Effect of Recoil on the Elastic Scattering of High-Energy Electrons by Zero-Spin Nuclei*

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The effect of nuclear recoil on the elastic scattering of high-energy electrons or muons by zero-spin nuclei is studied by adapting the Breit two-particle Hamiltonian to the case that one of the two particles is of finite size, is spinless, and is nonrelativistic, the other being a normal point Dirac particle. A radial and angular separation of the Dirac equation is still possible. To leading order in the parameter (electron energy)/(nuclear mass), the effect of the dynamic recoil terms is to rotate the scattering amplitude vectors in the complex plane without changing their magnitudes, a result which is independent of the shape and size of the nuclear charge distribution. To this order, the cross section is affected only by the kinematic recoil corrections. The dynamic recoil terms also influence the scattering amplitudes through terms of order (electron mass)/(nuclear mass). These corrections, owing to large amplification factors in going from phase shifts to cross section, may be of some significance in muon scattering, but are probably of no importance in the analysis of high-energy electron scattering. The dynamic effect is proportional to nuclear charge and therefore nearly as great for heavy as for light nuclei.

I. INTRODUCTION

DETAILED calculations of the elastic scattering of high-energy electrons by nuclei[1,2] have in the past usually treated the nucleus as a rigid charge distribution. Other effects on the scattering, all of which become increasingly important as the electron energy increases, are unresolved inelastic scattering to low nuclear excited states, unresolved inelastic scattering due to emission of low-energy bremsstrahlung, nuclear polarization effect, magnetic scattering from nuclear currents, both direct and exchange, and the dynamic effect of nuclear recoil (as well, of course, as the purely kinematic effect). None of these effects are expected to be very important for electron energies of several hundred Mev.[3] Their omission from calculations has been partially justified theoretically,[1,4–8] and justified also in practice by the fact that the same static nuclear charge distribution leads to a good fit to the experimental cross sections at several different energies.

Because the detailed numerical analysis and the determination of parameters of the nuclear charge distribution have been carried out to such high accuracy, however, it is possible that even rather small correction effects may be significant in modifying the conclusions drawn from electron scattering. Especially this could be true at the higher energies at which experiments are now being carried out.[1]

We have chosen to investigate the effect of the recoil of zero-spin nuclei, not because recoil should be the dominant correction effect, but because recoil can be treated in an unambiguous way, without reference to details of the nuclear energy level structure, and without the introduction of any further parameters or form factors beyond those associated with the static charge distribution. Inclusion of the recoil effect in the numerical analysis was also of interest for the extension of the calculations to the scattering of μ mesons by light nuclei.

We find that the dynamic recoil effect influences the cross section only through terms of order (electron mass)^2/[(nuclear mass)$\times$(electron energy)], the leading order terms proportional to (electron energy)/(nuclear mass) having no effect.

The theory of the nuclear recoil effect for spinless nuclei is presented in Sec. II. This investigation was initiated in connection with the calculations of high-energy electron scattering at Los Alamos, and the presentation of the theory in Sec. II is accordingly slanted toward the numerical calculations, most of the relevant equations for nonrecoil calculations, as well as recoil corrections, being included. The effect of the recoil correction on the phase shifts and scattering amplitude is discussed in Sec. III. Some numerical results are referred to as corroboration of the perturbation theory results of this section. We shall speak

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[1] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956); and *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1957), Vol. 7, p. 231. References to earlier work are given in these comprehensive review articles.
[2] Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).
[3] This statement applies only to spherical nuclei. For the significant effects of nuclear deformation, see Yennie, Ravenhall, and Downs, Phys. Rev. **106**, 1285 (1957).
[4] L. I. Schiff, Phys. Rev. **98**, 756 (1955); Nuovo cimento **5**, 1223 (1957).
[5] B. W. Downs, Phys. Rev. **101**, 820 (1956).
[6] R. R. Lewis, Phys. Rev. **102**, 544 (1956).
[7] L. R. B. Elton and H. H. Robertson, Proc. Phys. Soc. (London) **A65**, 145 (1952).
[8] K. W. Ford and D. L. Hill, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), Vol. 5, p. 25.

of the bombarding particles as electrons, but the theory will apply without modification to muons as well.

II. THEORY

A. Summary

Our object is to construct a Hamiltonian describing the electron-nucleus interaction in the center-of-momentum frame, and to derive from this Hamiltonian radial equations suitable for numerical integration. The Hamiltonian will be accurate to order Ze^2 and to order m/M, where m is the mass of the electron, and M the mass of the nucleus. We at first adopt the fiction that the nucleus is a Dirac particle, and generalize the two-particle Breit Hamiltonian[9] to a spread-out charge for one particle. A nonrelativistic reduction for the nucleus is then performed,[10,11] and terms involving the nuclear spin are discarded. A radial and angular separation of the resulting wave equation is possible, and a change of dependent variable finally brings the radial equations into a form close to their form in the absence of recoil. The scattering problem is solved in the center-of-momentum frame, and the calculated cross section is transformed to the laboratory frame.

B. Breit Hamiltonian for Spread-Out Charge

For two Dirac point particles, the Hamiltonian is given by[9]

$$H=-\beta_1 m_1-\beta_2 m_2-\boldsymbol{\alpha}_1\cdot\mathbf{p}_1-\boldsymbol{\alpha}_2\cdot\mathbf{p}_2+V+\Delta H, \quad (1)$$

where V is the electrostatic interaction energy and ΔH is given by

$$\Delta H^{(\text{point})}=-\tfrac{1}{2}e_1e_2\left[\frac{\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2}{r_{12}}+\frac{(\boldsymbol{\alpha}_1\cdot\mathbf{r}_{12})(\boldsymbol{\alpha}_2\cdot\mathbf{r}_{12})}{(r_{12})^3}\right]. \quad (2)$$

We let index 1 refer to the electron and index 2 to the nucleus, and generalize to a finite-sized nucleus by the transformation $e_1\rightarrow -e$ (e is the magnitude of the electron charge), and $e_2\rightarrow\int\rho d\mathbf{s}$ (ρ is the nuclear charge density and $d\mathbf{s}$ the volume element). Letting $\mathbf{r}$ be the vector from the center of the nucleus to the electron, $\mathbf{s}$ the vector from the center of the nucleus to any element of charge in the nucleus, $\mathbf{r}'=\mathbf{r}-\mathbf{s}$, and $r'=|\mathbf{r}'|$, we have for the Breit interaction term

$$\Delta H=\tfrac{1}{2}e\left[\int(r')^{-1}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2\rho(\mathbf{s})d\mathbf{s}+\int(r')^{-3}(\boldsymbol{\alpha}_1\cdot\mathbf{r}')(\boldsymbol{\alpha}_2\cdot\mathbf{r}')\rho(\mathbf{s})d\mathbf{s}\right], \quad (3)$$

and for the electrostatic potential

$$V=-e\int(r')^{-1}\rho(\mathbf{s})d\mathbf{s}. \quad (4)$$

We specialize to the center-of-momentum frame, and set $\mathbf{p}_1=-\mathbf{p}_2\equiv\mathbf{p}$. Also for convenience we set $m_1=m$, $m_2=M$. The Hamiltonian is then,

$$H=-\beta_1 m-\beta_2 M-\boldsymbol{\alpha}_1\cdot\mathbf{p}+\boldsymbol{\alpha}_2\cdot\mathbf{p}+V(r)+\Delta H, \quad (5)$$

with $V(r)$ given by (4) and ΔH by (3).

Before performing a nonrelativistic reduction with respect to particle 2 on the Hamiltonian (5), however, we reduce the interaction ΔH to simpler form. ΔH may be written

$$\Delta H=-\tfrac{1}{2}[2\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2V(r)-(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla})W(r)], \quad (6)$$

where $W(r)$ is defined by

$$W(r)=-e\int r'\rho(\mathbf{s})d\mathbf{s}, \quad (7)$$

and has the property $\nabla^2W=2V$. We now define a quantity U with the dimensions of energy which we shall call the "pseudopotential":

$$U(r)=2r^{-3}\int_0^r r^2V(r)dr. \quad (8)$$

The last term in (6) may be expressed in terms of U and V:

$$(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla})W(r)=\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2U+r^{-2}(\boldsymbol{\alpha}_1\cdot\mathbf{r})(\boldsymbol{\alpha}_2\cdot\mathbf{r})(2V-3U). \quad (9)$$

Finally, we introduce two "auxiliary potentials," V_1 and V_2, which are simple linear combinations of U and V:

$$V_1=2V-U, \quad (10)$$
$$V_2=3U-2V=-r(dU/dr).$$

In terms of the auxiliary potentials the generalized Breit interaction term takes on its simplest form,

$$\Delta H=-\tfrac{1}{2}[\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2V_1(r)+r^{-2}(\boldsymbol{\alpha}_1\cdot\mathbf{r})(\boldsymbol{\alpha}_2\cdot\mathbf{r})V_2(r)]. \quad (11)$$

For a pure Coulomb field (point particles), $U=V=V_1=V_2$. Note that in the Coulomb field outside a finite-sized particle, these four potentials are not equal, but approach equality only at great distance. Figure 1 illustrates the forms of U, V, V_1, and V_2 for a uniform charge distribution. They differ by terms of order $(R/r)^3$ outside the nucleus. For all values of r, V is equal to the weighted average of the auxiliary potentials, $\frac{1}{4}(3V_1+V_2)$.

C. Nonrelativistic Approximation for the Nucleus

The nonrelativistic reduction with respect to the nucleus is achieved to first order in m/M by the method of Foldy and Wouthuysen.[10,11] Since the term $-\beta_2M$ in the Hamiltonian goes to a constant nonrelativistically, we discard it at once, and write for the approximate Hamiltonian

$$H=-\beta_1 m-\boldsymbol{\alpha}_1\cdot\mathbf{p}+V(r)+(2M)^{-1}(\boldsymbol{\alpha}_2\cdot\mathbf{p}+\Delta H)^2. \quad (12)$$

[9] G. Breit, Phys. Rev. **34**, 553 (1929).
[10] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[11] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953).

Moreover, since we wish to specialize to zero-spin nuclei, it is understood that all spin-dependent terms resulting from the expansion of the last term in (12) are to be discarded. When the quantity $(\boldsymbol{\alpha}_2\cdot\mathbf{p}+\Delta H)$ is squared, there result terms (a) quadratic in $\mathbf{p}$, (b) bilinear in $\mathbf{p}$ and the auxiliary potentials V_1 and V_2, and (c) quadratic in the potentials. The last terms, which are $\frac{1}{4}(3V_1^2+V_2^2+2V_1V_2)$, we discard on the grounds that they are of second order in the interaction, whereas the Breit interaction term, ΔH, is accurate only to first order in αZ. As a practical matter, one may also note that for electron energies in the hundred Mev region, the discarded terms are also small compared to the terms linear in $\mathbf{p}$ and very small compared to the term quadratic in $\mathbf{p}$.

With the omission of nuclear spin terms and quadratic potential terms, the Hamiltonian (12) may be written

$$\begin{aligned}H=&-\beta m-\boldsymbol{\alpha}\cdot\mathbf{p}+V(r)\\&+(2M)^{-1}[p^2-\tfrac{1}{2}\boldsymbol{\alpha}\cdot(\mathbf{p}V_1+V_1\mathbf{p})\\&-\tfrac{1}{4}r^{-2}(\boldsymbol{\alpha}\cdot\mathbf{r})\mathbf{r}\cdot(\mathbf{p}V_2+V_2\mathbf{p})\\&-\tfrac{1}{4}(\mathbf{p}V_2+V_2\mathbf{p})\cdot\mathbf{r}(\boldsymbol{\alpha}\cdot\mathbf{r})r^{-2}],\end{aligned}\quad(13)$$

in which the subscript 1 has been dropped from the electron matrices α and β. The interaction terms in this Hamiltonian are exactly what one would arrive at by taking a symmetrized form of the classical Darwin interaction,[12] suitably generalized for a finite-sized nucleus.

D. Radial Separation

The wave equation, $H\psi=E\psi$, with the Hamiltonian (13) is separable in the same way as for a central field without recoil. We use the representation and notation of Schiff,[13] defining $\alpha_r=r^{-1}\boldsymbol{\alpha}\cdot\mathbf{r}$, and $p_r=\mathbf{p}\cdot(\mathbf{r}/r)=r^{-1}\mathbf{r}\cdot\mathbf{p}-ir^{-1}$, and defining the matrix k by $\boldsymbol{\alpha}\cdot\mathbf{p}=\alpha_rp_r+ir^{-1}\alpha_r\beta k$. In the correction term which multiplies $(2M)^{-1}$ it is permissible to substitute the solution of the wave equation without recoil. Operating on a solution of the equation without recoil, $\mathbf{p}^2$ is equivalent to $-m^2+(V-E)^2-i\alpha_r(dV/dr)$, and $\boldsymbol{\alpha}\cdot\mathbf{p}$ is equivalent to $-\beta m+V-E$. We therefore obtain the approximate Hamiltonian

$$\begin{aligned}H=&\{V+(2M)^{-1}[-m^2+(V-E)(V-E-V_1-V_2)]\}\mathbf{1}\\&+\{-m+(2M)^{-1}[m(V_1+V_2)]\}\beta\\&+\left\{ip_r+(2M)^{-1}\left[\frac{d}{dr}(\tfrac{1}{2}V_1+\tfrac{1}{2}V_2-V)\right]\right\}i\alpha_r\\&+\{-r^{-1}+(2M)^{-1}[r^{-1}V_2]\}i\alpha_r\beta k.\end{aligned}\quad(14)$$

[12] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

[13] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 322.

For eigenstates of k, the radial equations follow from $(H-E)\psi=0$, with

$$\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad i\alpha_r=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad\psi=\begin{pmatrix}r^{-1}F\\r^{-1}G\end{pmatrix}.\quad(15)$$

In the radial equations, it is no longer more convenient to use the auxiliary potentials V_1 and V_2. We re-express these in terms of U and V [Eqs. (10)], and find the coupled equations

$$\begin{aligned}&\{E+m-V-(2M)^{-1}(E+m-V)(E-m-V+2U)\}F\\&\quad-\{(d/dr)+kr^{-1}-(2M)^{-1}[dV/dr\\&\qquad+3(1+k)r^{-1}U-2(1+k)r^{-1}V]\}G=0\\&\{E-m-V-(2M)^{-1}(E-m-V)(E+m-V+2U)\}G\\&\quad+\{(d/dr)-kr^{-1}-(2M)^{-1}[dV/dr\\&\qquad+3(1-k)r^{-1}U-2(1-k)r^{-1}V]\}F=0.\end{aligned}\quad(16)$$

Terms of order V^2 or VU in these equations are not accurate, in view of the earlier approximations, and may be dropped.

These equations are satisfactory for numerical integration, but are not analytically tractable at large r, where $U\cong V=-\alpha Zr^{-1}$, because of terms proportional to r^{-2}. However, a simple transformation converts the equations into a form whose large-r behavior is identical in form to the Coulomb equations without recoil. We define new functions $\mathfrak{F}$ and $\mathfrak{G}$ by means of the transformation

$$\begin{aligned}F&=\{1+(2M)^{-1}[V-(1-k)U]\}\mathfrak{F},\\G&=\{1+(2M)^{-1}[V-(1+k)U]\}\mathfrak{G}.\end{aligned}\quad(17)$$

The radial equations for the new functions are

$$\begin{aligned}&\{\mathcal{E}+m-V+M^{-1}[\mathcal{E}V+(\mathcal{E}+m)(k-1)U]\}\mathfrak{F}\\&\qquad-[(d/dr)+kr^{-1}]\mathfrak{G}=0,\\&\{\mathcal{E}-m-V+M^{-1}[\mathcal{E}V-(\mathcal{E}-m)(k+1)U]\}\mathfrak{G}\\&\qquad+[(d/dr)-kr^{-1}]\mathfrak{F}=0,\end{aligned}\quad(18)$$

where

$$\mathcal{E}=E-(2M)^{-1}(E^2-m^2).\quad(19)$$

Note that E is the total energy in the center-of-momentum frame, and $\mathcal{E}$ is the electron energy in this frame. The electron energy in the laboratory frame is $E_{\text{lab}}=\mathcal{E}+M^{-1}(E^2-m^2)$.

The final radial equations used in the calculation are cast in a dimensionless form:

$$\begin{aligned}&v=V/m,\quad u=U/m,\quad\epsilon=\mathcal{E}/m,\\&\mu=m/M,\quad x=mr(\epsilon^2-1)^{\frac{1}{2}}.\end{aligned}\quad(20)$$

For convenience in joining to the Coulomb functions, we also introduce the further change of variable

$$\mathfrak{R}=\tfrac{1}{2}(\epsilon+1)^{-\frac{1}{2}}\mathfrak{G},\quad\mathfrak{g}=\tfrac{1}{2}(\epsilon-1)^{-\frac{1}{2}}\mathfrak{F}.\quad(21)$$

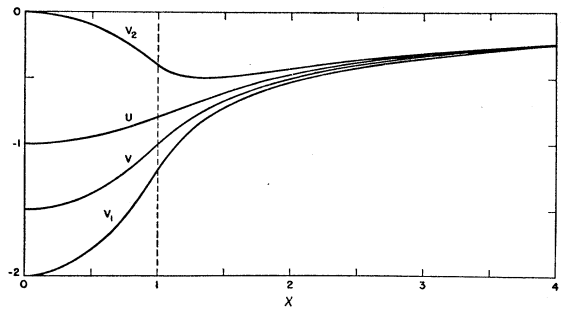

FIG. 1. The electrostatic potential, V; the pseudopotential, U, defined by Eq. (8); and the auxiliary potentials, V_1 and V_2, defined by Eqs. (10), illustrated for a uniform charge distribution. The horizontal scale is in units of the nuclear radius, R. The vertical scale is in units of Ze^2/R.

The radial equations then take the form

$$\begin{aligned}\frac{d\mathfrak{R}}{dx}&=-\frac{k}{x}\mathfrak{R}+\left\{1-\frac{v(x)}{\epsilon+1}+\mu\left[\frac{\epsilon v(x)}{\epsilon+1}+(k-1)u(x)\right]\right\}\mathfrak{g},\\ \frac{d\mathfrak{g}}{dx}&=-\frac{k}{x}\mathfrak{g}-\left\{1-\frac{v(x)}{\epsilon-1}+\mu\left[\frac{\epsilon v(x)}{\epsilon-1}-(k+1)u(x)\right]\right\}\mathfrak{R}.\end{aligned}\tag{22}$$

The recoil effect thus makes itself felt in several ways: (a) through the transformation $F\rightarrow\mathfrak{F}$, $G\rightarrow\mathfrak{G}$; (b) through the replacement of E by $\mathcal{E}$; and (c) through the extra terms multiplying μ in Eqs. (22).

E. Coulomb Equations

At large r, $u(x)\cong v(x)=-\alpha z(\epsilon^2-1)^{\frac{1}{2}}x^{-1}$. The radius at which $u=v$ to suitable accuracy will depend upon the overall importance of the recoil correction terms, but will normally be several times the nuclear radius (see Fig. 1). Outside the nucleus, approximately $|u-v|/v\approx\frac{1}{5}(R/r)^2$. We shall mean by the Coulomb region the radii for which the difference between u and v may be neglected.

In the Coulomb region, the recoil equations (22) take on the same form as the nonrecoil equations,

$$\begin{aligned}d\mathfrak{R}/dx&=-kx^{-1}\mathfrak{R}+(1+\beta_1x^{-1})\mathfrak{g},\\ d\mathfrak{g}/dx&=kx^{-1}\mathfrak{g}-(1+\beta_2x^{-1})\mathfrak{R},\end{aligned}\tag{23}$$

where

$$\begin{aligned}\beta_1&=\alpha z[(\epsilon-1)/(\epsilon+1)]^{\frac{1}{2}}[1+\mu(1-k-k\epsilon)],\\ \beta_2&=\alpha z[(\epsilon+1)/(\epsilon-1)]^{\frac{1}{2}}[1-\mu(1+k-k\epsilon)].\end{aligned}\tag{24}$$

Now set[14] $\mathfrak{R}=\frac{1}{2}(\sigma_1+\sigma_2)$, $\mathfrak{g}=-\frac{1}{2}i(\sigma_1-\sigma_2)$. The equations for σ_1 and σ_2 are

$$\begin{aligned}d\sigma_1/dx&=-i(1+\gamma x^{-1})\sigma_1-x^{-1}(k+i\gamma')\sigma_2,\\ d\sigma_2/dx&=i(1+\gamma x^{-1})\sigma_2-x^{-1}(k-i\gamma')\sigma_1,\end{aligned}\tag{25}$$

whose regular solutions are[14]

$$\begin{aligned}\sigma_1{}^R(k)&=(\epsilon+1)^{-\frac{1}{2}}N_k{}^R(2x)^{\rho_k}e^{-ix}(\rho_k-i\gamma)\\ &\quad\times F(\rho_k+i\gamma,\,2\rho_k+1,\,2ix),\\ \sigma_2{}^R(k)&=-(\epsilon+1)^{-\frac{1}{2}}N_k{}^R(2x)^{2\rho_k}e^{-ix}(k-i\gamma_k')\\ &\quad\times F(\rho_k+1+i\gamma,\,2\rho_k+1,\,2ix),\end{aligned}\tag{26}$$

where the F's are confluent hypergeometric functions, and the normalization constant is

$$N_k{}^R=|\Gamma(\rho_k+1+i\gamma)|\,e^{\frac{1}{2}\pi\gamma}/2\Gamma(2\rho_k+1)\times(i\gamma'-k)^{+\frac{1}{2}}(\rho_k-i\gamma)^{\frac{1}{2}}.\tag{27}$$

The irregular solutions, $\sigma_1{}^I(k)$ and $\sigma_2{}^I(k)$, are obtained by reversing the sign of ρ_k (but not of k) everywhere in (26) and (27).

These solutions are identical in form to the usual Coulomb solutions, differing only in the definitions of the constants γ and γ'. In this case,

$$\begin{aligned}\gamma&=\tfrac{1}{2}(\beta_1+\beta_2)=\alpha z\epsilon(\epsilon^2-1)^{-\frac{1}{2}}(1-\mu/\epsilon),\\ \gamma_k&=\tfrac{1}{2}(\beta_2-\beta_1)=\alpha z(\epsilon^2-1)^{-\frac{1}{2}}[1-\mu\epsilon+\mu k(\epsilon^2-1)],\end{aligned}\tag{28}$$

going over to the usual definitions in the case $\mu\rightarrow0$. The quantity ρ_k is defined by

$$\rho_k=+[k^2-\gamma^2+(\gamma_k')^2]^{\frac{1}{2}}.\tag{29}$$

The same definition holds in the nonrecoil case, but is then more simply written as $\rho_k=+[k^2-(\alpha z)^2]^{\frac{1}{2}}$. For $\mu\neq0$, however, $\gamma^2-(\gamma_k')^2\neq(\alpha z)^2$.

F. Evaluations of Phase Shifts and Cross Section

The total phase shifts of the functions $\mathfrak{F}$ and $\mathfrak{G}$ are the same as for the functions F and G, since asymptotically, $\mathfrak{F}\rightarrow F$, $\mathfrak{G}\rightarrow G$. It is therefore not necessary to transform back to F and G. The numerical solutions of Eqs. (22) may be joined to the known regular solutions (26) and the corresponding irregular solutions, and the phase shifts evaluated in the same way as without recoil. The solution of the scattering problem likewise follows in the same way as without recoil. The radial variable x is equivalent to kr, where k is the wave number of the electron in the center-of-momentum frame. In this frame, the formulation of the scattering problem in terms of an incident distorted plane wave and outgoing scattered waves is formally equivalent to the formulation for a rigid scattering center.

At some joining radius, a, the wave functions $\mathfrak{R}$ and $\mathfrak{g}$ obtained by integrating Eqs. (22) are joined to the regular and irregular Coulomb functions (for each value of k):

$$\begin{aligned}\mathfrak{R}(a)&=C\mathfrak{R}^R(a)+D\mathfrak{R}^I(a);\\ \mathfrak{g}(a)&=C\mathfrak{g}^R(a)+D\mathfrak{g}^I(a).\end{aligned}\tag{30}$$

The phase shift, ζ_k, of the regular Coulomb function is given by

$$\exp(2i\zeta_k)=\frac{k-i\gamma_k'}{\rho_k-i\gamma}\frac{\Gamma(\rho_k+1-i\gamma)}{\Gamma(\rho_k+1+i\gamma)}\exp[-\pi i(\rho_k-l)],\tag{31}$$

[14] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition, p. 79.

where l, corresponding to orbital angular momentum, is equal to k for $k>0$, and is equal to $-k-1$ for $k<0$. The definition of the irregular Coulomb phase shift, ζ_k', is obtained from (31) by reversing the sign of ρ_k everywhere. The difference, δ_k, between the total phase shift, η_k, and the regular Coulomb phase shift, ζ_k, is given by

$$\tan\delta_k=\sin(\zeta_k'-\zeta_k)/[(C/D)+\cos(\zeta_k'-\zeta_k)]. \quad (32)$$

Finally, the cross section is defined by $\sigma=|f|^2+|g|^2$, with

$$f=(\lambda\!\!\!^{-}/2i)\sum\{(l+1)[\exp(2i\eta_{-l-1})-1] +l[\exp(2i\eta_l)-1]\}P_l(\cos\theta), \quad (33)$$

$$g=(\lambda\!\!\!^{-}/2i)\sum[-\exp(2i\eta_{-l-1})+\exp(2i\eta_l)]P_l^1(\cos\theta).$$

The polarization induced in an unpolarized beam is given by

$$P=2(\mathrm{Re}f\,\mathrm{Im}g-\mathrm{Re}g\,\mathrm{Im}f)/(|f|^2+|g|^2), \quad (34)$$

and is entirely negligible at energies much greater than mc^2.

III. THE EFFECT OF RECOIL

A. Neglect of Electron Mass

Consider the dimensionless equations (22) for $\mathfrak{R}$ and $\mathfrak{g}$ in the center-of-momentum frame, and the corresponding equations $(22)_0$ for $\mathfrak{R}_0$ and $\mathfrak{g}_0$ defined by setting μ equal to zero. In order to isolate the dynamic recoil effect, we let the energy ϵ be the same in both equations. In addition, as a matter of convenience, we at first neglect the mass of the electron ($\epsilon\gg1$). This permits the derivation of an unusually simple result, expressed by (38) and (39) below.

From (22) and $(22)_0$ it follows that

$$\frac{d}{dx}(\mathfrak{g}_0\mathfrak{R}-\mathfrak{R}_0\mathfrak{g}) =\mu[(v-u)(\mathfrak{g}_0\mathfrak{g}+\mathfrak{R}_0\mathfrak{R})+ku(\mathfrak{g}_0\mathfrak{g}-\mathfrak{R}_0\mathfrak{R})]. \quad (35)$$

We choose unit normalization for the asymptotic solutions $\mathfrak{g}$ and $\mathfrak{R}$: $\mathfrak{g}\to\cos\varphi$, $\mathfrak{R}\to\sin\varphi$, where $\varphi=x+\gamma\ln x-\frac{1}{2}\pi l+\eta$; and similar forms for $\mathfrak{g}_0$ and $\mathfrak{R}_0$. Then (35) may be written

$$\eta-\eta_0=\mu\int_0^\infty[(v-u)(\mathfrak{g}_0\mathfrak{g}+\mathfrak{R}_0\mathfrak{R}) +ku(\mathfrak{g}_0\mathfrak{g}-\mathfrak{R}_0\mathfrak{R})]dx. \quad (36)$$

To first order in μ, replace $\mathfrak{g}$ and $\mathfrak{R}$ by $\mathfrak{g}_0$ and $\mathfrak{R}_0$ on the right of (36). But from $(22)_0$ and (10),

$$(v-u)(\mathfrak{g}_0^2+\mathfrak{R}_0^2)+ku(\mathfrak{g}_0^2-\mathfrak{R}_0^2) =\frac{1}{2}\frac{d}{dx}[xu(\mathfrak{g}_0^2+\mathfrak{R}_0^2)]. \quad (37)$$

At large x, $xu\to-\alpha Z\epsilon$, $\mathfrak{g}_0^2+\mathfrak{R}_0^2\to1$. Therefore,

$$\eta-\eta_0\cong-\tfrac{1}{2}\mu\alpha Z\epsilon, \quad (38)$$

i.e., the total phase shift is changed by the recoil terms by a constant amount, independent of k and independent of the shape and size of the charge distribution. In particular therefore for the pure Coulomb phase shifts, $\zeta-\zeta_0\cong-\frac{1}{2}\mu\alpha Z\epsilon$, as could be inferred also directly from (28) and (31). Note in (36) that for the pure Coulomb field, $u=v$ and the first term in the integrand vanishes. For a distributed charge the contribution of the first term exactly balances the decrease in the contribution of the second term (see Fig. 1). From (38) and (33) it follows at once for the scattering amplitudes that

$$(f-f_0)/f_0=(g-g_0)/g_0=-i\mu\alpha Z\epsilon. \quad (39)$$

(In the present approximation of neglecting the electron mass, $g=f\tan\frac{1}{2}\theta$). Therefore to first order in μ, the cross section is unaffected by the dynamic recoil effect.

The fact that this first order result is quite accurate has been verified by numerical integration of Eqs. (22) and $(22)_0$ for realistic charge distributions. For 420-Mev electrons incident on C^{12}, the theoretical value of $\eta-\eta_0$ is -8.2×10^{-4}. The values of the same difference obtained numerically ranged from -7.9×10^{-4} for $|k|=1$ to -8.2×10^{-4} for $|k|\geqslant10$. For 420-Mev electrons incident on Pb^{208}, the theoretical value of $\eta-\eta_0$ is -6.5×10^{-4}. The values of the same difference obtained numerically ranged from -6.3×10^{-4} for $|k|=1$ to -6.5×10^{-4} for $|k|\geqslant20$. This weak k-dependence of the phase shift difference is barely significant numerically.

It is of some interest to note that a small energy change is amenable to the same sort of perturbation treatment. For example, let ϵ_0 be the (dimensionless) laboratory energy, and $\epsilon=\epsilon_0-\mu\epsilon_0^2$. Again form the derivative of $\mathfrak{g}_0\mathfrak{R}-\mathfrak{R}_0\mathfrak{g}$ as in (35). Then, in analogy with (38), the combined effect of the kinetic and dynamic recoil corrections on the phase shift is given by

$$\eta-\eta_0\cong-\tfrac{1}{2}\mu\alpha Z\epsilon_0-\mu\int_0^\infty(\mathfrak{g}_0^2+\mathfrak{R}_0^2)\frac{d}{dx}(xv_0)dx. \quad (40)$$

For a point nucleus, the second term vanishes, i.e., a small energy change has no effect on the Coulomb phase shifts. For a finite nucleus the two terms are of the same order of magnitude for small $|k|$, and the second term vanishes for large $|k|$.

B. Effect of Finite Electron Mass

If the finite mass of the electron is taken into account, (38) is replaced by

$$\eta-\eta_0\cong-\tfrac{1}{2}\mu\alpha Z\epsilon-\mu\epsilon^{-1}\int_0^\infty(\mathfrak{g}_0^2-\mathfrak{R}_0^2)vdx, \quad (41)$$

to first order in αZ. The last term in (41) is of order ϵ^{-1} relative to the leading term, but it is k-dependent and hence capable of changing the value of the cross section. The k-dependent part of $\eta-\eta_0$ for electrons is of order 10^{-6} (it is 200 times larger for muons). However, in the zero-mass approximation, $\mathcal{G}_0^2-\mathcal{R}_0^2$ for given k is equal to the negative of the same quantity for $-k$. Consequently, the first order contribution of the last term in (41) vanishes. This may be seen as follows. The radial wave equations (16) are invariant under the simultaneous substitutions

$$k\rightarrow -k, \quad m\rightarrow -m,$$
$$F\rightarrow G, \quad G\rightarrow -F.$$

This implies

$$\eta_k(m)=\eta_{-k}(-m). \tag{42}$$

Now, expanding the scattering amplitudes (33) to order m, it is readily found that

$$f=f|_{m=0}+m(\partial f/\partial m)|_{m=0}+\cdots,$$
$$g=\tan\tfrac{1}{2}\theta\, f|_{m=0}-\cot\tfrac{1}{2}\theta\, m(\partial f/\partial m)|_{m=0}+\cdots. \tag{43}$$

To first order in m, these contributions cancel completely in the cross section. The residual contributions will therefore be of order ϵ^{-2}, $\epsilon^{-1}\alpha Z\mu$, and $(\mu\alpha Z)^{+2}$ relative to the cross section in which electron mass is neglected. The first of these residual contributions (ϵ^{-2}) is present even if recoil is neglected; the second ($\epsilon^{-1}\alpha Z\mu$) is an interference between the last term of (41) and the finite mass contribution without recoil; the third $[(\mu\alpha Z)^{+2}]$ arises from the last term of (41) by itself. For 400-Mev electrons, all of these corrections seem to be entirely negligible; they are approximately 10^{-6}, 10^{-9}, and 10^{-12} respectively, relative to the zero-mass, non-recoil result. Of course the delicate cancellations that take place in summing the series to obtain the first term of (43) may not take place in the later terms. This may lead to results considerably larger than just estimated (say 100 or 1000 times larger), but still entirely negligible. In the numerical calculations referred to above, a given phase shift change, $\Delta\eta/\eta$, produced typically a cross-section change, $\Delta\sigma/\sigma$, of about $10^3\,\Delta\eta/\eta$.

We conclude that for energies and angles which have been studied so far in electron scattering, the dynamic recoil correction is probably of no significance. It may be of significance in the scattering of muons.

Because of the extremely high accuracy required in the phase shifts, and because of the great cancellations occurring in the partial-wave sum, the numerical techniques have not as yet been sufficiently refined to give a quantitatively reliable prediction of these small recoil effects. Results of the numerical analysis will be presented later.[15]

[15] Ford, Hill, Hill, and Wills (to be published).

C. Perturbation Theory Without Partial Wave Expansion

It is interesting to see how the results of the preceding subsections can be understood in terms of a perturbation treatment of the complete scattering wave function. Ravenhall[16] has pointed out that in Born approximation the recoil effects are purely kinematic. In the present discussion we shall go beyond the Born approximation in that the nonrecoil problem will be considered exactly and the recoil terms will be treated as a perturbation.

We may dispense with the term involving the electron mass. This follows from the fact that a change in sign of m can be exactly compensated for by a change in representation of the Dirac matrices. Thus the cross section cannot contain odd powers of m; the results of subsection B follow immediately.

It is convenient to rewrite the Hamiltonian (13) as

$$H=-\boldsymbol{\alpha}\cdot\mathbf{p}+V+(2M)^{-1}\{p^2-\boldsymbol{\alpha}\cdot\mathbf{p}V-V\boldsymbol{\alpha}\cdot\mathbf{p} +\tfrac{1}{2}i[\boldsymbol{\alpha}\cdot\mathbf{p},(\mathbf{p}\cdot\boldsymbol{\nabla}W+\boldsymbol{\nabla}W\cdot\mathbf{p})]\}. \tag{44}$$

The unperturbed scattering problem is then given by the equation

$$(-\boldsymbol{\alpha}\cdot\mathbf{p}+V-\mathcal{E})\varphi^{(\pm)}=0. \tag{45}$$

To order $(1/M)$ the perturbing Hamiltonian may be written

$$H_{\text{eff}}'=-(2M)^{-1}(V^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}W) +(2M)^{-1}(\boldsymbol{\alpha}\cdot\mathbf{p}-V+\mathcal{E})(\boldsymbol{\alpha}\cdot\mathbf{p}-V-\mathcal{E}) +(i/4M)[(\boldsymbol{\alpha}\cdot\mathbf{p}-V+\mathcal{E}),(\mathbf{p}\cdot\boldsymbol{\nabla}W+\boldsymbol{\nabla}W\cdot\mathbf{p})]. \tag{46}$$

For consistency with the approximations already made, the first term must be neglected. The second term vanishes in lowest order perturbation theory because of the equation satisfied by the unperturbed function. The last term also *appears* to vanish in lowest order; however, neither the operator nor the scattering states approach zero sufficiently rapidly at large distances to justify an integration by parts and the hermitian property of p is not valid. Instead, the matrix element of the last term may be written

$$(4M)^{-1}\int\nabla\cdot\{\varphi_f^{(-)\dagger}\boldsymbol{\alpha}(\mathbf{p}\cdot\boldsymbol{\nabla}W+\boldsymbol{\nabla}W\cdot\mathbf{p})\varphi_i^{(+)}\}d\tau. \tag{47}$$

Clearly the value of this expression can depend only on the asymptotic forms of the wave functions, which are

$$\varphi_i^{(+)}\sim e^{i\mathbf{k}_i\cdot\mathbf{r}}+f(\mathbf{k}_i\cdot\mathbf{r}/kr)e^{ikr}/r,$$
$$\varphi_f^{(-)}\sim e^{i\mathbf{k}_f\cdot\mathbf{r}}+f^*(-\mathbf{k}_f\cdot\mathbf{r}/k_fr)e^{-ikr}/r, \tag{48}$$

where $\mathbf{k}_i$ and $\mathbf{k}_f$ are the initial and final wave number vectors. Although the integral is still ambiguous, we may obtain a reasonable interpretation by the following procedure. Take the volume of integration to be a sphere centered about the potential and evaluate the

[16] D. Ravenhall (private communication).

resulting surface integral by the method of stationary phase. Neglecting terms which oscillate with R (the radius of the sphere) or decrease with increasing R, the result turns out to be

$$\Delta f = -i\mu\alpha Z\epsilon f, \tag{49}$$

in agreement with that obtained by the partial wave analysis. This also confirms Ravenhall's result that recoil effects do not contribute in Born approximation (note that the first term of H_{eff}' corresponds to second Born approximation in the electromagnetic interaction).

IV. CONCLUSIONS

To the extent that the finite mass of the electron or muon may be neglected, the dynamic recoil effect does not influence the cross section. To this approximation, recoil is taken into account by calculating with an energy $\mathcal{E}=E_{\text{lab}}(1-E_{\text{lab}}/M)$, and by transforming calculated cross sections and angles from the center-of-momentum frame to the laboratory frame. Consideration of the finite mass of the electron or muon leads to a dynamic recoil effect on the cross section which is $\Delta\sigma/\sigma = (\alpha Z)(m/M)(m/E)\gamma$, where γ is an amplification factor (arising from the great cancellations in the partial wave sum) whose magnitude is very uncertain, but might be as great as 10^4 at the highest energies and greatest angles where electron scattering data are available.

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Alternative Method for Comparing Pion-Proton Scattering Data with Dispersion Equations*

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A method for comparing pion-proton scattering experiments with the predictions of the forward angle scattering dispersion equations is proposed, which allows the usual statistical measure (χ^2) of the agreement. A slight discrepancy is found between negative pion-proton data and the theory; however, the over-all agreement is considered satisfactory. Values of the coupling constant and S-wave zero-energy scattering lengths are determined. They are $f^2=0.08\pm0.01$, $a_1=0.193\pm0.050$, and $a_3=-0.089\pm0.048$.

1. INTRODUCTION

SINCE the analysis of the pion-nucleon scattering by use of forward scattering dispersion equations was made by Puppi and Stanghellini,[1] several authors have discussed the lack of agreement between the theory and experiments.[2] It is desirable to make the comparison in a way more easily analyzed statistically than the Puppi-Stanghellini method. One such method is presented here,[3] and the results, which include a determination of the pion-nucleon coupling constant and the zero-energy S-wave scattering lengths, are reported.

2. METHOD OF ANALYSIS

One can write the forward scattering pion-proton dispersion equations as follows[4]:

$$\tfrac{1}{2}[D_+(1)+D_-(1)]\pm\tfrac{1}{2}\omega[D_+(1)-D_-(1)]$$
$$=D_\pm(\omega)-\frac{k^2}{4\pi^2}\,\text{P.V.}\int_1^\infty\frac{d\omega'}{k'}\frac{\sigma_\pm^{\text{tot}}(\omega')}{\omega'-\omega}-\frac{k^2}{4\pi^2}$$
$$\times\int_1^\infty\frac{d\omega'}{k'}\frac{\sigma_\mp^{\text{tot}}(\omega')}{\omega'+\omega}\mp\frac{2k^2}{\omega\mp(1/2M)}f^2\equiv J_\pm(\omega),\quad(1\pm)$$

where the notation and units are the same as in SS.[2] $D_\pm(\omega)$ is the real part of the $\pi^\pm$-p forward scattering amplitude in the laboratory system at pion energy ω. Define

$$C_1=\tfrac{1}{2}[D_+(1)+D_-(1)],$$
$$C_2=\tfrac{1}{2}[D_+(1)-D_-(1)]. \tag{2}$$

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[1] G. Puppi and A. Stanghellini, Nuovo cimento **5**, 1305 (1957).
[2] H. J. Schnitzer and G. Salzman, Phys. Rev. **112**, 1802 (1958). This will be referred to as SS. Additional references may be found in this paper.
[3] H. P. Noyes and D. N. Edwards (to be published) recast the comparison to facilitate statistical analysis. The analyses differ in that we use "experimental" cross sections, with no errors assigned to the integrals, while they use "theoretical" ones to evaluate the integrals, with an associated error. They find an f^2 for each energy while we require a fit to all energies with the same f^2.
[4] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).