# Conductivity of Plasmas to Microwaves

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Plasma conductivities for electrons with a Maxwellian energy distribution are evaluated for the cases in which the collision cross section is (i) velocity independent and (ii) inversely proportional to the velocity. The corresponding distribution functions of relaxation times are discussed.

## 1. INTRODUCTION

In a recent paper Margenau[1] derived the frequency spectra of the complex conductivities of plasmas to microwaves by considering different statistical energy distributions of the electrons. In the case where the energy distribution of electrons is Maxwellian, the resultant complex conductivity is given in an implicit form which involves two integrals. Some asymptotic expansions of these integrals are obtained by the saddle-point method. In the present paper, explicit calculations of the complex conductivities are made for a Maxwellian distribution in which the collision cross section $q$ is (i) velocity independent, (ii) inversely proportional to the velocity. The corresponding distribution functions of the relaxation times are also discussed.

## 2. CALCULATIONS OF THE SPECTRA OF COMPLEX CONDUCTIVITIES

For electrons with Maxwellian distribution in their energy, i.e.,

$$f_0^0=(m/2\pi kT)^{\frac{3}{2}}\exp(-mv^2/2kT), \qquad (1)$$

where $mv^2/2$ is the kinetic energy of the electrons, and other symbols have their usual meanings, Margenau gives the current density $I$:

$$I=ne\gamma(J_1\cos\omega t+J_2\sin\omega t), \qquad (2$$

$$J_1=\frac{8}{3\sqrt{\pi}}\int_0^\infty \exp(-u^2)\frac{\nu u^4}{\omega^2+\nu^2}du, \qquad (3a)$$

$$J_2=\frac{8}{3\sqrt{\pi}}\int_0^\infty \exp(-u^2)\frac{\omega u^4}{\omega^2+\nu^2}du, \qquad (3b)$$

where $n$ is the number density of electrons, $\gamma=eE/m$, $E$ is the electric field strength of the microwaves, $\omega$ is the frequency of this field, $\nu$ is the collision frequency, and $u=v/v_0$ with $v_0=(2kT/m)^{\frac{1}{2}}$. The asymptotic expressions of $J_1$ and $J_2$ are obtained by Margenau by the saddle-point method. We would like to evaluate the $J$'s directly for two special cases: (1) $\nu$ is a constant, i.e., the collision cross section $q$ depends inversely on the velocity; (2) $\nu=u\nu_0$, where $\nu_0$ is a constant, i.e., $q$ is independent of $v$.

For the first case the integrals are simply gamma functions and the result is

$$J_1=\nu/(\omega^2+\nu^2), \qquad (4a)$$

$$J_2=\omega/(\omega^2+\nu^2). \qquad (4b)$$

This has the same form as that for Lorentz dispersion and is similar to the case of electrons with uniformly distributed energy, as given by Eq. (21) of Margenau.[1]

In the second case, the $J$'s can be conveniently expressed in terms of the function introduced by Dingle *et al.*[2]:

$$(p!)\mathfrak{A}_p(x)=\int_0^\infty \exp(-\epsilon)\frac{\epsilon^p}{x+\epsilon}d\epsilon. \qquad (5)$$

In this way we obtain

$$J_1=\frac{8}{3(\pi)^{\frac{1}{2}}\nu_0}\mathfrak{A}_2(x), \qquad (6a)$$

$$J_2=\frac{1}{\nu_0}\mathfrak{A}_{1.5}(x), \qquad (6b)$$

where $x=(\omega/\nu_0)^2$. This result is simpler for the purpose of numerical evaluation than that obtained by Altshuler and Molmud.[3] These authors express the $J$'s in terms of exponential integrals $\mathrm{Ei}(x)$ and error integrals $\Phi(x)$,

$$J_1=\frac{4}{3\sqrt{\pi}}\frac{x^{\frac{1}{2}}}{\omega}[1-x-x^2\exp(x)\,\mathrm{Ei}(-x)], \qquad (7a)$$

$$J_2=\frac{4}{3}\frac{x}{\omega}\left\{(\tfrac{1}{2}-x)+\frac{1}{\sqrt{\pi}}x^{\frac{3}{2}}\exp(x)[1-\Phi(x^{\frac{1}{2}})]\right\}. \qquad (7b)$$

Complex conductivities for some cases of semiconductors lead also to this form of Eq. (7) and have been derived by many authors.[4]

From the numerical table given by Dingle, $\nu_0J_1$ and $\nu_0J_2$ are calculated as a function of $\omega/\nu_0$. The table of Dingle *et al.* gives $\mathfrak{A}_p(x)$ for values of $x$ from 0.1 to 20. For $x>20$, $\mathfrak{A}_p(x)$ can be calculated from formula (19);

[1] H. Margenau, Phys. Rev. **109**, 6 (1958).

[2] Dingle, Arndt, and Roy, Appl. Sci. Research **B6**, 144 (1957). The numerical tables are given in this paper, as well as some asymptotic forms of expansion.

[3] S. Altshuler and P. Molmud (unpublished results quoted by H. Margenau[1]).

[4] See M. Bronstein, Physik. Z. Sowjetunion. **2**, 28 (1932), A. H. Wilson, *The Theory of Metals* (Cambridge University Press, Cambridge, 1953), p. 235.

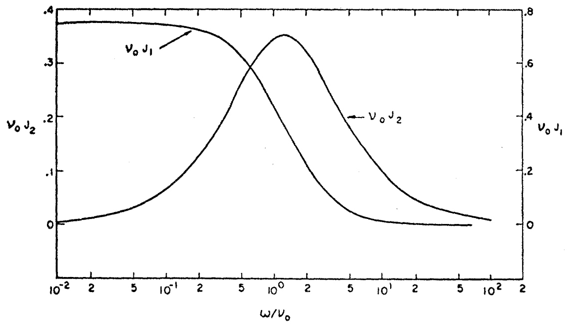


FIG. 1. The frequency spectrum of $\nu_0 J_1$ and $\nu_0 J_2$.

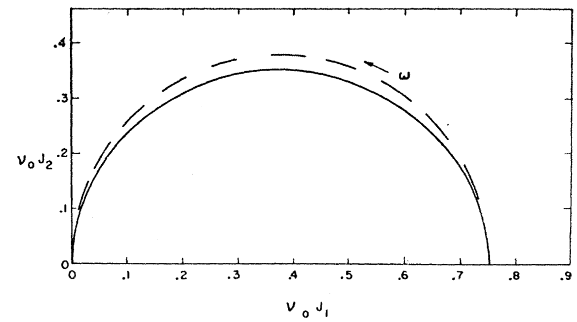


FIG. 2. Argand diagram of the complex conductivity of the plasma. The solid line is for the case in which the collision cross section depends inversely on the velocity; the broken line is for the case in which the collision cross section is velocity independent.

for $x<0.1$, $\mathfrak{A}_{1.5}(x)$ is calculated from formula (17), the formulas referring to the work of Dingle. On the other hand, it is easier to obtain $\mathfrak{A}_2(x)$ for $x$ from 0.1 to 0.01 from the mathematics table of Placzek.[5] The numerical results are given in Fig. 1, which shows the logarithmic frequency spectra of $\nu_0 J_1$ and $\nu_0 J_2$.

The relation between $J_1$ and $J_2$ can also be represented by an Argand diagram which is the frequency trace in the space formed by $J_1$ and $J_2$ as coordinates. This method of representation is very similar to the so-called Cole and Cole diagram[6] in the literature of dielectrics. In the Cole and Cole diagram, the coordinates are formed by the real and imaginary parts of the complex permittivity, and in the present case, the coordinates are formed by those of the complex conductivity. The result is shown in Fig. 2, where the solid curve is for Eqs. (6a) and (6b). This curve is almost identical to the curve given by the dispersion function of Cole and Cole[6] with $\alpha=4°$, where $\alpha$ is the measurement of the depression of the center of a semicircle below the real axis. An appreciable difference occurs only at the extreme frequencies where the arc intercepts on the real-axis; while the intercepts form right angles in the present case, oblique angles are formed in the case of Cole and Cole.

Figure 2 also shows a broken curve which is a semicircle. This curve is the diagram of Eqs. (4a) and (4b), and is identical to the Debye dispersion curve in dielectrics.[6]

### 3. DISTRIBUTION FUNCTION OF THE RELAXATION TIMES

A distribution function of the relaxation times can be formally introduced in the following way: $J_1$ and $J_2$, as given by Eqs. (3a) and (3b), can be immediately written in the form of Stieltjes' transform, i.e.,

$$J^*=J_1-iJ_2$$
$$=\frac{8}{3\sqrt{\pi}}\int_0^\infty \exp(-u^2)\frac{u^4}{\nu+i\omega}du. \quad (8)$$

This is to be compared with the usual definition of the distribution function of the relaxation time $g(\tau)$,[7] which is connected to $J^*$ in the following way:

$$J^*=\int_0^\infty \frac{g(\tau)}{1+i\omega\tau}d\tau. \quad (9)$$

In the case in which $\nu=k^{-1}$, a constant, $g(\tau)$ is a Dirac $\delta$ function of argument $(\tau-k)$.

In the case in which $\nu=u\nu_0$,

$$g(\tau)=\frac{8}{3\sqrt{\pi}}(\tau_0/\tau)^5\exp[-(\tau_0/\tau)^2], \quad (10)$$

where the relaxation time variables are defined at $\tau=\nu^{-1}$, $\tau_0=\nu_0^{-1}$. This $g(\tau)$ approaches zero at both $\tau=0$ and $\tau\rightarrow\infty$, and has a maximum at $\tau=0.633\tau_0$.

While many distribution functions have been introduced phenomenologically to describe the broadening effect in the relaxation dispersion, $g(\tau)$ given by Eq. (10) may be of special interest because this is based on a physical model.


### ACKNOWLEDGMENT

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[5] G. Placzek, National Bureau of Standards Applied Mathematics Series No. 37 (U. S. Government Printing Office, Washington, D. C.).

[6] K. S. Cole and R. H. Cole, J. Chem. Phys. 2, 341 (1941).

[7] See, for example, J. R. Macdonald and M. K. Brachman, Revs. Modern Phys. 28, 393 (1956).