Theoretical Total-Energy Distribution of Field-Emitted Electrons*

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The Fowler-Nordheim equation is derived in terms of total electron energy so as to obtain the total-energy distribution of field-emitted electrons. At 0°K the new distribution width is less than $\frac{1}{3}$ of that obtained from the previous "normal-energy" theory. A surprising mirror-image symmetry is observed between the zero-temperature total-energy field emission and the zero-field normal-energy thermionic emission distributions. The total-energy distribution is also derived for the zero-field thermionic case and this is found to be the mirror image of the normal-energy zero-temperature field emission distribution.

The new distribution applies to problems involving the total energy of electrons before and after emission.

INTRODUCTION

A SENSITIVE test of the validity of any model for the emission of electrons is the energy distribution measurement. Information about the origin of emitted electrons and the energy dependence of electron emission probability can also be expected from this source.

In view of these facts steps were undertaken to improve the resolution of the field-emission retarding-potential analyzer with the hope of using the improved tube in the study of the electron energy band structure in conductors and semiconductors. The improved tube, discussed in the following paper, revealed a much narrower energy distribution than predicted by the "normal-energy" distribution theory. Further analysis indicated that the improved analyzer measures total electron energy rather than the energy associated with the component of velocity normal to the emitting surface. For this reason the Fowler-Nordheim equation is derived in this paper in such a way as to preserve the distribution in total energy of field-emitted electrons.

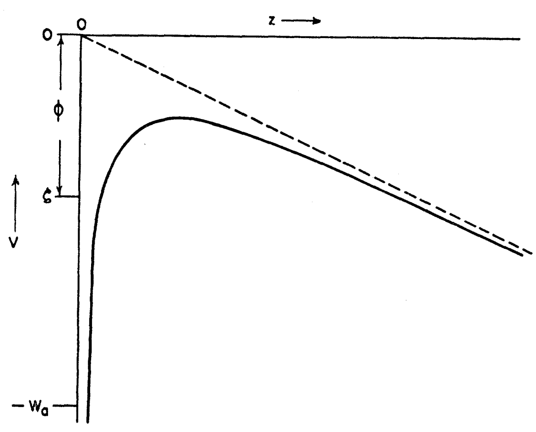

FIG. 1. One-dimensional potential energy $V(z)$ of an electron near a metal surface as given by Eqs. (1) and (2).

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DERIVATION OF THE TOTAL-ENERGY DISTRIBUTION OF FIELD-EMITTED ELECTRONS

I. Fowler-Nordheim Model[1,2]

According to the Fowler-Nordheim model, electrons arrive at the surface of a metal according to Fermi-Dirac statistics and penetrate the potential hump in front of the surface with a probability which is predicted by a solution of the Schrödinger equation.

Figure 1 shows the one-dimensional potential energy of an electron near the metal surface at $z=0$. Without field, the potential energy of an electron far outside the metal surface is chosen to be zero. $V(z)$ is then[3]

$$V(z)=-W_a \quad \text{where} \quad z<0, \tag{1}$$

$$=-\frac{e^2}{4z}-eFz \quad \text{where} \quad z>0. \tag{2}$$

The z part of energy is defined by the equations

$$W=E-\frac{P_x^2}{2m}-\frac{P_y^2}{2m} \tag{3}$$

$$=\frac{P_z^2}{2m}+V(z). \tag{4}$$

In the usual Fowler-Nordheim derivation[3,4] a supply function $N(W)dW$ equal to the number of electrons with the z part of their energy within the range W to $W+dW$ incident on the surface per second per area is multiplied by a barrier penetration probability $D(W)$ to obtain the number of electrons within the range W to $W+dW$ that emerge from the metal per second per unit area. In this paper the number in the range W to $W+dW$ emerging per second per unit area, $P(W)dW$, will be called the normal-energy distribution. The Fowler-Nordheim equation is obtained by integrating $P(W)dW$ over all W.

[1] R. H. Fowler and L. Nordheim, Proc. Roy. Soc. (London) **A119**, 173 (1928).
[2] L. Nordheim, Proc. Roy. Soc. (London) **A121**, 626 (1928).
[3] See for example R. H. Good and E. W. Müller, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 21, p. 181.
[4] E. L. Murphy and R. H. Good, Jr., Phys. Rev. **102**, 1464 (1956).

In the present derivation of the Fowler-Nordheim equation the above definitions hold, and in addition the following:

$N(W,E)dWdE \equiv$ number of electrons with energy within the range E to $E+dE$ whose z part of the energy lies in the range W to $W+dW$, incident upon the surface $z=0$ per area per time. (5)

$D(W) \equiv$ probability that an electron with energy W will penetrate the barrier. (6)

$P(W,E)dWdE = N(W,E)D(W)dWdE$ = number of electrons in the given energy ranges penetrating the barrier. (7)

$P(E)dE = \int_W P(W,E)dWdE$ = total-energy distribution. (8)

$j = e\int P(E)dE$ = electric current per unit area = Fowler-Nordheim equation. (9)

II. Calculation of Supply Function $N(W,E)dWdE$

Let $n(E) \equiv$ number of electrons per unit volume within the metal between E and $E+dE$. In Fermi-Dirac statistics the number of electrons with energy E is uniform in solid angle ω. The number of electrons with energy between E and $E+dE$ incident between θ and $\theta+d\theta$ and between ϕ and $\phi+d\phi$ on the unit surface at $z=0$ per unit time is

$$N(\omega,E)d\omega dE = n(E)dE\frac{|v|\cos\theta}{4\pi}\sin\theta d\theta d\phi$$

= (number arriving at θ per solid angle)

$\times$ (differential solid angle). (10)

where

$|v|$ = magnitude of electron velocity $= [2(E-V)/m]^{\frac{1}{2}}$; (11)

θ = angle between the electron velocity vector and the normal to the surface;

ϕ = azimuthal angle.

From Eqs. (3), (4), and (11), we have

$$|v|\cos\theta\sin\theta d\theta = \frac{-dW}{[2m(E-V)]^{\frac{1}{2}}}. \quad (12)$$

Substituting (12) in (10) and integrating on ϕ, we get

$$N(W,E)dWdE = \frac{-n(E)dWdE}{2[2m(E-V)]^{\frac{1}{2}}}, \quad (13)$$

where

$n(E)dE$ = energy distribution in Fermi-Dirac electron gas with energy measured relative to an electron at rest at ∞. (14)

$$n(E)dE = \frac{4\pi(2m)^{\frac{3}{2}}(E-V)^{\frac{1}{2}}dE}{h^3\exp[(E-\zeta)/kT]+1}, \quad (15)$$

where ζ is the Fermi energy. The supply function is obtained by substituting (15) into (13):

$$N(W,E)dWdE = -\frac{4\pi m}{h^3}\frac{dWdE}{\exp[(E-\zeta)/kT]+1}. \quad (16)$$

III. Transmission Coefficient $D(W)$

The transmission coefficient $D(W)$ is the one used in the normal-energy derivation and is discussed in reference 3. For $W \ll V_{\max}$ and for emission in range $W \sim \zeta$, it can be shown[3] that

$$D(W) \cong \exp[-c+(W-\zeta)/d], \quad (17)$$

where

$$c = \frac{4(2m\phi^3)^{\frac{1}{2}}}{3\hbar eF}v((e^3F)^{\frac{1}{2}}/\phi), \quad (18)$$

$$d = \frac{\hbar eF}{2(2m\phi)^{\frac{1}{2}}t((e^3F)^{\frac{1}{2}}/\phi)}, \quad (19)$$

$$\phi = -\zeta = \text{work function}; \quad (20)$$

$t(y)$ and $v(y)$ are slowly varying functions.[3]

IV. Calculation of $P(E)dE$

It follows from Eqs. (7) and (8) that

$$P(E)dE = \int_{W=E}^{-W_a} N(W,E)D(W)dWdE, \quad (21)$$

where the original limits of integration from 0 to $\pi/2$ have been transformed to the range from $W=E$ to $-W_a$. When $W=-W_a$ the integrand is essentially zero and the integration is facilitated by setting the $-W_a$ limit equal to $-\infty$. Substituting (16) and (17) into (21) and integrating on W, one obtains

$$P(E)dE = \frac{4\pi md}{h^3}\exp\left(-c-\frac{\zeta}{d}\right) \times \frac{e^{E/d}}{\exp[(E-\zeta)/kT]+1}dE. \quad (22)$$

Equation (22) is the total-energy distribution for field-emitted electrons. The energy-dependent portion of this function is the product of a field- and work function-dependent barrier penetration probability and the Fermi-Dirac distribution function.

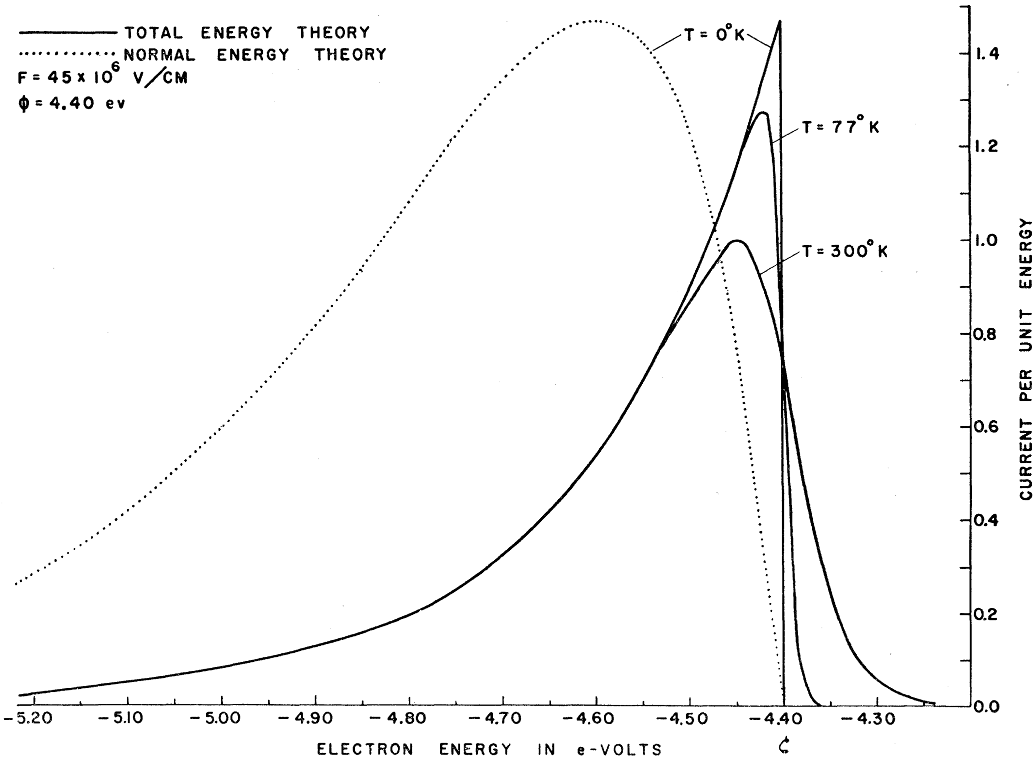

FIG. 2. Total- and normal-energy distributions for field emitted electrons. The 300°K curve is arbitrarily normalized.

V. Fowler-Nordheim Equation

The emission current density is found by substituting (22) into (9):

$$j=e\int_{-\infty}^{\infty}P(E)dE=\frac{4\pi mde}{h^3}\exp\left(-c-\frac{\zeta}{d}\right)\times\int_{-\infty}^{\infty}\frac{e^{E/d}}{\exp[(E-\zeta)/kT]+1}dE. \quad (23)$$

After some manipulation this can be put in standard form.[5] The solution, good only when $d>kT$, is

$$j=\frac{4\pi mde}{h^3}\exp\left(-c-\frac{\zeta}{d}\right)\times\left[\frac{akT\Gamma(kT/d)\Gamma(1-kT/d)}{a^{(1-kT/d)}}\right], \quad (24)$$

where $a=\exp(\zeta/kT)$. After transforming from gamma functions to factorials and recalling that[6] $\alpha!(-\alpha!)=\pi\alpha/\sin\pi\alpha$, Eq. (24) becomes

$$j=\frac{e^3F^2}{8\pi h\phi t^2((e^3F)^{\frac{1}{2}}/\phi)}\times\exp\left[\frac{-4(2m)^{\frac{1}{2}}\phi^{\frac{3}{2}}}{3\hbar eF}v((e^3F)^{\frac{1}{2}}/\phi)\right]\frac{\pi kT/d}{\sin(\pi kT/d)}. \quad (25)$$

Equation (25) is the standard Fowler-Nordheim equation in the higher temperature approximation and is limited to the region where $kT<d$.

FURTHER DISCUSSION OF THE TOTAL-ENERGY DISTRIBUTION

Figure 2 shows total-energy distribution curves for three different temperatures and typical conditions of $F=45\times10^6$ v/cm and $\phi=4.40$ ev. The normal-energy distribution is plotted for comparison. The steeply rising edge of the zero-temperature curve depicts the beginning of the Fermi sea at the Fermi energy. The exponential drop at lower energies is due to the decrease of barrier penetration probability. The zero-temperature total-energy distribution is seen to have a half-width of 0.14 ev as opposed to the 0.48 ev half-width of the normal-energy curve. The maximum value of the total-energy distribution decreases rapidly with temperature whereas the maximum value of the normal-energy distribution remains constant over a wide

[5] Gröbner und Hofreiter, *Integraltafeln* (Springer-Verlag, Berlin, 1950), Part II, p. 53, formula 9.

[6] Harold Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1956), p. 464.

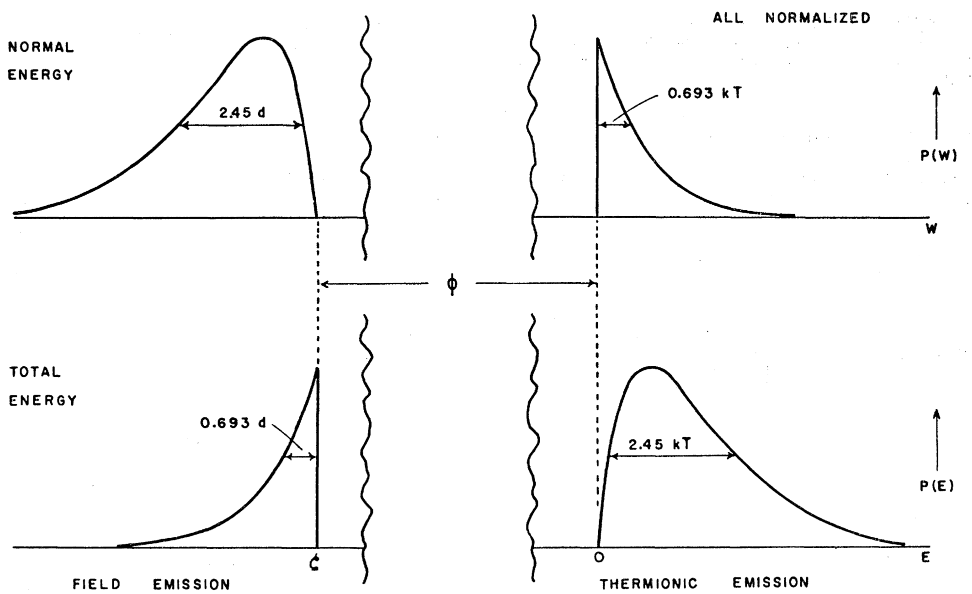

FIG. 3. Comparison of the total- and normal-energy distributions for field and thermionically emitted electrons. The diagonal mirror-image symmetry of the curves extends even to the equal range of values for d and kT.

temperature range. The 300°K curve is arbitrarily normalized.

It is easily shown that the energy at which the maximum in the total-energy distribution occurs is

$$E_{\max}=\zeta-kT\ln(d/kT-1), \qquad (26)$$

which reduces to $E_{\max}=\zeta$ when $T=0$°K.

The half-width of the total-energy distribution at zero temperature is

$$\sigma(0)=0.693d. \qquad (27)$$

An expression for the half-width at higher temperatures is too complex to be useful.

COMPARISON OF THE TOTAL-ENERGY DISTRIBUTION AND THE NORMAL-ENERGY DISTRIBUTION

1. *Total-Energy Distribution*

(a) The total-energy distribution represents the distribution in total energy of all the electrons brought to a single potential anywhere outside of the metal.

(b) It reveals details in the supply function, for example the top of the Fermi sea in field emission, but smears out details of the barrier.

(c) It permits a clear mental picture of energy distribution as a product of the well-known Fermi-Dirac distribution function and an exponentially decreasing penetration probability.

(d) Its half-width at low temperatures is only about $\frac{1}{3}$ of the half-width of the normal-energy distribution.

2. *Normal-Energy Distribution*

(a) The normal-energy distribution represents the distribution in energy associated with the normal component of velocity during the emission process. In special cases such as planar geometry the distribution is preserved far from the emitting surface.

(b) It reveals details in the potential barrier at the surface of the metal, for example the peak of the barrier in thermionic emission (see following section), but smears out details in the supply function.

TOTAL-ENERGY DISTRIBUTION FOR THERMIONIC ELECTRONS

The same supply function (16) applies to thermionic and field emission. In the zero-field approximation a transmission coefficient of zero is assumed for electrons with normal energy less than zero and unity for electrons with normal energy greater than zero. Proceeding as in the field-emission case, the total-energy distribution is seen to be

$$P(E)dE=\frac{4\pi m}{h^3}E\exp\left(\frac{-E+\zeta}{kT}\right)dE \quad \text{(thermionic).} \qquad (28)$$

Integration of Eq. (28) over all energies from zero to infinity gives the Richardson-Dushman equation.

The half-width of this distribution is found to be $2.45kT$ and the maximum in the distribution occurs at $E_{\max}=kT$.

TABULATION OF NORMAL-ENERGY DISTRIBUTION FUNCTIONS FOR FIELD AND THERMIONIC EMISSION[3,4]

For thermionic emission,

$$P(W)dW=\frac{4\pi mkT}{h^3}e^{\zeta/kT}e^{-W/kT}dW. \qquad (29)$$

For field emission in the higher temperature approxi-

TABLE I. Range of values of half-width for typical field, work function, and temperature ranges for the total-energy field emission and normal-energy thermionic emission energy distributions.

Work function range	4.50	to	6.00	ev
Field range	20×10^6	to	60×10^6	v/cm
Temperature range	1000°	to	3000°	K
Range of both d and kT	0.09	to	0.26	ev
Range of half-widths	0.062	to	0.180	ev

mation,[3]

$$P(W)dW=\frac{4\pi mkT}{h^3}\left[\exp\left(-c+\frac{W-\zeta}{kT}\right)\right]\times\ln\left[1+\exp\left(\frac{-W+\zeta}{kT}\right)\right]dW. \quad (30)$$

For field emission in the zero-temperature approximation[3],

$$P(W)dW=\frac{4\pi m}{h^3}\exp\left(-c-\frac{\zeta}{d}\right)e^{W/d}(\zeta-W)dW \quad \text{for } W<\zeta,$$
$$=0 \quad \text{for } W>\zeta. \quad (31)$$

COMPARISON OF TOTAL- AND NORMAL-ENERGY DISTRIBUTIONS FOR FIELD AND THERMIONIC EMISSION

Figure 3 shows the total- and normal-energy distributions in the zero-temperature field emission and zero-field thermionic emission approximations. A striking mirror-image symmetry is observed in the patterns when $d=kT$. It is shown in Table I that the same range of half-widths is found in the total-energy field emission and normal-energy thermionic emission distributions if values for work function, field, and temperature are considered which apply to practical working conditions. This is also true for the other pair of distributions.

The peak of the potential barrier is clearly depicted in the normal-energy thermionic distribution whereas the top of the Fermi sea is depicted in the total-energy field emission distribution.

The total-energy distribution is most useful in matters pertaining to

(1) the distribution in total energy of electrons removed in the emission process—for example, Nottingham heating and cooling and depletion of the Fermi sea;

(2) electron ballistics and electron optics problems where spherical geometry suggests emphasis of total-energy considerations; some vacuum tube noise problems[7];

(3) field-emission energy distribution measurement as discussed in the following paper.

The normal-energy distribution is most useful in matters pertaining to

(1) analysis of quantum-mechanical barrier penetration problems;

(2) electron ballistics and electron optics problems where planar geometry suggests emphasis of normal-energy considerations.[8]

In the following paper the measured total-energy distributions in the field-emission case are compared with the above theory.

ACKNOWLEDGMENTS

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[7] For example, R. W. Degrasse and G. Wade [Inst. Radio Engrs. **44**, No. 8, 1048 (1956)] have predicted a large inherent noise for a field emission cathode on the basis of the relatively wide normal-energy distribution. However, the appropriate distribution to be considered is the narrow total-energy distribution which brings the field cathode on an equal footing with the thermionic cathode.

[8] H. Shelton, Phys. Rev. **107**, 1553 (1957).