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Coulomb Scattering in a Very Strong Magnetic Field

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The spiralling of charged particles in an intense magnetic field is taken into account in the description of their collisions involving small fractional momentum transfer. The transition probability to the continuum of possible states is given. In addition, the transition probability is given for a particle's orbit center to be displaced from one magnetic line of force to another, with accompanying momentum change, as a result of scattering by a fixed charge. The equivalent results are derived for the scattering of two identical particles in their relative coordinate system.

As the result of scattering, the momentum change along the magnetic field in a uniform, collimated beam of spiralling particles is found to be very much smaller than the sum of the magnitudes of the individual momentum changes in that direction. In contrast to ordinary Coulomb scattering, one finds that there is a lower limit to the momentum transfer to individual particles and that there is an adiabatic cutoff distance associated with the interaction which, in some plasma situations, can be shorter than the value of the Debye distance.

The WKB approximation for *generalized* Laguerre polynomials is appended.

I. INTRODUCTION

THE scattering problems that will be described in this work can be outlined, as it is commonly done, by ordering their collisions according to the fractional momentum and energy transfer. For large fractional transfer, in probably all so far realizable situations, an imposed magnetic field can only play an unimportant role in Coulomb scattering of charged particles—even for (transient) fields of around 10^6 gauss that are now coming into use. However, for small fractional momentum transfers, the role of the magnetic field becomes increasingly significant with increasing distances of closest approach.

The spiralling of charged particles, being a form of binding, causes their energy values to be quantized. It should therefore be investigated what conditions must exist for the consequences of this effect to become observable. A quantum-mechanical treatment of the scattering is therefore desirable.

Although it is not the aim of this paper to emphasize applications, it should be noted that knowledge of the transition probabilities for encounters involving small fractional momentum and energy transfer is essential, for example, to the computation of transport properties of a highly ionized plasma; there such encounters predominate over the more violet ones because the long range of the Coulomb force makes the former much more probable.[1]

Wherever the presently available literature of the Boltzmann and Fokker-Planck equations deals with the transport properties of plasmas in very strong magnetic fields, no account is taken of the spiralling of the charged particles in computing the relevant transition probabilities; the Rutherford formula is always used, even though in many interesting situations the reliability of this procedure is problematical. But even if it does turn out, contrary to doubts expressed in what follows, that the spiralling does not have effects worth exploring further, then one may still find the set of transition probabilities presented in this paper to be more natural to the description of the problem than the one resulting from the Rutherford cross section.

Apart from its applicability to plasma physics, a large part of the calculation is an instructive example of a situation where it is undoubtedly advantageous[2] (and in general necessary[3]) to find physical observables (certain transition probabilities, orbit centers, etc.) independent of $\hbar$, via the Schrödinger equation. There are two reasons for this; the basis of both are in distinct contrast to what is the case in ordinary Coulomb scattering. The first reason is the difficulty of integrating the classical equations of motion even after making small momentum transfer approximations. The second one is the much greater ease with which the expression for the transition probability can be evaluated using quantum mechanics instead of classical mechanics.

This problem, moreover, has the characteristic feature that (orbit centers of) particles approach and recede from the scattering center along magnetic lines of force. Therefore, the next section contains information about the properties of a free particle in a magnetic field which are needed to calculate the scattering phenomenon by means of the first Born approximation.

In Sec. III, various transition probabilities will be given that are applicable either to the scattering of one spiralling particle by a arbitrary fixed charge, or to the scattering of two identical particles as described in their relative coordinate system.

Possible implications of the results of Sec. III are discussed in Sec. IV. A by-product of Sec. II is the

[1] See L. Landau, Physik. Z. Sowjetunion **10**, 154 (1936).

[2] Before this work was begun, William M. MacDonald conjectured that this would turn out to be true.

[3] Analogous to $e^2/\hbar v \ll 1$ case in ordinary Coulomb scattering where quantum effects cannot be disregarded.

WKB approximation for generalized Laguerre polynomials which is given in an Appendix.

II. UNPERTURBED PROBLEM

The Hamiltonian of a free particle of mass m and charge e in a uniform magnetic field B parallel to the z direction can be taken to have the form

$$\mathcal{H}_0=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)-\frac{eB}{2mc}L_z+\frac{e^2B^2}{8mc^2}\rho^2$$

in cylindrical coordinates $\{\rho,\varphi,z\}$. The vector potential associated with this $\mathcal{H}_0$ is

$$A_\rho=0,\quad A_\varphi=\tfrac{1}{2}B\rho,\quad A_z=0.$$

L_z is the z component of the angular momentum operator, which (unlike L^2) commutes with $\mathcal{H}_0$; another constant of the motion is the momentum p_z.

The Hamiltonian $\mathcal{H}_0^{(2)}$ for the unperturbed system of two identical particles (mass m, charge e) in a magnetic field is expressible in terms of $\mathcal{H}_0$ in the following way:

$$\mathcal{H}_0^{(2)}(x_1,x_2)=\mathcal{H}_0(\tfrac{1}{2}(x_1+x_2),\,2m,\,2B)+\mathcal{H}_0(x_2-x_1,\,\tfrac{1}{2}m,\,\tfrac{1}{2}B).$$

Without loss of generality, all analysis in this section can, therefore, be limited to $\mathcal{H}_0$. It may be noted here that the introduction of relative coordinates is in general not a useful step in achieving the solution to two-particle problems involving a magnetic field.

Taking the eigenvalues of L_z and p_z to be $\alpha\hbar$ and $p'\hbar$, respectively, one is led to the equation[4]

$$\frac{d^2v}{d\rho^2}+\left(\lambda-\frac{\alpha^2-\frac{1}{4}}{\rho^2}-\rho^2\right)v\equiv\frac{d^2v}{d\rho^2}+\Phi(\rho^2)v=0, \tag{1}$$

for $\rho^{\frac{1}{2}}$ times the radial wave function if the unit of length is chosen to be

$$\rho_0=(2\hbar c/eB)^{\frac{1}{2}}=3.6\times10^{-4}B^{-\frac{1}{2}}.$$

It will be convenient, moreover, to introduce a dimensionless quantum number for p_z, related to it by $p_z=p\hbar/\rho_0$. The necessary and sufficient condition for Eq. (1) to have a polynomial (factor) solution and thus keep $v(\infty)$ finite is

$$\lambda=\frac{4}{\hbar\Omega}\left(E-\frac{p'^2\hbar^2}{2m}+\tfrac{1}{2}\alpha\Omega\hbar\right)=4n+2\alpha+2,$$

with $n=0,1\cdots$. Therefore,

$$E=E_{\parallel}+E_{\perp},$$

with

$$E_\perp=(n+\tfrac{1}{2})\hbar\Omega,\qquad E_{\parallel}=p'^2\hbar^2/2m=\tfrac{1}{4}p^2\hbar\Omega, \tag{2}$$

and $\Omega=|e|B/mc$.

Equation (1) has the solution

$$v_{n\alpha}=C_{n\alpha}\rho^{\alpha+\frac{1}{2}}\exp(-\rho^2)L_n^{(\alpha)}(\rho^2), \tag{3}$$

where

$$L_n^{(\alpha)}(x)=\sum_{\sigma=0}^{\sigma=n}\binom{n+\alpha}{n-\sigma}\frac{(-x)^\sigma}{\sigma!}$$

are the generalized Laguerre polynomials, and

$$C_{n\alpha}^2=2[n!/(n+\alpha)!].$$

It is evident from Eq. (2) that n determines the particle energy perpendicular to the magnetic field. In most problems of interest $n\gg1$; for example, a 10-kev electron (with $p=0$) in a field of 10^6 gauss has $n\sim10^6$. The physical meaning of α is that it determines the radial position of the orbit (center), for fixed $E_\perp$. This can be expressed by

$$\langle n\alpha|\rho^2|n\alpha\rangle=2n+\alpha+1, \tag{4}$$

or classically by

$$\hat{\rho}^2=(4E_\perp+2\Omega L_z)/m\Omega^2=\rho_0^2(2n+\alpha+1), \tag{4'}$$

where $\hat{\rho}$ is the square root of the time average of the square of the particle radius. For α, much smaller values than n are usually the more significant (see Sec. III).

It will be sufficient to describe the charge density of a particular $|pn\alpha\rangle$ eigenstate as being confined within a cylindrical shell that is the locus of all cyclotron orbits having the orbit center coordinate, $\hat{\rho}$. In this quantum-mechanical treatment one is therefore left with an averaging over the phase angle by which a classical calculation identifies the initial position of the spiralling particle in some reference plane; it is felt that this is a very significant feature of the quantum-mechanical approach.

The (two) classical turning points of the motion are where

$$\Phi(\rho^2)=4n+2\alpha+2-(\alpha^2-\tfrac{1}{4})/\rho^2-\rho^2=0. \tag{5}$$

The classical values of ρ^2 possessed by the points on the cyclotron orbit furtherest from and nearest to the ρ origin are taken to be, respectively, ρ_+^2 and ρ_-^2. The distance of closest approach is, therefore, at $\rho=\rho_-$, $z=0$. If $\alpha\neq0$, Φ may be written as

$$x\Phi(x)=(x_+-x)(x-x_-), \tag{6}$$

with

$$x_+=\rho_+^2\cong4n+2\alpha+2+\cdots,\qquad x_-=\rho_-^2\cong(\alpha^2-\tfrac{1}{4})/4n+\cdots, \tag{7}$$

if $n\gg1$ and $n\gg|\alpha|$; similarly, if $\alpha\gg n\gg1$, then

$$x_\pm\cong\alpha+2n+1\pm2[n(\alpha+n)]^{\frac{1}{2}}+\cdots. \tag{7'}$$

[4] Leigh Page, Phys. Rev. **36**, 444 (1930).

From (3) it follows that $\alpha \geqslant -n$; $\alpha = -n$ is the angular momentum of a positive particle centered at the origin; for $\alpha=0$ the particles will spend more time near the origin than for any other angular momentum. For negatively charged particles $\alpha \leqslant n$. All formulas independent of the sense of rotation remain unchanged in value if e and α are simultaneously replaced by $(-e)$ and $(-\alpha)$.

Before going on to the scattering problem, it appears necessary to simplify the solution of the $\mathcal{H}_0$ problem somewhat further so that the perturbation calculation becomes tractable. One of the apparently few ways to find a simpler expression for the solution of Eq. (1) is to note that Laguerre polynomials have the asymptotic expansion[5,6]

$$L_n^{(\alpha)}(x) = \pi^{-\frac{1}{2}} n^{\frac{1}{2}\alpha-\frac{1}{4}} x^{-\frac{1}{2}\alpha-\frac{1}{4}} e^{\frac{1}{2}x} \cos[2(nx)^{\frac{1}{2}} - \tfrac{1}{2}\pi(\alpha+\tfrac{1}{2})], \quad (8)$$

in the limit $n \to \infty$ [therefore $|\alpha| \ll n$ in (8)]. From the behavior of $v_{n\alpha}$ [Eq. (3)] it is clear that relation (8) must not be trusted to represent the wave function outside its classical region, *viz.*, $\rho \lesssim n^{-\frac{1}{2}}$ and $\rho \gtrsim \rho_+$ if $\alpha=0$, and $\rho \lesssim \rho_-$ and $\rho \gtrsim \rho_+$ otherwise. Moreover, if $\alpha \simeq -n$, relation (8) is again not applicable.

For $\alpha \neq 0$ $(\alpha+n \gg 1)$ the solution of Eq. (1) by means of the WKB method is presumably more accurate than the one obtained from (8) because it takes a more detailed account of the variation of Φ; numerical work supporting this statement is given in an Appendix. In the classical region of Φ, i.e. $\Phi > 0$, the WKB approximation for v is

$$v_{n\alpha} = (2/\pi^{\frac{1}{2}}) \Phi_{n\alpha}^{-\frac{1}{4}} \cos[S_{n\alpha} - \tfrac{1}{4}\pi], \quad (9)$$

with

$$S_{n\alpha} = \int_{\rho_-}^{\rho} \Phi^{\frac{1}{2}}(\rho^2) d\rho. \quad (10)$$

It turns out that in the evaluation of the relevant matrix elements for our problem, only differences $S_{n'\alpha} - S_{n\alpha}$ will be needed with $|n'-n| \ll n$. Therefore, it is sufficient to give the value of

$$\frac{\partial S}{\partial n} = 2 \arctan\left(\frac{\rho^2 - \rho_-^2}{\rho_+^2 - \rho^2}\right)^{\frac{1}{2}}. \quad (11)$$

Nevertheless, S is calculated in the Appendix because with it one can immediately obtain, over a wide range of x, what seems to be a better approximation to $L_n^{(\alpha)}(x)$ (for $\alpha \neq 0$) than expression (8). For use in Sec. III, other expressions for $L_n^{(\alpha)}(x)$ cited in the mathematical literature[5,6] do not appear as tractable as the WKB result.

III. SCATTERING PROBLEM

This section gives the first Born approximation of various transition probabilities. The scattering of one particle (charge e and mass m) by a fixed charge Ze (at the origin), and the scattering of two identical particles will be described.

In analogy with situations without magnetic fields, one may expect the reliability of the Born approximation to improve with increasing values of n and p. The range over which the Born approximation is expected to be useful appears to become larger with increasing strength of the external magnetic field. This idea suggests itself because the motion at large distances from the scattering center then becomes less and less determined by the perturbing Coulomb field.

A restriction to what follows is met in the limit of adiabatic behavior; this fact has as a consequence that there will exist an upper limit to α, beyond which the expression for the transition probability should not be assumed to be valid. Moreover, as is almost implied by the previous paragraph, taking the limit $B \to 0$ is to be eschewed. The necessity for this exists already in the corresponding classical treatment.

The matrix element of interest is

$$M = Ze^2 \langle n+\Delta n, p+\Delta p, \alpha | (\rho^2+z^2)^{-\frac{1}{2}} | np\alpha \rangle. \quad (12)$$

The final state has been characterized by $n' = n + \Delta n$, $p+\Delta p$, and α; the initial state by n, p, and α. M reduces to

$$M = \frac{2Ze^2}{L} \int_0^\infty d\rho v_{n'\alpha}(\rho) K_0(|\Delta p|\rho) v_{n\alpha}(\rho), \quad (12')$$

after integration over z and φ. The former integration contributes $2K_0$ because[7]

$$K_n(x) = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{x}\right)^n \int_0^\infty \frac{\cos(xs)}{(s^2+1)^{n+\frac{1}{2}}} ds. \quad (13)$$

$K_n(x)$ is a Bessel function of purely imaginary argument related to the Hankel function $H_n^{(1)}(ix)$.[7] L, in (12′), is the length of the cylindrical quantization volume, V, of radius R whose axis is along the direction of the magnetic field.

Two important general features of the scattering can be inferred already from the form (12′) for M. Firstly, the oscillatory nature of the eigenfunctions (3) in M cause it to be relatively small unless $|\Delta n| \ll n$. Secondly, M will become extremely small if there is little overlap between the region where both $v_{n\alpha}$ are essentially localized ($\rho_- < \rho < \rho_+$), and the interval where $K_0(|\Delta p|\rho)$ is large ($|\Delta p|\rho < 1$). The interaction will therefore cease rapidly beyond where

$$|\Delta p|\rho_- \sim 1,$$

or

$$\rho_- \sim (n^{\frac{1}{2}}/|\Delta n|)(E_{\parallel}/E_{\perp})^{\frac{1}{2}}.$$

The last relation delineates the onset of adiabatic conditions. In our units, it is the same, well-known

[5] G. Szego, *Orthogonal Polynomials* (Am. Math Soc. Coll. Pub. Vol. XXIII, 1939).

[6] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, New York, 1953), Vol. 2.

[7] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1952).

relation that one finds by saying that if the closest distance of approach between an oscillator, having a natural frequency Ω, and a particle moving with velocity v (parallel to B) is greater than $b_{\max}\approx v/|\Delta n|\Omega$, then the interaction will cease to induce transitions between levels $\Delta n\Omega$ apart.

For electrons (mass m_e) moving with a velocity v corresponding to E ev in a field of B gauss, one finds

$$b_{\max}\simeq 3.4(E)^{\frac{1}{2}}/|\Delta n|B,$$

and the corresponding expressions for heavier particles of mass m_H is

$$b_{\max}=\frac{3.4}{|\Delta n|B}\left(\frac{m_H}{m_e}E\right)^{\frac{1}{2}}.$$

One can easily show from the general expression for $\rho_-(\alpha,n)/n^{\frac{1}{2}}$ that if one deals with n, $\Delta n\gg 1$, then the onset of the adiabatic phenomenon already occurs at $|\alpha|\ll n$ unless $(\Delta n)^2E_\perp\lesssim E_{\parallel}$. Therefore, in a plasma at high temperatures the collisions involving $|\alpha|\ll n$ will be by far the most significant; this is the first and main reason why the $\alpha\gg n$ case will be treated in less detail.

Combining (8), (3), and (12′) gives

$$M=\tfrac{1}{2}\pi(Ze^2/L)(nn')^{-\frac{1}{4}}[(\Delta p)^2+4(n'^{\frac{1}{2}}-n^{\frac{1}{2}})^2]^{-\frac{1}{2}}, \quad (14)$$

to a first approximation, provided, as is very often the case,[8] $|\Delta p|\rho_+\gg 1$ and $|\Delta p|\rho_-\ll 1$. It is necessary that $\rho_+\gg\rho_-$ and $\Delta n(E_\perp/E_{\parallel})^{\frac{1}{2}}\gg 1$ for these inequalities to be satisfied. M may be simplified to

$$M\cong\tfrac{1}{2}\pi\frac{Ze^2}{L}\frac{1}{n^{\frac{1}{2}}|\Delta p|}\left(1+\frac{E_{\parallel}}{E_\perp}\right)^{-\frac{1}{2}}, \quad (14')$$

because

$$\Delta n+\tfrac{1}{2}p\Delta p\cong 0. \quad (15)$$

The expressions (14) and (14′) are of special value for $\alpha=0$ because for only this eigenvalue is $\Phi\sim+\rho^{-2}$ as $\rho\rightarrow 0$, which implies in turn, that no WKB treatment exists for Eq. (1). For $0<|\alpha|\ll n$ (14′) will serve as a comparison to the corresponding results found by means of the WKB method that are given next.

One finds for $|\Delta n|\ll n$ that

$$M_1=\frac{M}{Ze^2}=\frac{8}{\pi}\int_{\rho_-}^{\rho_+}d\rho(\Phi_{n\alpha}\Phi_{n'\alpha})^{-\frac{1}{4}}$$
$$\times K_0(|\Delta p|\rho)\cos(S_{n'\alpha}-\tfrac{1}{4}\pi)\cos(S_{n\alpha}-\tfrac{1}{4}\pi)$$
$$\simeq\frac{4}{\pi}\int_{\rho_-}^{\rho_+}d\rho\,\Phi_{n\alpha}^{-\frac{1}{2}}K_0(|\Delta p|\rho)\cos\left(\frac{\partial S}{\partial n}\Delta n\right), \quad (16)$$

by substituting (9) into (12), and then retaining only the largest contribution to the integrand. Finally one gets

$$M_1=4I_{\Delta n}(|\Delta n|\tau_2)K_{\Delta n}(|\Delta n|\tau_1), \quad (17)$$

after putting (11) and (5) into (16). In (17), $I_{\Delta n}(x)$ is the Bessel function of purely imaginary argument and

$$\tau_1\pm\tau_2=|\Delta p/\Delta n|\rho_\pm(\alpha,n)\cong 2\rho_\pm/p.$$

Relation (17) follows from (16) in a straightforward manner after one changes variables from ρ to

$$\beta=2\arctan\left(\frac{\rho^2-\rho_-^2}{\rho_+^2-\rho^2}\right)^{\frac{1}{2}}.$$

This substitution allows one to write

$$M_1=\frac{4}{\pi}\int_0^\pi d\beta\cos(\Delta n\beta)K_0(|\Delta n|(\tau_1^2+\tau_2^2-2\tau_1\tau_2\cos\beta)^{\frac{1}{2}}). \quad (16')$$

Replacement of K_0 in (16′) by an equivalent expression obtained from the addition theorem of Bessel functions,[7] *viz.*,

$$K_0(|\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\beta|^{\frac{1}{2}})=\sum_{s=-\infty}^{s=\infty}I_s(\rho_2)K_s(\rho_1)e^{is\beta}, \quad (\rho_1>\rho_2)$$

then yields (17).

Most of the information in this paper comes from use of (14) or (17) in special cases, but additional knowledge will be obtained by giving a discussion of the behavior of M_1, as it depends on $\tau_1(\alpha,n)$ and $\tau_2(\alpha,n)$.

A general decrease in M_1 can be demonstrated by proving* that, for all τ,

$$M_1=4I_{\Delta n}(|\Delta n|(\tau-\delta\tau))K_{\Delta n}(|\Delta n|\tau)$$

decreases monotonically with increasing $\delta\tau(>0)$, which is related to ρ_- by

$$\delta\tau=|\Delta p/\Delta n|\rho_-.$$

The power series representation[7] of $I_{\Delta n}$ makes the proof of its monotonic behavior self-evident. Part of this fall-off is of classical origin, but for sufficiently large ρ_- values, it is always a quantum-mechanical phenomenon.

It will be shown next that there can be a range of ρ_- (and α) values, where M decreases relatively slowly at first with increasing ρ_-. One may, therefore, refer to M as staying rather close to a certain "plateau" value. Outside of this region, one expects and finds that the scattering decreases rapidly as the minimum distance of closest approach, ρ_-, increases beyond the outer edge of the "plateau." This, it will be seen, can occur at distances much less than $\rho_-\sim|\Delta p|^{-1}$.

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[8] For ρ_+ and ρ_- one takes, in principle, $\min\{\rho_+(n'),\rho_+(n)\}$ and $\max\{\rho_-(n'),\rho_-(n)\}$, respectively. For the case of small $\Delta n/n$ treated in this paper, the $\{\rho_\pm(n'),\rho_\pm(n)\}$ difference is usually not important.

* *Note added in proof.*—The proof given is only appropriate if $\alpha<0$. However, the proof for $\alpha>0$ is nearly the same. One must then merely consider variations $\tau+\delta\tau$ in K instead of variations $\tau-\delta\tau$ in I.

"On the plateau," $\delta\tau$ is such that*

$$|\Delta n|\delta\tau I_{\Delta n}'(\Delta n\,\tau)\ll I_{\Delta n}(|\Delta n|\tau).$$

If $|\Delta n|\tau\ll 1$, the foregoing inequality becomes approximately

$$\tfrac{1}{2}|\Delta n|\delta\tau\frac{I_{|\Delta n|-1}(|\Delta n|\tau)}{I_{\Delta n}(|\Delta n|\tau)}\approx|\Delta n|\delta\tau/\tau\ll 1,$$

or $2|\Delta n|\rho_-\ll\rho_+$; i.e., $|\alpha|\ll 2n/|\Delta n|$ for $E_{\parallel}>E_{\perp}$. Similarly if $\tau\gg 1$, one finds

$$\Delta n\delta\tau\ll 1;$$

again it follows that $\rho_-\ll\rho_+$; i.e., $|\alpha|\ll n$ for $E_\perp\gg E_\parallel$.

On the other hand, if $\alpha\gg n$, the situation is the following. According to (6) and (7′), one can express M_1 roughly as

$$M_1\simeq 4I_{\Delta n}(|\Delta n|2n^{\frac{1}{2}}/p)K_{\Delta n}(2|\Delta n|(\alpha+2n)^{\frac{1}{2}}/p),$$

if $n\gg|\Delta n|$.

First of all, it is interesting to point out again the asymptotic behavior of M_1 as a function of α for

$$\tau_1=(2/p)(\alpha+2n)^{\frac{1}{2}}\gg 1.$$

Under these conditions M_1 is proportional to $(\Delta n\tau_1)^{-\frac{1}{2}}\exp(-\Delta n\tau_1)$ because of the asymptotic behavior of $K_{\Delta n}(\Delta n\,x)$.[7]

At ρ_- distances much less than $b_{\max}$ (which corresponds to $\rho_-\sim|\Delta p|^{-1}$) but still such that $\alpha\gg n$, one finds M_1 is smaller, approximately by a factor

$$e\left(\frac{\tau_2}{\tau_1}\right)^{|\Delta n|}\approx e\left(\frac{n}{\alpha+2n}\right)^{|\Delta n|/2},$$

than M_1 given by (17″) if $|\Delta n|\gg 1$. The factor demonstrates just how rapidly M_1 falls with increasing distance of closest approach $\rho_-(\alpha,n)$ beyond "the plateau."

Therefore, except for a relatively brief reference to M for $\alpha\gg n$ in Sec. III B, we shall confine ourselves to the $|\alpha|\ll n$ interval; for these values of α, the cyclotron radius is much greater than the distance of closest approach.

One may also, of course, consider Δn and Δp as variable and keep α fixed. Then one can use the above inequalities as a means of sorting out the set of important final states $|n+\Delta n, p+\Delta p, \alpha\rangle$. No unique final state can be expected because we are dealing with an ensemble of classical particles with the same $\rho_\pm$ but different trajectories. The loss of information about individual trajectories thus suffered is of no consequence in applications to kinetic theory.

The scattering that arises from an incident beam of spiralling particles, and from an incident beam propagating in the direction of the magnetic field, will be described in the remainder of this section. The former is more interesting from the point of view of the kinetic theory of plasmas. The latter, however, is probably the only situation where a strict comparision with the $B=0$ case is possible because then the initial conditions can be made identical for the particles[9]—in particular, the preferred directions with and without B field become identical; the direction of the magnetic field is then that of the incident beam.

A. Spiralling Incident Beam of Particles

The density of final states for a given angular momentum, ρ_F, and the density of initial angular momentum states for a monoenergetic, collimated (so as to obtain particles of only one pitch angle) beam of particles, $P_I(\alpha)$, are all that remain to be added before listing various transition probabilities.[10]

$$\rho_F=\frac{d}{dE}\int_E dn\,dN_{\parallel},$$

where $2\pi N_\parallel=p'L$. With a change of variable from n and p to

$$\epsilon=E/\hbar\Omega=n+\tfrac{1}{4}p^2,$$

and

$$\gamma=\arc\tan(2(n)^{\frac{1}{2}}/p),$$

ρ_F becomes easily convertible into the expression

$$\rho_F=\frac{L}{\pi\hbar^2\Omega}\left(\frac{mE}{2}\right)^{\frac{1}{2}}\sin\gamma d\gamma = \frac{L}{2\pi\hbar^2\Omega}dp_z|_E. \tag{18}$$

In a uniform beam, the distribution function of the square of the orbit center, $Q(\hat{\rho}^2)$, is a constant, Q_0. Therefore, if the beam is also monoenergetic, it follows from (4′) that $P_I(\alpha)$ is constant.

$$P_I(\alpha)=\rho_0^2/R^2. \tag{19}$$

Some special cases will now be dealt with—partly in order to compare the WKB result, (17), with (14′). We note for example that for $|\Delta p|\rho_+\gg 1$ and $|\Delta p|\rho_-\ll 1$, (17) becomes

$$M_1=2/|\Delta p|n^{\frac{1}{2}}\quad\text{if}\quad E_\perp\gg E_\parallel, \tag{17′}$$

and

$$M_1=2/\Delta n\quad\text{if}\quad E_\parallel\gg E_\perp. \tag{17″}$$

Results equivalent to (17′) and (17″) found by using the trial wave functions based on approximation (8) differ, therefore, principally, in that they yield matrix elements roughly smaller in the ratio $\frac{1}{2}\pi:2$. Such a difference should not be surprising; as already mentioned, (9) takes a more detailed account of the spatial variation of $v_{n\alpha}$ for $\alpha\neq 0$; for these values of α, the WKB approximation appears more reliable. (See Appendix.)

$W_{np\alpha}\sin\gamma d\gamma$, the transition probability per unit time

[9] Observation of S. Gartenhaus.

[10] In cgs units.

for going from a single state characterized by (n,p,α) to a set of states consistent with energy conservation, having γ in the interval between γ and $\gamma+d\gamma$, and $(\Delta p)n^{\frac{1}{2}}\gg 1$, is

$$W_{np0}\simeq\frac{\pi^2Z^2e^4}{4(\Delta p_z)^2\rho_0^2L}\left(\frac{2m}{E}\right)^{\frac{1}{2}}, \tag{20'}$$

if $\alpha=0$, while if $\alpha\neq 0$ ($|\alpha|\ll n$) one gets

$$W_{np\alpha}=\frac{4Z^2e^4(2mE)^{\frac{1}{2}}}{(\Delta p_z)^2E_\perp\rho_0^2L}. \tag{21'}$$

For a collimated, uniform beam of particles, one computes the transition probability from $|np\alpha\rangle$, now considered part of a continuum of initial states, to be

$$W_{np\alpha}{}^c\equiv W_{np\alpha}P_I(\alpha)=\frac{4\pi Z^2e^4(2mE)^{\frac{1}{2}}}{(\Delta p_z)^2E_\perp V}f_1(\alpha), \tag{22'}$$

with $f_1(0)=(\pi^2/16)(E_\perp/E)$, and $f_1(\alpha)=1$ if $\alpha\geqslant 1$. The expected absence of separate factors of $\hbar$ (nor n or α) from (22′) should be noted. For $E_{\shortparallel}\gg E_\perp$ the formulas corresponding to the last three are respectively

$$W_{np0}=\frac{\pi^2Z^2e^4}{2L(\Delta E_\perp)^2\rho_0^2}\left(\frac{2E}{m}\right)^{\frac{1}{2}}=\frac{\pi^2Z^2e^4}{2L(\Delta E_{\shortparallel})^2\rho_0^2}\left(\frac{2E}{m}\right)^{\frac{1}{2}}, \tag{20''}$$

$$W_{np\alpha}=\frac{8Z^2e^4}{(\Delta E_{\shortparallel})^2\rho_0^2L}\left(\frac{2E}{m}\right)^{\frac{1}{2}}, \tag{21''}$$

$$W_{np\alpha}{}^c\equiv W_{np\alpha}P_I(\alpha)=\frac{8\pi Z^2e^4}{(\Delta E_{\shortparallel})^2V}\left(\frac{2E}{m}\right)^{\frac{1}{2}}f_2(\alpha), \tag{22''}$$

with $f_2(0)=\pi^2/16$, and $f_2(\alpha)=1$ if $\alpha\geqslant 1$.

Various kinds of transition probabilities may be derived from (20′)–(22″). For example, a transition probability $\mathcal{W}d(\hat{\rho}^2)$ can be defined which gives the probability for the square of the final orbit center coordinate to lie between $\hat{\rho}^2$ and $\hat{\rho}^2+d\hat{\rho}^2$; it can be obtained from the set W by the relation

$$\mathcal{W}=\sin\gamma\left(\frac{d\hat{\rho}^2}{d\gamma}\right)_\epsilon^{-1}W=\frac{m\Omega^2}{8(E_{\shortparallel}E_\perp)^{\frac{1}{2}}}W,$$

because

$$\hat{\rho}^2=\rho_0^2(\alpha+2\epsilon\sin^2\gamma+1).$$

In harmony with expectations, the dependence of $\mathcal{W}^c$ on the magnetic field is inversely proportional to the square of the magnetic field and $|\Delta\hat{\rho}^2|$. This result is seen at once, following replacement of $\Delta E_{\shortparallel}$ or $\Delta E_\perp$ in W, and thus $\mathcal{W}$, with the difference

$$\Delta\hat{\rho}^2=4\Delta E_\perp/m\Omega^2=-4\Delta E_{\shortparallel}/m\Omega^2,$$

obtainable from (4) or (4′). $\mathcal{W}$ may be said to be the transition probability for an orbit-center shift from one magnetic field line to another because each field line may be labeled with orbit-center coordinates of the initial or final state.

The interesting absence of $|\alpha|$ (>0), to first order, from the transition probabilities, is undoubtedly related to the fact that a quantum state with a fixed α corresponds classically to a set of particle orbits with different distances of closest approach but the same $\hat{\rho}$ and $\rho_\pm$. This helps explain why the dependence of $W_{np\alpha}$ on the minimum distance of closest approach, ρ_-, and thus α, is expected to be not very strong for the angular momenta considered above, i.e., those on "the plateau."

Just as the Born approximation to the Rutherford cross section does not determine the sign of the momentum change, so do our formulas for W leave the sign of $\Delta\rho^2$, Δp, Δn, and other interrelated quantities undetermined; this is so in spite of the fact that these differences have a unique sign, except in the $\alpha=0$ case, depending on that of the charges and their relative angular momenta. If one deals, for example, with two identical charged particles, then (in their relative coordinate system) $\Delta\hat{\rho}^2\geqslant 0$ if $\alpha>0$; $\Delta\hat{\rho}^2\leqslant 0$ if $\alpha<0$; and if $\alpha=0$, $\Delta\hat{\rho}^2>0$ or $\Delta\hat{\rho}^2<0$. For a fixed positive charge and spiralling electron the signs of $\Delta\hat{\rho}^2$ (Δn, etc.) are the opposite of those just listed.

An interesting feature can be inferred from those of the last paragraph by considering a uniform, collimated, monoenergetic beam of particles carrying one sign of the charge. Particles in such a beam may be paired off so that they have the same ρ_-, which for the range of α values of greatest interest ($|\alpha|\ll n$) essentially amounts to pairing particles having opposite signs of α and the same n and p. According to the W formulas, the probabilities for the two components of beam with $\pm\alpha$ to have the opposite momentum *change* is therefore the same. Moreover, since the density of states for $\alpha>0$ and $\alpha<0$ is the same, we conclude that the net momentum change of a beam of particles along the magnetic field is very much smaller than the sum of the magnitudes of the individual momentum changes in that direction. It is not apparent that there should be a meaningful analogy to this feature in scattering without a magnetic field, though one may remember in this connection that in ordinary Coulomb scattering the total momentum change in a beam perpendicular to the initial direction of motion is zero.

B. Incident Beam Parallel to Magnetic Field

The eigenfunctions of particles which move in the direction of the magnetic field are those having $n=0$. Therefore, for a transition from an initial state having quantum numbers $n=0$, p, α to a state with quantum numbers n, $p+\Delta p\simeq p-2\Delta n/p$, α the relevant matrix element is

$$M_1=4\left(\frac{n!}{(n+\alpha)!\alpha!}\right)^{\frac{1}{2}}\int_0^\infty\exp(-\rho^2)\rho^{2\alpha+1}L_n{}^{(\alpha)}(\rho^2)\times K_0(|\Delta p|\rho)d\rho,$$

because $L_0{}^\alpha(x)=1$. With the use of[6]

$$\exp(-\rho^2)\rho^\alpha L_n{}^{(\alpha)}(\rho^2)=\frac{2}{n!}\int_0^\infty \exp(-s^2)s^{2n+\alpha+1}J_\alpha(2s\rho)ds,$$

and[7]

$$\int_0^\infty d\rho\, \rho^{\alpha+1}K_0(|\Delta p|\rho)J_0(2s\rho)=\frac{s^\alpha \alpha!}{4[s^2+(\Delta p/2)^2]^{\alpha+1}},$$

one can convert M_1 to

$$M_1=\left(\frac{\alpha!}{n!(\alpha+n)!}\right)^{\frac{1}{2}}\Delta_n{}^n\int_0^\infty e^{-\Delta_n t}\frac{t^{n+\alpha}}{(1+t)^{\alpha+1}}dt,$$

where $\Delta_n=(\Delta p/2)^2\simeq(n/p)^2$; from this form for M_1, one can readily show that for $\Delta_n\ll 1$, M_1 decreases as n increases, and it does this more sharply, the greater α becomes. This behavior is easily understood in terms of cancellations in the integrand of M_1, introduced by the sign changes (in number $=n$) of the final state wave function.

Later on, we shall be primarily interested in estimating the forward momentum change in situations where $n\ll\alpha$ and therefore we shall work only with

$$M_1=(\alpha+1)^{-\frac{1}{2}}\Delta_1\int_0^\infty e^{-\Delta_1 t}\left(\frac{t}{1+t}\right)^{\alpha+1}dt$$

$$\approx\frac{\sqrt{\pi}}{2}\frac{\exp\{-[(\alpha+1)\Delta_1]^{\frac{1}{2}}\}}{[\Delta_1(\alpha+1)]^{\frac{1}{4}}}\left(\frac{[\Delta_1(\alpha+1)]^{\frac{1}{2}}-\Delta_1/2}{[\Delta_1(\alpha+1)]^{\frac{1}{2}}+\Delta_1/2}\right)^{\alpha+1}\times\{1+\operatorname{erf}([\Delta_1(\alpha+1)]^{\frac{1}{2}})\},$$

and the cross section associated with it, which is

$$\sigma(B)=\pi(e^2/E)(e^2/\hbar\Omega)|M_1|^2. \qquad (22''')$$

Here one obviously deals with a quantum phenomenon. An important manifestation of this is that M_1 decreases exponentially with $(\alpha+1)^{\frac{1}{2}}$ if

$$[\Delta_1(\alpha+1)]^{\frac{1}{2}}=(\alpha+1)^{\frac{1}{2}}/p\gtrsim 1.$$

The onset of adiabatic condition is therefore again apparent. It should be realized here that in a large portion of the interval, the α dependence of M_1, although explicit, is weak. The cross section therefore shows "plateau behavior" for the corresponding large interval of impact parameters (just as in Sec. II A).

Now one can compare the forward momentum transfer, ΔP_z, with and without a magnetic field. The latter is

$$\Delta P_z\equiv 2\pi\int_{b-}^{b+} bdb\,\Delta p_z(b)$$

$$=-2\pi\int_{\chi_-}^{\chi_+} p_z(1-\cos\chi)\sigma(\chi)\sin\chi d\chi$$

$$=-4\pi p_z r_0{}^2\ln\left(\frac{\sin(\chi_+/2)}{\sin(\chi_-/2)}\right),$$

with b, p_z, $\chi_\pm$, $\sigma(\chi)$, and r_0 being respectively the impact parameter, the initial momentum, the maximum and minimum scattering angles, the Rutherford cross section, and me^2/p_z. Since we shall only be concerned with small-angle scattering (in large-angle scattering the magnetic field is a perturbation), we may approximate ΔP_z as

$$\Delta P_z=\Delta P_z(B=0)=4\pi p_z r_0{}^2\ln(\chi_-/2),$$

by a suitable choice for χ_+. χ_- is taken to be the scattering angle associated with the Debye length,[11] h, for applications to plasmas—in that case therefore

$$\chi_-/2\cong r_0/h.$$

Because of the aforementioned plateau behavior, the magnitude of the equivalent result valid in very strong magnetic fields is roughly the product of the magnitude of the cross section $\sigma(B)$, the momentum change $m\hbar\Omega/p_z$, and the width of the α interval; thus

$$\Delta P_z(B)\approx 2\pi p_z e^4/E\hbar\Omega.$$

The ratio of the two ΔP_z's is therefore

$$\mathcal{R}=\frac{\Delta P_z(B)}{\Delta P_z(0)}\approx\frac{1}{2}\left(\frac{E}{\hbar\Omega}\right)\frac{1}{|\ln(\chi_-/2)|}.$$

It decreases, as it must, with increasing B. The expression for $\mathcal{R}$ however, should not be assumed to have validity except as an asymptotic formula in B.

As was already pointed out before, the main value in having $\mathcal{R}$ is to enable one to make a meaningful comparison with the $B=0$ situation. In addition, manifestations of quantum phenomenon at very high field strengths is again shown.

It should be clear that relations (20′)-(22‴) have application in those situations where electrons have such high velocities in comparison to those of the ions that it is reasonable to consider the ions at rest.

This completes the description of the scattering by the field of a fixed point charge. The next section will discuss the identical-particle problem.

C. Identical Particles

With simple modifications the previously obtained transition probabilities may therefore be used to solve the classical problem of the scattering of two identical particles in the relative coordinate system; the quantum-mechanical interference terms will, therefore, be ignored. Symbols of physical quantities referring to the laboratory system will now be taken to represent the equivalent quantities in the relative coordinate system. The mass m therefore becomes the reduced mass, and p shall represent the relative momentum. Therefore $2p=p_2-p_1$, where the momenta p_1 and p_2

[11] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

are those of the individual particles. The transition probabilities for identical particles are expressible in terms of the ones given above by means of the relations

$$W_{np\alpha}{}^{(2)}=W_{np\alpha}(p-p')+W_{np\alpha}(p+p'), \qquad (23)$$

where p and p' respectively represent the relative momenta along the field, before and after collision, For small momentum changes (which are the ones of interest), one has

$$W_{np\alpha}{}^{(2)}\cong W_{np\alpha}(p-p'). \qquad (24)$$

The calculation of transition probabilities $W_{n_1p_1\alpha_1n_2p_2\alpha_2}^{n_1'p_1'\alpha_1'n_2'p_2'\alpha_2'}$ between states stipulated by (and not just expressed in terms of) individual-particle quantum numbers will be considered next. The most important reason for doing this is that, in kinetic theory, changes in the distribution function are most easily computed from the type of transition probabilities just mentioned.

Without a magnetic field, in scattering problems where the wave function obeys the plane-wave boundary condition at $z=-\infty$, a distinction between transition probabilities involving eigenstates of relative and center-of-mass momenta, (p and P, respectively) and those concerning eigenstates of the individual-particle momenta (p_1p_2) is indeed simple because the bracket,

$$\langle p_1p_2|pP\rangle=\delta(p_1+p-\tfrac{1}{2}P)\delta(p_2-p-\tfrac{1}{2}P), \qquad (25)$$

relating the corresponding matrix elements reduces the sum,

$$\langle p_1p_2|\mathcal{V}|p_1'p_2'\rangle = \sum_{pp'PP'}\langle p_1p_2|pP\rangle\langle p|\mathcal{V}|p'\rangle\delta_{PP'}\langle p'P'|p_1'p_2'\rangle,$$

to a single term, $\langle p_2-p_1|\mathcal{V}|p_2'-p_1'\rangle$.

In a magnetic field, however, the unperturbed eigenstates in relative and center-of-mass coordinates have their $\langle\ \ |\mathcal{V}|\ \ \rangle$ matrix elements related to the corresponding ones of the individual particles by the sum

$$\langle\Gamma|\mathcal{V}|\Gamma'\rangle=\textstyle\sum_{\Delta\Delta'}\langle\Gamma|\Delta\rangle\langle\Delta|\mathcal{V}|\Delta'\rangle\langle\Delta'|\Gamma'\rangle, \qquad (26)$$

where Γ symbolizes all the individual-particle quantum numbers $\{n_1,p_1,\alpha_1;n_2,p_2,\alpha_2\}$, and Δ represents the quantum numbers $\{np\alpha;NPA\}$ for the center-of-mass and relative coordinate system.

Instead of approaching the calculation of $W_{n_1p_1\alpha_1n_2p_2\alpha_2}^{n_1'p_1'\alpha_1'n_2'p_2'\alpha_2'}$ from (26), one may, perhaps, accomplish more by working directly with the wave functions of the two individual particles; however, this appears to have no obvious advantages because one must then deal with sizable difficulties in reliably evaluating certain types of iterated double integrals that are needed to find the Born approximation.

Despite the aforementioned difficulties, the work presented here still leads one to expect that the quantum-mechanical description of two-particle scattering in a magnetic field is more tractable than its classical counterpart. In the case of a slow ion and fast electron, this has been demonstrated.

IV. DISCUSSION

The interest in the material presented in the preceding sections will inevitably be related to how nearly correct the already published work on plasma properties is in taking into account the influence of very intense magnetic fields. Therefore, some comparisons will be made of quantities which enter into the calculation of plasma properties insofar as they do or do not ignore spiralling and (thus) the cutoff in momentum transfer, etc. At least two basic attributes of plasma must be kept in mind for this. One already mentioned is that the so-called long range of the Coulomb interaction makes the multitude of distant collisions involving small momentum transfer more influential than close, more violent ones. The second, very characteristic property of a plasma is the existence of a cutoff length beyond which the field of a particular (positive) ion is screened out by the plasma electrons; it is customarily (even in magnetic fields) taken to be the Debye length

$$h=(kT/4\pi Ne^2)^{\frac{1}{2}}=6.9(T/N)^{\frac{1}{2}},$$

where the symbols k, T, and N are, respectively, the Boltzman constant$=1.38\times10^{-16}$ erg/°K, the plasma temperature measured in °K, and the number of electrons per cc.

It follows at once, therefore, that $b_{\max}$ can be made smaller than h in situations which are within the capabilities of the present day laboratory, e.g., $T\gtrsim10^7$ °K, $N\lesssim10^{14}$/cc, $E\lesssim10^5$ ev and $B\gtrsim10^6$ gauss. From this alone one may conclude that the concept of screening distance in a magnetic field deserves careful investigation.

Furthermore, if one assumes for a moment that the cutoff phenomenon is the most important consequence of the external field, which somehow crops up only—as it does in the B field-free case—in a logarithmic factor, then one may conjecture that there exist at least logarithmic uncertainties in transport coefficients[12] over a range of wide interest.

Three reasons readily suggest themselves why there are likely to be additional uncertainties in the transport coefficient. First of all, one does not know whether the functional dependence on cutoff distance will reside in a logarithmic factor; because of the innate anisotropy that stems from the magnetic field, this functional dependence of at least some tensor components of the transport coefficients could be stronger than logarithmic. Secondly, the work in Sec. III shows that in many situations the transition probability already takes a substantial drop at ρ values smaller than $v/\Delta n\Omega$, which implies that the cutoff distance of the plasma in a magnetic field may be considerably shorter than previously estimated. Lastly, there ought to be ob-

[12] C. L. Longmire and M. N. Rosenbluth, Phys. Rev. **103**, 507 (1956).

servable consequences of the feature that a scattered beam of particles suffers little net momentum change in the direction of the magnetic field. Because of the neglect of ion motion in ion-electron collisions, this phenomenon may, however, turn out to be relatively somewhat less consequential (by some power of the mass ratio) in affecting those transport properties of plasmas which are governed by ion-electron collisions.

The effects of quantum phenomenon on individual scattering events for small n values (initial motion very nearly parallel to B), is, on the other hand, of no consequence in thermal plasmas because states with very small values of n contain such a small fraction of the total number of particles.

APPENDIX. THE WKB APPROXIMATION TO LAGUERRE POLYNOMIALS

In addition to being the solution to Eq. (1), Eq. (3) yields the WKB approximation to generalized Laguerre polynomials.

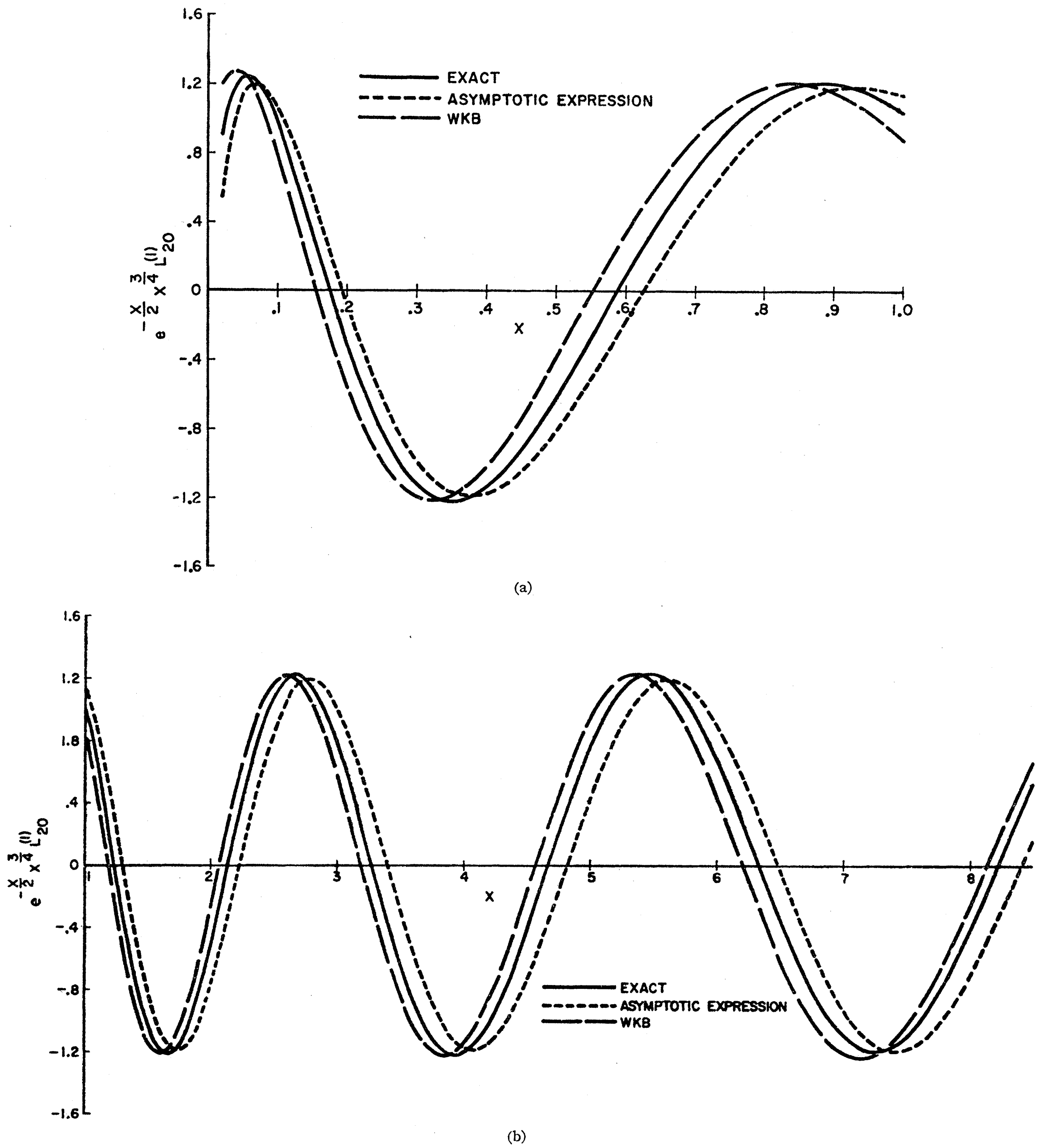

Fig. 1. Comparison of Laguerre polynomial $L_{20}^{(1)}$ to its approximations.

One obtains

$$\left[\frac{(n+\alpha)!}{n!}\right]^{\frac{1}{2}} L_n^{(\alpha)}(x) = \left(\frac{8}{\pi}\right)^{\frac{1}{2}} \Phi_{n\alpha}^{-\frac{1}{4}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha-\frac{1}{4}} \cos(S_{n\alpha}-\tfrac{1}{4}\pi), \quad (27)$$

where

$$S(x)=\tfrac{1}{2}\Bigg[(x_+-x)^{\frac{1}{2}}(x-x_-)^{\frac{1}{2}} - \tfrac{1}{2}(x_++x_-)\arcsin\left(\frac{x_++x_--2x}{x_+-x_-}\right)+\tfrac{1}{4}\pi(x_+^{\frac{1}{2}}-x_-^{\frac{1}{2}})^2 - (x_+x_-)^{\frac{1}{2}}\arcsin\left(\frac{(x_++x_-)x-2x_+x_-}{x(x_+-x_-)}\right)\Bigg], \quad (28)$$

and $x\Phi=(x_+-x)(x-x_-)$, by combining (3) and (9).

It is useful now to compare the accuracy of the approximations (8) and (27). The following conclusions were based on considerable computational work. For $\alpha>1$ the WKB approximation for $L_n^{(\alpha)}$ or $v_{n\alpha}$ [see Eq. (3)] becomes greatly superior to the usual asymptotic expression before the second node of $L_n^{(\alpha)}$. Moreover, even for $\alpha=1$ there is a strong trend for the WKB approximation to have more over-all accuracy than (8); the further away $L_n^{(\alpha)}$ is computed from the classical turning point, the more accurate its WKB approximation becomes relative to (8).

$L_n^{(1)}$ was computed for $n=15$, 20, 30, and 50 up to values of x where the nesting scheme used by the UNIVAC to generate the "exact" Laguerre polynomials was suspected of becoming subject to round-off error. For higher values of n, the first few oscillations of $L_{100}^{(2)}$, $L_{200}^{(2)}$, and $L_{100}^{(3)}$ were computed by hand.

The plots of $L_{20}^{(1)}$ shown in Fig. 1 demonstrate the typical behavior of the two approximations being compared for $\alpha=1$—the value of α apparently least favorable to the WKB method. To be more quantitative, we have computed

$$\sigma_1^2\equiv\sum_{x_1=d}^{x_i=c_id}[\text{exact}L_n^{(1)}(x_i)-\text{WKB}L_n^{(1)}(x_i)]^2,$$

and

$$\sigma_2^2\equiv\sum_{x_1=d}^{x_i=c_id}[\text{exact}L_n^{(1)}(x_i)-\text{asymptotic}L_n^{(1)}(x_i)]^2,$$

where $c_i=1, 2, 3\cdots$ and $d=0.02$, for $n=20$, 30, etc.

For $n=20$, for example, it turns out that individual cumulative (error) contributions to σ_1^2 are less than those to σ_2^2 except for x_i less than the position of the second node of $L_{20}^{(1)}$; for x_i shortly thereafter $\sigma_1^2<\sigma_2^2$ remains true.

For $\alpha>1$ (and strictly, only for $x<1$) the values of L from the WKB approximation were found to be so much better than those from the asymptotic expansion (8) that further comparison can add little useful information.