$$\frac{\partial}{\partial t}(\rho s) = -\nabla\cdot\mathbf{J}_s, \tag{50}$$

$$\frac{\partial}{\partial t}(\rho\epsilon) = -\nabla\cdot\mathbf{J}_E. \tag{51}$$

In (49) we adopt the convention of summing over the repeated index j ($j=x, y, z$).

Using the explicit forms of $\mathbf{J}_\rho$, $\mathbf{J}_s$, $\mathbf{J}_E$, and T_{ij}, these transport equations can be reduced to

$$\frac{\partial}{\partial t}(\rho_n+\rho_s) = -\nabla\cdot(\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s), \tag{52}$$

$$\frac{\partial}{\partial t}(\rho_n\mathbf{k}_n+\rho_s\mathbf{k}_s)+\frac{\partial}{\partial x_i}[\rho_n\mathbf{k}_n(\mathbf{v}_n)_i+\rho_s\mathbf{k}_s(\mathbf{v}_s)_i] = -\nabla p, \tag{53}$$

$$\frac{\partial}{\partial t}(\rho s) = -\nabla\cdot(\rho s\mathbf{v}_n), \tag{54}$$

$$\mathbf{u}\cdot\left\{\rho_s\left[\frac{\partial\mathbf{k}_s}{\partial t}+(\mathbf{v}_s\cdot\nabla)\mathbf{k}_s\right]+b\rho_n\left[\frac{\partial\mathbf{k}_s}{\partial t}+(\mathbf{v}_n\cdot\nabla)\mathbf{k}_s\right]\right\} = -\rho_s\mathbf{u}\cdot\nabla\mu, \tag{55}$$

where b is defined by

$$(1-b)(\mathbf{v}_n-\mathbf{v}_s) = 2(\mathbf{k}_n-\mathbf{k}_s) \tag{56}$$

and serves as a measure of the difference of inertia between a phonon and a free hard sphere.

It should be remarked that unlike the thermodynamical discussions of the quasi-equilibrium states, the validity of these transport equations rests heavily on a rather heuristic extension of kinetic considerations to the present system.

If one puts $b=0$ in (52)–(55), the resultant equations are consistent[5] with the usual two-fluid hydrodynamical equations.[3]

7. SOUND VELOCITIES

Let us consider a simple one-dimensional motion in which $\mathbf{v}_n$, $\mathbf{v}_s$ are all parallel to the z axis and all the dynamical variables ρ_n, ρ_s, $\mathbf{v}_n$, $\mathbf{v}_s$ are functions of z and t only. Furthermore, we shall consider only small deviations from the state $\mathbf{v}_n=0$, $\mathbf{v}_s=0$. Equations (52)–(55), then, become the following linear differential equations:

$$\dot{\rho} = -\rho_n\frac{\partial v_n}{\partial z}-\rho_s\frac{\partial v_s}{\partial z}, \tag{57}$$

$$\dot{\rho}s+\rho\dot{s} = -\rho s\frac{\partial v_n}{\partial z}, \tag{58}$$

$$[1+b(\rho_n/\rho_s)]\dot{k}_s = -\rho^{-1}\frac{\partial p}{\partial z}+s\frac{\partial T}{\partial z}, \tag{59}$$

$$\rho_n\dot{k}_n+\rho_s\dot{k}_s = -\partial p/\partial z, \tag{60}$$

where the dot means partial derivative with respect to t. Except for the term $b\rho_n/\rho_s$ in (59), these equations are the same as the usual equations[3] for small oscillations in the two-fluid model. In the present case, however, the thermodynamical functions are explicitly known, as displayed in Sec. 3.

For small oscillations we keep only terms linear in the disturbances. Now the *scalar* quantities p, T, ρ_n, ρ_s, and s depend on the *vectors* $\mathbf{v}_n$ and $\mathbf{v}_s$ at least quadratically. In the approximation here, therefore, they must be regarded as independent of $\mathbf{v}_n$ and $\mathbf{v}_s$. Using (17) and (56) and using the linear approximation, one reduces (57)–(60) to a set of four homogeneous linear equations in $\delta\rho$, δT, $\mathbf{v}_n$, and $\mathbf{v}_s$ with coefficients which are given by the thermodynamical functions (and the function b) for the case $\mathbf{v}_n=\mathbf{v}_s=0$, i.e., for absolute equilibrium, a case discussed in paper I.

The eigensolutions of these equations have a periodic space-time dependence of the form

$$\exp[i\mathcal{K}(z-Ct)], \tag{61}$$

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[5] The usual equations, however, are more strict than (52)–(55) in the approximation $b=0$ in two ways. First, while (55) asserts that $\mathbf{u}$ is perpendicular to $\mathbf{A}=\rho_s[\partial\mathbf{k}_s/\partial t+(\mathbf{v}_s\cdot\nabla)\mathbf{k}_s]+\rho_s\nabla\mu$, the usual equation asserts that $\mathbf{A}=0$. Second, the usual equations[3] include $\nabla\times\mathbf{v}_s=0$, which is not one of the equations (52)–(55) expressing the conservation of mass, momentum, entropy, and energy.

where the sound velocities C are determined by the algebraic equations

$$\begin{aligned}[(1-b)C^2-A_{11}]v_n&=A_{12}v_s,\\ A_{21}v_n&=\{[1+b(\rho_n/\rho_s)]C^2-A_{22}\}v_s.\end{aligned}\tag{62}$$

The elements A_{ij} are given by

$$\begin{aligned}A_{11}&=2\frac{\rho_n}{\rho}\left(\frac{\partial p}{\partial\rho}\right)_s+2\frac{\rho_s^2}{\rho\rho_n}\left(\frac{s^2T}{c_v}\right)+4\left(\frac{\rho_s}{\rho^2}\right)\left(\frac{\partial p}{\partial T}\right)_\rho\left(\frac{sT}{c_v}\right),\\ A_{22}&=2\frac{\rho_s}{\rho}\left(\frac{\partial p}{\partial\rho}\right)_s+2\left(\frac{\rho_s}{\rho}\right)\left(\frac{s^2T}{c_v}\right)\\ &\qquad-4\left(\frac{\rho_s}{\rho^2}\right)\left(\frac{\partial p}{\partial T}\right)_\rho\left(\frac{sT}{c_v}\right),\\ A_{12}&=2\frac{\rho_s}{\rho}\left(\frac{\partial p}{\partial\rho}\right)_s-2\frac{\rho_s^2}{\rho\rho_n}\left(\frac{s^2T}{c_v}\right)\\ &\qquad-2\left(\frac{\rho_s}{\rho^2}\right)\left(\frac{\rho_n-\rho_s}{\rho_n}\right)\left(\frac{sT}{c_v}\right)\left(\frac{\partial p}{\partial T}\right)_\rho,\\ A_{21}&=2\frac{\rho_n}{\rho}\left(\frac{\partial p}{\partial\rho}\right)_s-2\left(\frac{\rho_s}{\rho}\right)\left(\frac{s^2T}{c_v}\right)\\ &\qquad-2\left(\frac{\rho_n-\rho_s}{\rho^2}\right)\left(\frac{sT}{c_v}\right)\left(\frac{\partial p}{\partial T}\right)_\rho,\end{aligned}\tag{63}$$

where c_v is the specific heat per particle.

Equation (62) yields two sound velocities C:

$$C_{\mathrm{I}}^2=(2K)^{-1}[L+(L^2-4KM)^{\frac{1}{2}}],\tag{64}$$

$$C_{\mathrm{II}}^2=(2K)^{-1}[L-(L^2-4KM)^{\frac{1}{2}}],\tag{65}$$

where

$$K=(1-b)[1+b(\rho_n/\rho_s)],\tag{66}$$

$$L=A_{11}[1+b(\rho_n/\rho_s)]+A_{22}(1-b),\tag{67}$$

$$M=A_{11}A_{22}-A_{12}A_{21}=4\frac{\rho_s}{\rho_n}\frac{s^2T}{c_v}\left(\frac{\partial p}{\partial\rho}\right)_T.\tag{68}$$

Following the usual definitions, we call the normal mode which has the larger velocity, C_{I}, the first sound, and the mode which has the smaller velocity, C_{II}, the second sound.

In the following we shall give the explicit values of C_{I} and C_{II} for this dilute system at moderate temperature $a\rho\lambda^2\ll 1$ [case (i)], at fairly low temperature $a\rho\lambda^2\sim 1$ [case (ii)], and at very low temperature $a\rho\lambda^2\gg 1$ [case (iii)]. The thermodynamical functions for these cases have been given explicitly in reference 1.

(1) At any moderate temperature, i.e., $a\rho\lambda^2\ll 1$, the coefficients A_{ij} and the parameter b are found to be [neglecting terms of higher orders in $(a\rho\lambda^2)$]

$$\begin{aligned}A_{11}&=2\frac{\rho}{\rho_n}\left(\frac{\partial p}{\partial\rho}\right)_s=1.713\kappa T,\\ A_{12}&=32\pi a\rho,\\ A_{21}&=32\pi a\rho_n,\\ A_{22}&=16\pi a\rho_s,\\ b&=O[(a\rho\lambda^2)^{\frac{1}{2}}].\end{aligned}\tag{69}$$

The two sound velocities C_{I} and C_{II} are, then, given by

$$C_{\mathrm{I}}^2=(1.713)\kappa T,\quad C_{\mathrm{II}}^2=16\pi a\rho_s.\tag{70}$$

(2) At much lower temperature, $a\rho\lambda^2\sim 1$, the coefficients A_{ij} and b are found to be [neglecting terms of higher orders in $(\rho a^3)^{\frac{1}{2}}$]

$$\begin{aligned}A_{11}&=2\frac{\rho_s^2}{\rho\rho_n}\frac{s^2T}{c_v}=16\pi a\rho\left(t^{-1}\mathcal{G}-\frac{d\mathcal{G}}{dt}\right)^2\left(t\mathfrak{F}\frac{d^2\mathcal{G}}{dt^2}\right)^{-1},\\ A_{12}&=16\pi a\rho\left[1+\left(\mathcal{G}-t\frac{d\mathcal{G}}{dt}\right)\left(-\tfrac{3}{2}\mathcal{G}+\tfrac{3}{2}t\frac{d\mathcal{G}}{dt}\right.\right.\\ &\qquad\left.\left.+t^2\frac{d^2\mathcal{G}}{dt^2}\right)\left(t^3\mathfrak{F}\frac{d^2\mathcal{G}}{dt^2}\right)^{-1}\right],\end{aligned}\tag{71}$$

$$\begin{aligned}A_{21}&=O[a\rho(\rho a^3)^{\frac{1}{2}}],\\ A_{22}&=16\pi a\rho_s,\\ (1-b)&=[\mathcal{H}(t)/\mathfrak{F}(t)],\end{aligned}$$

where

$$t=2a\rho\lambda^2,$$

$$\mathfrak{F}(t)=\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{x}{e^{tx}-1}\left[\frac{(x^2+1)^{\frac{1}{2}}-1}{x^2+1}\right]^{\frac{1}{2}}dx,$$

$$\mathcal{G}(t)=-\frac{2}{\sqrt{\pi}}\int_0^\infty x\ln(1-e^{-tx})\left[\frac{(x^2+1)^{\frac{1}{2}}-1}{x^2+1}\right]^{\frac{1}{2}}dx,$$

$$\begin{aligned}\mathcal{H}(t)=\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{dx}{e^{tx}-1}&\left[\frac{(x^2+1)^{\frac{1}{2}}-1}{x^2+1}\right][\tfrac{2}{3}(1+x^2)^{-\frac{1}{2}}\\ &-\tfrac{2}{3}(1+x^2)^{-1}+x^2(1+x^2)^{-\frac{1}{2}}].\end{aligned}$$

Using (62), we find that the two sound velocities become degenerate,

$$C_{\mathrm{I}}^2=C_{\mathrm{II}}^2=16\pi a\rho_s,$$

at a temperature

$$T=T_1=21.7(a\rho/\kappa).\tag{72}$$

For temperature $T>T_1$, we have

$$C_{\mathrm{I}}^2=A_{11}(1-b)^{-1},\quad C_{\mathrm{II}}^2=16\pi a\rho_s.\tag{73}$$

TABLE I. Sound velocities in a region $a\rho\lambda^2 \sim 1$ where the two velocities are nearly degenerate.

$t=2a\rho\lambda^2$	$C_{\mathrm{I}}^2/(16\pi a\rho_s)$	$C_{\mathrm{II}}^2/(16\pi a\rho_s)$
∞	1.00	$\frac{1}{3}$
1.20	1.00	0.978
1.16	1.00	1.00
1.10	1.05	1.00
1.00	1.13	1.00
0.80	1.35	1.00

For the temperature range $T < T_1$, we find

$$C_{\mathrm{I}}^2 = 16\pi a\rho_s, \quad C_{\mathrm{II}}^2 = A_{11}(1-b)^{-1}. \tag{74}$$

The numerical values of C_{I}^2 and C_{II}^2 at different temperatures ($a\rho\lambda^2 \sim 1$) are tabulated in Table I.

(3) At extremely low temperature, $a\rho\lambda^2 \gg 1$, the A_{ij}'s and b become

$$\begin{aligned} A_{11} &= (16/135)\pi^5(1.202\lambda^2)^{-1}, \\ A_{12} &= 16\pi a\rho, \\ A_{21} &= 16\pi a\rho_n, \\ A_{22} &= 16\pi a\rho, \\ b &= 1-(1/45)\pi^4(1.202a\rho\lambda^2)^{-1}. \end{aligned} \tag{75}$$

Thus, the two sound velocities are given [neglecting $O(\lambda^{-2})$] by

$$C_{\mathrm{I}}^2 = 16\pi a\rho, \quad C_{\mathrm{II}}^2 = \tfrac{1}{3}C_{\mathrm{I}}^2. \tag{76}$$

Figure 1 illustrates the variations of these two sound velocities in different temperature ranges.

8. COMPARISONS BETWEEN VELOCITIES OF SOUND VIBRATIONS AND PHONONS

In the previous paper it is shown [Eq. (85) of reference 1] that at any temperature T below the transition point the velocities of long-wavelength phonons have a statistical value

$$\bar{v}_{\text{phonon}} = (16\pi a\rho_s)^{\frac{1}{2}}. \tag{77}$$

While the phonons represent microscopic excitations, it is nevertheless expected that by superimposing together a large number of phonons of the same wave number we should obtain a corresponding macroscopic vibration. Thus, $\bar{v}_{\text{phonon}}$ must also be the same as one of the macroscopic sound velocities. A direct comparison of (77) with (70), (73), (74), and (76) shows indeed that

$$\begin{aligned} &\text{at} \quad T > T_1, \quad \bar{v}_{\text{phonon}} = C_{\mathrm{II}}, \\ &\text{and at} \quad T < T_1, \quad \bar{v}_{\text{phonon}} = C_{\mathrm{I}}, \end{aligned} \tag{78}$$

where T_1 is given by (72).

This result throws new light on the relations between microscopic and macroscopic excitations. In the present case, (78) can be understood physically by examining the different characteristics of these two macroscopic sound vibrations.

Let us consider first the extremely-low-temperature region: $a\rho\lambda^2 \gg 1$. Using (62) and (75), we find the eigensolutions v_n and v_s for the first sound vibration to satisfy

$$(v_n/v_s)_{\mathrm{I}} = (135/2\pi^4)(1.202a\rho\lambda^2), \tag{79}$$

and for the second sound to satisfy

$$(v_n/v_s)_{\mathrm{II}} = -\tfrac{2}{3}\rho/\rho_n.$$

Correspondingly, if we denote by $\delta\rho_n$ and $\delta\rho_s$ the fluctuations of ρ_n and ρ_s, then for the first sound vibrations we have

$$(\delta\rho_n/\delta\rho_s)_{\mathrm{I}} = 0.565(a/\lambda)(a\rho\lambda^2)^{-\frac{3}{2}}, \tag{80}$$

while for the second sound vibration we have

$$(\delta\rho_n/\delta\rho_s)_{\mathrm{II}} = -\tfrac{2}{3}. \tag{81}$$

Equations (80) and (81) show that *at very low temperatures the first sound represents a density fluctuation of the superfluid alone* while the second sound represents an oscillation together with the normal fluid (or phonons). Thus, we expect the phonon velocity to be the same as the first sound velocity which is, in turn, identical with

$$[2(dp/d\rho)]_{T=0}{}^{\frac{1}{2}} = (16\pi a\rho)^{\frac{1}{2}},$$

since the density ρ consists of essentially only ρ_s at very low temperature. Furthermore, if we regard these phonons as "molecules" then, as was pointed out[3] by Landau and others, the macroscopic oscillations of the thermal excitations of these phonons should travel with a velocity equal to $(1/\sqrt{3})$ times the first sound velocity. All these expectations are confirmed by (78) and (76).

On the other hand, at moderate temperatures ($a\rho\lambda^2 \ll 1$), the physical characteristics of these two sound vibrations seem to be reversed. Using (62) and (69), we find for the first sound

$$(\delta\rho_n/\delta\rho_s)_{\mathrm{I}} = 0.214(a\rho_s\lambda^2)^{-1} \gg 1, \tag{82}$$

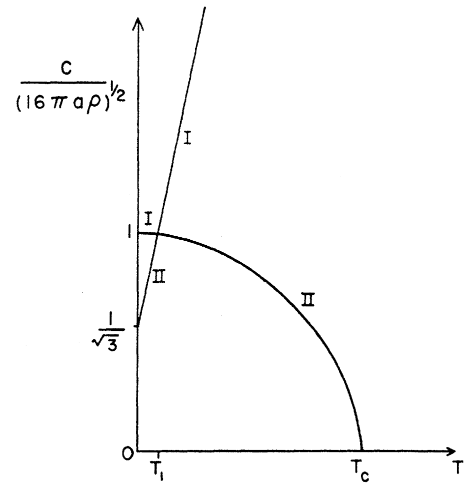

FIG. 1. Sound velocities as functions of T.

while for the second sound

$$-(\delta\rho_n/\delta\rho_s)_{\rm II}=12.2(a/\lambda)\ll 1. \tag{83}$$

Thus *at moderate temperatures the second sound represents the oscillation of the superfluid;* instead, the first sound is essentially the oscillations of the normal fluid (or phonons). Consequently, one expects the velocity of phonons, $\bar{v}_{\rm phonon}$, to be the same as $C_{\rm II}$, and not $C_{\rm I}$.

It may be emphasized that the customary definition of first and second sound is quite arbitrary. The above apparent change of the physical characteristics of these two sound vibrations at very low temperatures and at moderate temperatures is essentially due to this choice of definition. For the present dilute system, at any temperature below the transition point there are two sound vibrations: one represents the oscillations of the density fluctuation of the superfluid alone and is directly related to the microscopic phonon excitations, while the other is connected with the macroscopic oscillations of the density fluctuations of the phonons. The velocity of the former type of oscillation (or long-wavelength phonon velocity) is

$$(16\pi a\rho_s)^{\frac{1}{2}},$$

and the oscillation is called first sound for $T<T_1$ and second sound for $T>T_1$.

9. REMARKS

A dilute system of hard spheres is, of course, quite different from a real system such as liquid He. They differ in two main aspects: that liquid He has a positive binding energy per particle and that liquid He is not a dilute system. Qualitatively, however, there are many similarities between these two systems. In particular, the explicit solution of the dilute system of hard spheres clarifies some of the physical concepts underlying the low-temperature behavior of liquid He and enables one to gain new understandings concerning the phenomena of superfluidity.

For example, both the binary collision method and the pseudopotential method can be applied to study the low-temperature behavior of a dilute system of hard spheres obeying Fermi statistics. These results show that such a system does not undergo any phase transition and it exhibits no superfluidity. Thus, these results confirm London's proposal that superfluidity is a result of Bose-Einstein statistics.

As remarked before, the explicit solution of the present dilute system of Bose hard spheres shows some interesting and somewhat unexpected relations between the velocity of phonons and the macroscopic sound velocities. While a direct extrapolation of these details to a dense and strongly interacting system such as liquid He II is not permissible, these results, even in their general outlines, do stimulate new thoughts as to the corresponding relations in He.

The dispersion relation of phonons in He has been measured by scattering experiments using slow neutrons. It would be of particular interest to know whether the phonon velocity in He II varies with temperature and to study the relationship between the phonon velocity and the sound velocities.

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