and P denotes the Panofsky ratio. Using the value $P=1.8$ and the experimental value of Sargent *et al.* for R, we obtain $\rho_{\gamma\text{exp}}=0.0089$. We see that theory and experiment are no longer in good agreement.

3. CONCLUSIONS

We have found that the nucleon structure contribution to the internal conversion coefficient, ρ_γ, for the process $\pi^-+p\rightarrow n+e+e^+$ is of the order of 2%. Present experimental data are not sufficient to detect this contribution. Indeed the most reliable value of the Panofsky ratio to date reveals a discrepancy of 25% between theoretical and experimental values of ρ_γ. This situation may be removed through the collection of more events, and indeed may be regarded as the result of a statistical fluctuation.

The stopping of high-energy π^- mesons with subsequent formation of pairs[10] would seem to be of more theoretical interest with regard to effects of nucleon structure. However, the application of the photomeson dispersion relations to this problem is not so straightforward.

ACKNOWLEDGMENTS

We would like to thank Leona Marshall for several helpful discussions. We also wish to thank Professor R. Oppenheimer and the Institute for Advanced Study for their hospitality. One of us (J.G.T.) would like to thank the Commonwealth Fund of New York for a grant.

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Perturbation Theory for an Infinite Medium of Fermions*

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The ground-state energy and low-energy excitations of single-particle character of an infinite medium of fermions are discussed with the aid of time-dependent Green's functions, which are convenient generalizations of the exact particle correlation functions in the ground state of N particles. The power series development for the one- and two-particle functions, under the restriction to two-body forces, is derived and described by means of Feynman diagrams. The derivation of the linked-cluster expansion for the energy then follows immediately. The equivalence to previous versions is established. The one-particle function is examined in particular detail, and it is shown that the poles of its space-time Fourier transform studied as a function of the energy variable, for fixed momentum, determined the $(N+1)$-particle and $(N-1)$-particle excited states which have single-particle character. For a reasonable assumption about the full spectrum of excited states, it is found that for the interacting system, single-particle excitations with a real energy occur only at the Fermi momentum. It is pointed out that the corresponding energy, termed the perturbed Fermi energy, equals the binding energy per particle in the ground state of N particles for a saturating system at equilibrium density.

It is shown, finally, that the entire structure of the theory may be carried over to the case of finite temperature, requiring only a redefinition of the Green's functions. The analogy is constructed from a discussion of the internal energy.

I. INTRODUCTION

THE important advances made recently in the understanding of many-body problems in quantum mechanics has resulted in the first instance from the realization of a suitable form of perturbation theory, the so-called "linked-cluster expansion," and in the second instance from the recognition that both for Fermi and Bose statistics certain subseries of terms, which can (and must) be summed, are most relevant to the extreme low- or high-density approximations.[1] By this time there exist several distinct derivations of the linked cluster property,[2] as well as fundamental applications to the properties of nuclear matter,[1] He_3,[3] the electron gas,[4] the hard-sphere Bose gas,[5] and the fundamental problem of statistical mechanics[6]—to name but a few.

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[6] R. P. Feynman, Phys. Rev. **91**, 1291 (1953); T. Matsubara, Progr. Theoret. Phys. Japan **14**, 351 (1955); K. Watson, Phys. Rev. **103**, 489 (1956); Ezawa, Tomozawa, and Umezawa, Nuovo cimento **5**, 810 (1957); E. Montroll and J. C. Ward, Phys. Fluids **1**, 55 (1958). C. Bloch and C. De Dominics, Nuclear Phys

With this paper we initiate yet another study of the same class of problems. The formal development of the subject has been marked by an increasing reliance on the method of second quantization, not at all surprising in view of the fact that we are dealing from one standpoint with a class of particularly simple field theories. What has seemed surprising to the authors is that the detailed treatments so far published have for the most part been either special to the problem at hand or else have taken over general techniques of quantum field theory which date from the "early part" of the last decade of profound development of methodology. It appears to us completely appropriate therefore to introduce current history into this class of problems by applying to it that method which has proved most satisfactory in the more difficult instance of the field theory of elementary particles both for the elucidation of general properties and also for many applications.

The method in question makes use of Green's functions or propagators which are suitably defined matrix elements of Heisenberg field operators taken between exact eigenstates of the system.[7] The utility of such constructs is at least twofold: If one studies the equation of motion of the "one-particle" Green's function, for instance, one finds that it describes only the interactions which the particle in question undergoes with other particles of the medium; in short, only connected or linked diagrams occur. Again, the "two-particle" Green's function describes either the interactions of each particle with the medium or of the two particles with each other. The second point is that these functions are most closely related to the computation of observables: From the one-particle Green's function we may compute all one-particle observables of the system such as the kinetic energy; from the two-particle Green's function we can compute that part of the potential energy due to two-body forces, etc. It may be inferred from these remarks that the N-particle Green's function is nothing more than a convenient generalization of the N-particle correlation function, chosen so as to facilitate a systematic construction of its equation of motion and of that equation's solution.

The present paper illustrates this general point of view by detailed study of a particular example, the ground state and restricted excited states of an infinite medium of fermions. We are dealing with the problem near zero temperature. The initial emphasis is also on formal questions of the structure of the perturbation series. In the latter portions of the paper, however, the discussion is carried beyond perturbation theory and in a final section we point out how our results may be generalized to finite temperature.

The detailed contents of this paper may now be described as follows: In Sec. II we define the one- and two-particle Green's functions appropriate to the study of the zero-temperature problem, and several formulas are given for the computation of the ground-state energy by their means. The analogy to the field theory of elementary particles is already sufficient to imply the linked-cluster expansion. The equations of motion of the various Green's functions are derived in Sec. III, and it is seen that they are coupled in virtue of the nonlinearity of the fundamental field equation. In particular it is shown that all properties of the two-particle Green's function required for the computation of the energy may be inferred from the structure of the one-particle Green's function through its dependence on the self-energy operator, analogous to the operator so designated in quantum electrodynamics,[8] for instance. The power series for this operator is derived and interpreted in terms of Feynman diagrams.[9]

In Sec. IV, we return to the problem of the ground-state energy. This fundamental quantity is now related to the self-energy operator, the follow-through on the diagrammatic analysis of the previous section constituting an explicit proof of the linked-cluster expansion in power series form. The equivalence to previous time-dependent expressions such as that of Goldstone[2] is demonstrated explicitly. We next turn to the time-independent form of the theory. The expression of the self-energy operator and of the energy as a series of terms of the type occurring in conventional perturbation theory is given in Sec. V and here interpreted by means of time-ordered diagrams. In carrying out this transformation, an explicit representation of the one-particle Green's function for a Fermi gas introduced.

The investigation shifts to more general ground in Sec. VI with a discussion of those properties of the exact one-particle Green's function which can be inferred from invariance and plausibility arguments alone. The altered structure of the self-energy operator when expressed in terms of the exact propagator is described. It is indicated that it has a branch point at the perturbed Fermi energy, defined as the (negative) binding energy of the $(N+1)$st particle added to the ground state of N particles, at constant volume. In Sec. VI an eigenvalue equation for this excitation energy is derived from the requirement that it be a pole of the Fourier transform of the one-particle Green's function. It is remarked that for large systems the Fermi energy equals the binding energy per particle. A proof is then outlined, with some details left to an appendix, that the binding energy per particle does indeed satisfy the given eigenvalue problem. (This is the content of a theorem due to Hugenholtz and Van

7, 459 (1958); P. C. Martin and J. Schwinger, Bull. Am. Phys. Soc. Ser. II, 3, 202 (1958).

[7] See P. C. Martin and J. Schwinger, reference 6, where the application to statistical mechanics has been carried out. Similar ideas have been studied by N. Fukuda and C. Zemach (private communications).

[8] F. J. Dyson, Phys. Rev. 75, 1736 (1949).

[9] R. P. Feynman, Phys. Rev. 76, 749, 769 (1949).

Hove.[10]) It is moreover the only real solution. We then discuss the essential limitation on the concept of real single-particle excitation energies implied by this result. There remains the possibility of defining metastable single-particle states under suitable conditions.

The final section contains an account of how the theory may be generalized to finite temperatures. Using the example of the internal energy, defined with respect to the grand canonical ensemble, it is proved that it is given by a power series formally identical with that for the ground-state energy, requiring only the insertion of a modified temperature-dependent one-particle Green's function. Finally, the form of this Green's function for the noninteracting case is derived.

II. FORMULATION OF THE MANY-BODY PROBLEM IN TERM'S OF GREEN'S FUNCTIONS

We are interested in the eigenfunctions and eigenvalues of the Hamiltonian

$$H=\int d^3x :\nabla\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}):+\tfrac{1}{2}\lambda\int d^3x'd^3x \times :\psi^\dagger(\mathbf{x}')\psi^\dagger(\mathbf{x})v(|\mathbf{x}-\mathbf{x}'|)\psi(\mathbf{x})\psi(\mathbf{x}'):=H_0+\lambda H_1, \quad (1)$$

describing a system of identical particles interacting pairwise through the potential energy function v. We choose units in which $\hbar=2m=1$. The colons indicate that the normal product of the operators is to be taken.[11] This means that in any such sequence of operators all referring to the same time, destruction operators are to be brought to the right of creation operators. In this or any subsequent rearrangement, the anticommutation relations of Eq. (2) below are to be used, except that δ-function contributions must be omitted. For the sake of definiteness, we shall throughout this paper study the case of Fermi-Dirac statistics neglecting spin. The $\psi(\mathbf{x})$, $\psi^\dagger(\mathbf{x})$ are then operators satisfying the algebra

$$\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')+\psi^\dagger(\mathbf{x}')\psi(\mathbf{x})=\{\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\}=\delta(\mathbf{x}-\mathbf{x}'), \quad \{\psi(\mathbf{x}),\psi(\mathbf{x}')\}=\{\psi^\dagger(\mathbf{x}),\psi^\dagger(\mathbf{x}')\}=0. \quad (2)$$

The Hamiltonian (1) conserves the number of particles, here represented by the operator

$$N=\int d^3x :\psi^\dagger(\mathbf{x})\psi(\mathbf{x}):, \quad (3)$$

whose eigenvalues will label the states of interest. In addition we shall restrict ourselves throughout this discussion to situations of complete translational invariance such as the infinite nuclear medium or the electron gas and shall therefore employ the total momentum operator

$$\mathbf{P}=\int d^3x :\psi^\dagger(\mathbf{x})(-i\nabla)\psi(\mathbf{x}): \quad (4)$$

for a further labeling of states.

We first derive an expression for the energy of the ground state of N particles, which we take to have zero total momentum and therefore to be designated $|N,0\rangle$. This will serve to introduce the objects of fundamental interest in the present investigation. Such an expression is

$$\begin{aligned}E(N,0)&=\langle N,0|H|N,0\rangle\\ &=\int d^3x\langle N,0|:\psi^\dagger(\mathbf{x})(-\nabla^2)\psi(\mathbf{x}):|N,0\rangle\\ &\quad+\tfrac{1}{2}\lambda\int d^3x\,d^3x'\,v(|\mathbf{x}-\mathbf{x}'|)\\ &\quad\times\langle N,0|:\psi^\dagger(\mathbf{x}')\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x}'):|N,0\rangle\\ &=-i\int d^3x\lim_{\mathbf{x}'\to\mathbf{x},\ t'\to t+}\nabla^2G(\mathbf{x},t;\mathbf{x}',t')\\ &\quad+(-i)^2\tfrac{1}{2}\lambda\int d^3x\,d^3x'\lim_{t''\to t+}v(|\mathbf{x}-\mathbf{x}'|)\\ &\quad\times G(\mathbf{x},t,\mathbf{x}',t;\mathbf{x},t'',\mathbf{x}',t''), \quad (5)\end{aligned}$$

where the new functions introduced, the one- and two-particle functions G_1 and G_{12}, are defined, respectively, as

$$G(x;x')=i\langle N,0|T(\psi(x)\psi^\dagger(x'))|N,0\rangle, \quad (6)$$

$$G(x_1,x_2;x_1',x_2')=i^2\langle N,0|T(\psi(x_1)\psi(x_2)\psi^\dagger(x_2')\psi^\dagger(x_1'))|N,0\rangle. \quad (7)$$

In these definitions x now represents a point in space and time, and T is the chronological ordering symbol of Wick[11] which directs that the operators be arranged in increasing time from the right and an over-all sign factor affixed according to the parity of the permutation from the standard order given. The generalization of the definitions (6) and (7) to any number of particles is obvious.

Though the introduction of the functions G_1 and G_{12} into Eq. (5) may appear forced, such should not be the reaction of those readers who are conversant with current practice in the quantum theory of fields, for they will recognize that Eqs. (6) and (7) define Green's functions of interacting fields whose properties have been widely studied. For example, G_1 provides a convenient basis for the study of the propagation of a single "particle" through the medium. As we shall derive anew below, such a propagation can be described in terms of Feynman diagrams in which all interactions

[10] N. M. Hugenholtz and L. van Hove, Physica **24**, 363 (1958).
[11] G. C. Wick, Phys. Rev. **80**, 268 (1950).

occurring are directly connected with the given particle. Also for G_{12} all interactions are linked with one or both of the "particles" which occur in this function. *It follows that Eq. (5) can be interpreted as a closed form of the linked-cluster expansion.*

As a matter of fact, Eq. (5) differs in an essential way from any recent versions of the theory of the many-body problem, for it contains the actual kinetic energy of the system. This has certain advantages which we plan to exploit in a future publication. For the present, our goal is to make contact with previous forms of the theory. For this purpose we require a formula in which the kinetic energy is evaluated in the unperturbed Fermi-gas state. To this end, we employ a familiar device based on the variational principle. Upon differentiation of Eq. (5) with respect to the coupling strength λ, remembering Eq. (1), we find

$$dE(N,0)/d\lambda=\langle N,0|H_1|N,0\rangle, \tag{8}$$

owing to the stationary character of the energy upon variation of the wave function. Upon integration, we now have

$$\begin{aligned}E(N,0)&=E_0(N,0)+\int_0^\lambda d\lambda'\langle N,0|H_1|N,0\rangle\\&=E_0(N,0)\\&\quad+(-i)^2\tfrac{1}{2}\int_0^\lambda d\lambda'\int d^3x\,d^3x'\lim_{t''\to t+}v(|\mathbf{x}-\mathbf{x}'|)\\&\quad\times G(\mathbf{x},t,\mathbf{x}',t;\,\mathbf{x},t'',\mathbf{x}',t''),\end{aligned} \tag{9}$$

where $E_0(N,0)$ is the kinetic energy of the Fermi gas, and it is understood that the integrand in the second term is taken at λ'.

For the sake of completeness, we note here that

$$E_0(N,0)=\sum_{\mathbf{p}}p^2=\frac{\Omega}{(2\pi)^3}\int_0^{p_F}4\pi p^2dp\;p^2, \tag{10}$$

where Ω is the volume of the system and p_F the Fermi momentum, is defined by the equation

$$N=\frac{\Omega}{(2\pi)^3}\int_0^{p_F}4\pi p^2dp. \tag{11}$$

For the particle density ρ, we therefore have

$$\rho=(N/\Omega)=(p_F{}^3/6\pi^2). \tag{12}$$

Both Eqs. (5) and (9) now suggest that further progress be predicated upon a study of the functions G_1 and G_{12}.

III. STRUCTURE OF THE ONE-PARTICLE GREEN'S FUNCTIONS

We consider initially the one-particle function $G(x;x')$. From Eqs. (1) and (2), there follows the field equation

$$\begin{aligned}i\partial_t\psi(x)&=[\psi(x),H]\\&=p^2\psi(x)+\lambda\int d^3x''\,v(|\mathbf{x}''-\mathbf{x}|)\\&\quad\times:\psi^\dagger(\mathbf{x}'',t)\psi(\mathbf{x}'',t)\psi(x):,\end{aligned} \tag{13}$$

where $p^2=-\nabla^2$. Upon this equation we carry out the operation indicated in the definition Eq. (6) and obtain the equation[12]

$$\begin{aligned}(-i\partial_t+p^2)G(x;x')+i\lambda\int d^3x''\,v(|\mathbf{x}''-\mathbf{x}|)\\\times G(\mathbf{x}'',t,x;\,\mathbf{x}'',t,x')=\delta^4(x-x'),\end{aligned} \tag{14}$$

with the four-dimensional unit source on the right-hand side. The latter enters because of the relation

$$T(\partial_t\psi(x)\psi^\dagger(x'))=\partial_tT(\psi(x)\psi^\dagger(x'))-\delta^4(x-x'), \tag{15}$$

which can be verified by applying the operation

$$\lim_{\epsilon\to0}\int_{t'-\epsilon}^{t'+\epsilon}dt \tag{16}$$

to (15), noting that the left-hand side contributes nothing, and making use of the anticommutation relations (2).

Equation (14) can be converted to an integro-differential equation for $G(x;x')$ by defining an integral operator $M(x;x')$ by means of the relation

$$\begin{aligned}-\int d^4x''\,M(x;x'')G(x'';x')\equiv i\lambda\int d^3x''\,v(|\mathbf{x}''-\mathbf{x}|)\\\times G(\mathbf{x}'',t,x;\,\mathbf{x}'',t,x').\end{aligned} \tag{17}$$

At the same time the right-hand side bears an obvious kinship to the energy expression, Eq. (9).

Through various techniques are available for the construction of the operator $M(x;x')$, the "irreducible self-energy operator," we shall first describe a method which suffices to express M as a power series in the coupling constant λ. Proceeding as in the derivation

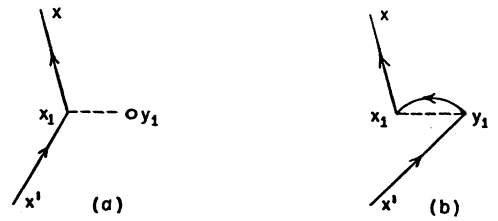

FIG. 1. Lowest order contributions to the single-particle Green's function.

[12] This method is due to J. Schwinger, Proc. Natl. Acad. Sci. U. S.37, 452, 455 (1951).

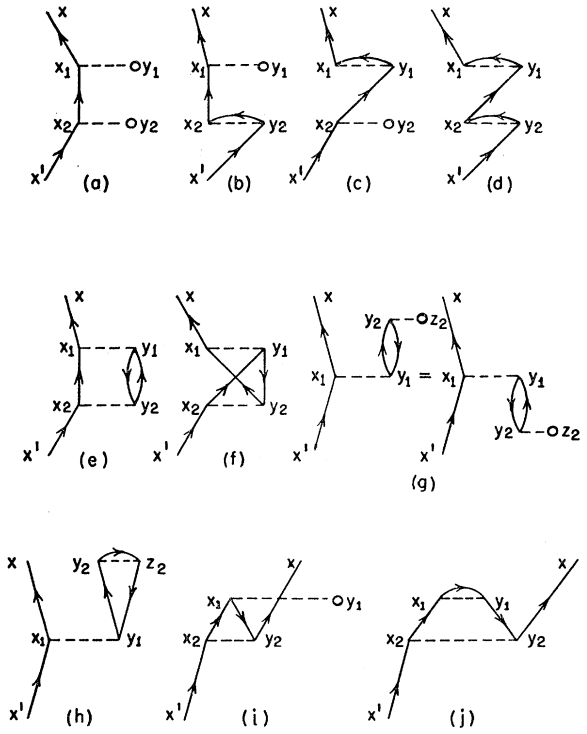

FIG. 2. Complete second-order contribution to the single-particle propagator.

of Eq. (14), we obtain the equation

$$[-i(\partial/\partial t_1)+p_1^2][-i(\partial/\partial t_2)+p_2^2]G_{12}(x_1,x_2;x_1',x_2')$$
$$=(-i)^2\lambda^2\int d^3y_1 d^3y_2\, v(|\mathbf{y}_1-\mathbf{x}_1|)v(|\mathbf{y}_2-\mathbf{x}_2|)$$
$$\times G_{1234}(x_1,y_1,x_2,y_2;x_1',y_1,x_2',y_2)$$
$$-i\delta^4(x_1-x_1')\lambda\int d^3y_2\, v(|\mathbf{y}_2-\mathbf{x}_2|)G_{12}(x_2,y_2;x_2',y_2)$$
$$+i\delta^4(x_1-x_2')\lambda\int d^3y_2\, v(|\mathbf{y}_2-\mathbf{x}_2|)G_{12}(x_2,y_2;x_1',y_2)$$
$$-i\delta^4(x_2-x_2')\lambda\int d^3y_1\, v(|\mathbf{y}_1-\mathbf{x}_1|)G_{12}(x_1,y_1;x_1',y_1)$$
$$+i\delta^4(x_2-x_1')\lambda\int d^3y_1\, v(|\mathbf{y}_1-\mathbf{x}_1|)G_{12}(x_1,y_1;x_2',y_1)$$
$$+i\lambda\int d^3y_1\,\delta^4(x_2-y_1)v(|\mathbf{y}_1-\mathbf{x}_1|)G_{12}(x_1,y_1;x_1',x_2')$$
$$+\delta^4(x_1-x_1')\delta^4(x_2-x_2')-\delta^4(x_1-x_2')\delta^4(x_2-x_1'). \quad (18)$$

Throughout, it is understood that whenever two points are connected by the interaction v they occur at the same time.

The structure of Eq. (17) will be sufficiently clear if we obtain the right-hand side to second order in λ. We therefore require the solution of Eq. (18) to first order in λ, which we obtain by neglecting the first term on the right-hand side of (18) and converting the differential equation to an integral equation by means of the noninteracting single-particle Green's function satisfying

$$(-i\partial_t+p^2)G^{(0)}(x;x')=\delta^4(x-x'). \quad (19)$$

(Boundary conditions will be studied below in Sec. V.) We then replace G_{12} in the linear terms by its zeroth approximation

$$G(x_1,x_2;x_1',x_2')\approx G^{(0)}(x_1;x_1')G^{(0)}(x_2;x_2')$$
$$-G^{(0)}(x_1;x_2')G^{(0)}(x_2;x_1'). \quad (20)$$

When we insert the result into (17), we obtain two terms linear in λ and ten quadratic in λ. By extension of the techniques described, we could generate the higher terms in the series.

We now attempt to survey the nature of the resulting expressions. For this purpose it is first convenient to obtain a formal expansion for $G(x;x')$ based on Eqs. (15) and (18). If we insert (17) into (14) and convert to an integral equation, we obtain

$$G(x;x')=G^{(0)}(x;x')+\int d^4x_1\, d^4x_2\, G^{(0)}(x;x_1)$$
$$\times M(x_1;x_2)G(x_2;x'), \quad (21)$$

or symbolically

$$G=G^{(0)}+G^{(0)}MG$$
$$=G^{(0)}+G^{(0)}MG^{(0)}+G^{(0)}MG^{(0)}MG^{(0)}+\cdots. \quad (22)$$

By comparison of the series (22) with the power series expression which corresponds to the right-hand side of (17), it is a straightforward matter to identify the structure of $M(x;x')$.

As usual, the results are most simply stated by the introduction of Feyman diagrams accompanied by the statement of rules for writing down the associated contributions. We proceed by induction. Writing M as a sum,

$$M=M^{(1)}+M^{(2)}+\cdots, \quad (23)$$

the first-order contribution, $G^{(0)}M^{(1)}G^{(0)}$, to Eq. (22) is represented by the two diagrams of Fig. 1, and as follows from Eqs. (17) and (18) by the expressions

$$G^{(0)}M_a{}^{(1)}G^{(0)}=-i\lambda\int d^4x_1\, d^3y_1\, v(|\mathbf{x}_1-\mathbf{y}_1|)$$
$$\times[G^{(0)}(x;x_1)G^{(0)}(x_1;x')]G^{(0)}(y_1;y_1), \quad (24)$$

$$G^{(0)}M_b{}^{(1)}G^{(0)}=i\lambda\int d^4x_1\, d^3y_1\, v(|\mathbf{x}_1-\mathbf{y}_1|)$$
$$\times[G^{(0)}(x;x_1)G^{(0)}(x_1;y_1)G^{(0)}(y_1;x')]. \quad (25)$$

In (24), (25), and henceforth in all subsequent expressions, it is understood that whenever a single-particle Green's function contains equal temporal arguments it represents the matrix element of the number density

$$G^{(0)}(y_1; y_1) = -i\langle N,0|\psi^\dagger(y_1)\psi(y_1)|N,0\rangle. \quad (26)$$

That this is correct stems directly from the occurrence of the normal product in the field equation (13).

In Fig. 2 are represented the ten diagrams quadratic in λ whose sum is $G^{(0)}M^{(2)}G^{(0)}+G^{(0)}M^{(1)}G^{(0)}M^{(1)}G^{(0)}$. The latter may be identified directly with diagrams (a)–(d). Of the remaining terms we record a few typical contributions:

$$G^{(0)}M_e{}^{(2)}G^{(0)} = -(i\lambda)^2\int d^4x_1 d^3y_1 d^4x_2 d^3y_2 \times v(|\mathbf{x}_1-\mathbf{y}_1|)v(|\mathbf{x}_2-\mathbf{y}_2|) \times[G^{(0)}(x; x_1)G^{(0)}(x_1; x_2)G^{(0)}(x_2; x')] \times[G^{(0)}(y_1; y_2)G^{(0)}(y_2; y_1)]. \quad (27)$$

$$G^{(0)}M_f{}^{(2)}G^{(0)} = (i\lambda)^2\int d^4x_1 d^3y_1 d^4x_2 d^3y_2 \times v(|\mathbf{x}_1-\mathbf{y}_1|)v(|\mathbf{x}_2-\mathbf{y}_2|) \times[G^{(0)}(x; x_1)G^{(0)}(x_1; y_2)G^{(0)}(y_2; y_1) \times G^{(0)}(y_1; x_2)G^{(0)}(x_2,x')]. \quad (28)$$

From these equations and the corresponding ones for the remaining diagrams, the general rules for $G^{(0)}M^{(n)}G^{(0)}$ are inferred without difficulty. Draw all topologically distinct diagrams consisting of the following elements: (i) A continuous open particle line which may reverse direction several times, is directed from the point x' to the point x, and traverses m interaction points, where m may take any of the values $m=1, 2\cdots 2n$, as illustrated in Fig. 2. (ii) Some number of closed loops including the degenerate one represented by Eq. (24). The sense of traversal, clockwise or counterclockwise is to be indicated. In most cases, for example, as in Figs. 3 (b), (c), the reversal of this sense yields a distinct contribution. The rule for this will be given below. (iii) A total of n horizontal (instantaneous) undirected dashed lines representing the interaction. (iv) Only connected diagrams are to be drawn. This is essentially the linked-cluster theorem. From the totality of such diagrams we then discard those that can be understood as the interates of lower order diagrams (the "reducible" ones) in the sense of Eq. (23). Thus in Fig. (3), where selected third-order diagrams are given, (a) is discarded and (b)–(d) retained.

To record the contribution of one of the remaining diagrams, we employ the following prescription: (i)

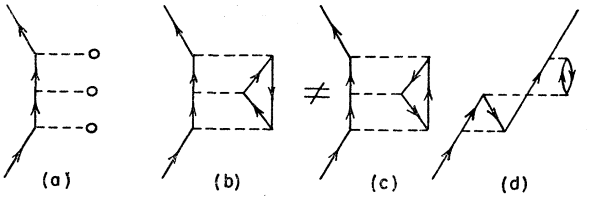

FIG. 3. Selected third-order contributions to the one-particle Green's function. Figures (b) and (c) here represent distinct contributions.

For every interaction connecting the points x_i, y_i there is a factor $(i\lambda)\int d^4x_i d^3y_i\, v(|\mathbf{x}_i-\mathbf{y}_i|)$. (ii) For the propagation of a particle from interactions point x_{i+1} to point x_i, there is a factor $G^{(0)}(x_i; x_{i+1})$. There are corresponding factors for the initial and final propagation of the particle. (iii) There is an over-all sign factor $(-1)^l$ where l is the number of closed loops.

Except for rule (iii), the others are obvious from the examples already given. The rule in question is most easily seen by consideration of an alternative mode of constructing the totality of diagrams (reducible and irreducible) in a given order. We illustrate this by consideration of Fig. (2). Let us refer to diagram (a) as the standard one. As a manner of speaking, we then say that all the remaining diagrams can be obtained from the standard one by a suitable number of exchanges. The parity of the latter determines the sign of a diagram relative to the standard one. Thus Fig. 2(b) results from the exchange $x_2\leftrightarrow y_2$ and carries a relative minus sign, Fig. 2 (e) from the exchange $y_1\leftrightarrow y_2$, and Fig. 2 (g) from the exchange $x_2\leftrightarrow z_2$, in Fig. 2 (e), i.e., in this case the replacement of x_2 by an arbitrary contemporaneous point of the "medium." Each such exchange, which brings in an additional minus sign, alters the number of closed loops by unity. Rule (iii) now follows if we note that the standard diagram carries a minus sign for each of its simple closed loops, as follows directly from Eq. (25). We may also remark that this method accounts for the nonidentity of contributions (b) and (c) of Fig. 3. They result from distinct permutations of the variables y_1, y_2, y_3, and all distinct permutations giving rise to linked diagrams are to be included.

We have thus arrived at a complete account of the power series for the integral operator $M(x; x')$. For many problems it is necessary to carry out at least a partial summation of the power series in question, the particular summation depending on the problem at hand. In fact one can also give a formally exact expression for this operator as has been done also in the more complicated field-theoretical cases. All these matters will be detailed at the apposite juncture. For the present we shall exploit the power series method already developed. We first return to the problem of computing the ground-state energy.

IV. GROUND-STATE ENERGY; LINKED-CLUSTER EXPANSION

It would be useful for future applications to carry out an analysis of the two-particle Green's function without the restriction on the coordinates implied in Eq. (17), For the present purpose, however, the comparison of (17) with (9) yields the immediate prescription

$$E(N,0)=E_0(N,0)-\tfrac{1}{2}i\int_0^\lambda \frac{d\lambda'}{\lambda'}\int d^3x d^4x'' \times M(x;x'')G(x'';x). \quad (29)$$

An alternate form, more appropriate when the direct utilization of the power series is contemplated, is obtained if we define the reducible self-energy operator M_R by the equation

$$M_R G^{(0)}\equiv MG =\{M+MG^{(0)}M+MG^{(0)}MG^{(0)}M+\cdots\}G^{(0)}, \quad (30)$$

as follows from Eq. (22). We thus find

$$E(N,0)=E_0(N,0)-\tfrac{1}{2}i\int_0^\lambda \frac{d\lambda'}{\lambda'}\int d^3x d^4x' \times M_R(x;x';\lambda')G^{(0)}(x';x). \quad (31)$$

In conjunction with the results of the previous section, Eq. (31) immediately settles all questions relating to the enumeration of independent contributions to the ground-state energy. It also constitutes an explicit proof of the linked-cluster expansion.

Though Eq. (31) may be used as is and for certain applications, as to the electron gas, is most useful in that form, it is a simple matter to show the equivalence of (31) to the formulas employed in Goldstone's treatment.[2]

The proof (for a summary see the last paragraph of this section) depends on a special property of the type of diagram which contributes to (31), namely a collection of closed loops connected by instantaneous interactions. Let us write

$$M_R(\lambda)G^{(0)}=\sum_{n=1}^{\infty}\lambda^n C^{(n)}. \quad (32)$$

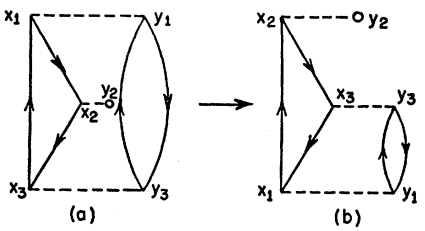

FIG. 4. Third-order contributions to the energy which indicate the redundancy of those contributions to Eq. (33) which differ only by a relabeling of coordinates.

The term of order n in (31) may therefore be written

$$\lambda^n C^{(n)}=-\tfrac{1}{2}i(\tau n)^{-1}\lambda^n\int d^4x_1 d^3y_1\cdots d^4x_n d^3y_n \times C^{(n)}(x_1\cdots x_n;y_1\cdots y_n). \quad (33)$$

Here, besides performing the λ integration, we have introduced an additional time integration for the sake of symmetry and divided by a "large" time interval τ as compensation. Again the coordinates with the same subscript are connected by the interaction.

We now divide the total n-dimensional region defined by the time integrations into n subregions obtained by choosing in turn each of the n times $t_1\cdots t_n$ to be the latest in any otherwise arbitrary time-ordered sequence, i.e., we divide the total of n time-ordered sequences into n classes. If the selected time integration is then reserved for last, the contribution to (33) of the particular class is independent of this coordinate, as follows from translational invariance, the integration merely cancelling the factor τ^{-1}. We may, in fact fix this last point and take it as the origin of time.

We now assert that each of the n classes yields *in toto* the same contribution to (33). For example, we compare class t_1 (t_1 latest) with class t_2. The contribution of any diagram of the first class corresponds in the sense illustrated in Fig. 4 to the contribution of some, in general different, diagram of the second class. By a simple relabelling of variables, however, the latter is identical with a contribution to class t_1. Moreover, the correspondence is both single-valued and complete. We may consequently choose the contribution of class t_1 and discard the factor n^{-1} in Eq. (33).

We summarize this result as follows: To find the nth order contribution to Eq. (31), we draw the same totality of graphs as contribute to $M_R(x;x'')G^{(0)}(x'';x)$, except that topologically equivalent distortions are now restricted to those for which all other coordinates chronologically precede x (which we identify with x_1 and take $t_1=0$). The time integrations over these coordinates extend from $-\infty$ to the origin. Corresponding to these restrictions, we define an operator $\mathfrak{M}_R(x;x'')$. It follows that the ground-state energy is given by the formula.

$$E(N,0)=E_0(N,0)-\tfrac{1}{2}i\int d^3x d^3x'\int_{-\infty}^{0}dt' \mathfrak{M}_R(\mathbf{x},0;x')G^{(0)}(x';x), \quad (34)$$

the result sought.

V. TIME-INDEPENDENT FORM OF THE THEORY

The discussion of this section will concern itself with the conversion of Eq. (34) and its predecessors to summations of the type which occur in time-independent perturbation theory. Toward this end we must finally

state explicitly the form of $G^{(0)}(x;x')$ which is required in order that we be able to perform the time integrations.

Introducing the Fourier transform

$$G^{(0)}(x;x')=\frac{1}{\Omega}\frac{1}{2\pi}\sum_{\mathbf{p}}\int_{-\infty}^{\infty}dp_0 \times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip_0(t-t')]\times G^{(0)}(p,p_0) \quad (35)$$

into the differential equation

$$(-i\partial_t+p^2)G^{(0)}(x;x')=\delta^4(x-x'), \quad (36)$$

we find that

$$G^{(0)}(p,p_0)=[-p_0+p^2]^{-1}. \quad (37)$$

We tentatively choose as path of integration for variable p_0 in (35) a contour which remains slightly below the real axis for $p_0<p_F^2$, crosses at this point, and remains above for $p_0>p_F^2$. Alternatively we integrate along the real axis and replace (37) by the formula

$$G^{(0)}(p,p_0)=[-p_0+p^2-i\eta\epsilon(p^2-p_F^2)]^{-1}, \quad (38)$$

with η arbitrarily small and positive and

$$\epsilon(p^2-p_F^2)=+1, \quad p^2>p_F^2 \qquad =-1, \quad p^2<p_F^2. \quad (39)$$

By elementary contour integration, we then find that[13]

$$G^{(0)}(x;x') =-\frac{i}{\Omega}\sum_{|\mathbf{p}|>p_F}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip^2(t-t')], \quad t>t'; \quad (40)$$

$$=-\frac{i}{\Omega}\sum_{|\mathbf{p}|<p_F}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip^2(t-t')], \quad t<t'.$$

That our choice (38) is correct is easily verified from the definition

$$G^{(0)}(x;x')=i\langle N,0|T(\psi(x)\psi^\dagger(x'))|N,0\rangle|_0. \quad (41)$$

For $t>t'$, we can write, for example (all states are now Fermi-gas states),[14]

$$G^{(0)}(x;x')=i\sum_{\mathbf{p}}\langle N,0|\psi(x)|N+1,\mathbf{p}\rangle \times\langle N+1,\mathbf{p}|\psi^\dagger(x')|N,0\rangle. \quad (42)$$

Equation (42) is evaluated by the introduction of the expansion

$$\psi(x)=\Omega^{-\frac{1}{2}}\sum_{\mathbf{p}}a_{\mathbf{p}}\exp[i\mathbf{p}\cdot\mathbf{x}-ip^2t], \quad (43)$$

and its Hermitian conjugate, both valid in the absence of interaction. Since the state $|N,0\rangle_0$ is that of N particles filling the Fermi sphere in momentum space, the only nonvanishing contributions to (42) occur when the state $|N+1,\mathbf{p}\rangle$ contains one additional particle of momentum $\mathbf{p}$, $|\mathbf{p}|>p_F$. This leads immediately to the first of Eqs. (40). For $t<t'$ a similar treatment results in the second of Eqs. (40).

As a first application we compute the Fourier transform

$$M(\mathbf{p},p_0)=\int d^4x \times\exp[-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+ip_0(t-t')]M(x;x') \quad (44)$$

of the irreducible self-energy operator. Here we consider a typical diagram of order n which will contain [including the integration in (44)] (i) $2n-1$ factors of $G^{(0)}$ for each of which we substitute (40); (ii) n interactions $v(|\mathbf{x}_i-\mathbf{y}_i|)$ for each of which we substitute the Fourier resolution

$$v(|\mathbf{x}|)=\Omega^{-1}\sum_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}v(|\mathbf{q}|); \quad (45)$$

(iii) $2n$ vertices of interaction, $2n-1$ spatial integrations, and $n-1$ temporal integrations. The spatial integrations insure momentum conservation at each vertex and are done once for all. We then divide the total time integration into $n!$ contributions, one for each relative ordering of the n times.

We content ourselves with a description of the contributions of the individual time-ordered parts, the rules to be given following easily from the ingredients described above. To each original "Feynman" diagram we draw $n!$ time-ordered diagrams which are merely different distortions of the same diagram, as required to achieve the given ordering. [The two diagrams resulting from Fig. 2(e) are shown in Fig. 5. The wavy lines are inserted to help keep track of the incident and emergent energy and momentum.] Momenta are assigned at each vertex with the help of momentum conservation. There will be n free momenta after this is done. It is important to remark that the directional structure of the original space-time diagram is maintained: In virtue of Eq. (40), an arrow directed upwards now represents a particle with momentum $\mathbf{p}$, $|\mathbf{p}|>p_F$ one directed downward a particle within the Fermi sphere, momentum $<p_F$ (actually a particle missing from the sphere, i.e., a "hole").

For the sake of a simple survey formula we leave the momentum conservation in the form of δ functions, and

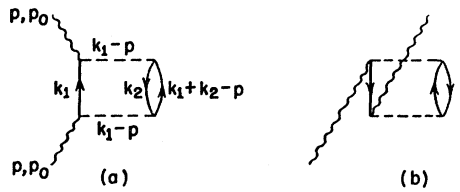

FIG. 5. Second-order contributions to the Fourier transform of the self-energy operator which arise from the same Feynman diagram.

[13] This propagator has also been discussed by J. Hubbard, reference 2.

[14] There is of course a whole spectrum of excited-state energies for given momentum $\mathbf{p}$ and more properly this should be indicated in Eq. (42). However, only the one involving single-particle excitation contributes to the unperturbed sum.

label the interaction momenta $q_1 \cdots q_n$ and the particle momenta $p_1 \cdots p_{2n-1}$. The general contribution of a single time-ordered diagram of order n may be written

$$\lambda^n(-1)^{l+m+1}\frac{1}{\Omega^n}\sum_{\mathbf{q}_1\cdots\mathbf{q}_n}\sum_{\mathbf{p}_1\cdots\mathbf{p}_{2n-1}}\prod_{j=1}^{2n-1}\delta_j \times v(|\mathbf{q}_1|)\cdots v(|q_n|)[D_{n-1}\cdots D_1]^{-1}. \quad (46)$$

Here l is still the number of closed loops and m is the number of holes in the Fermi sea, the residual factor of minus one arising from the definition, Eq. (17), of the operator M; δ_j is the δ function for conservation of momentum at the jth vertex. To complete the specification of Eq. (46) we require merely the energy denominators. Let us consider the ith energy denominator D_i. A horizontal line drawn between the ith interaction time and the one immediately following will cut an odd number of "particle" lines counting the undulating lines for the initial and final states. For every such line directed upwards, of momentum p_+, there occurs in D_i^{-1} a term $-(p_+^2-i\eta)$. For every such line propagating backwards in time with momentum p_- there is a term $(p_-^2+i\eta)$. In all we then find by carrying out the time integrations that

$$D_i^{-1}=p_0-\nu_c p_0+\sum\nolimits_-(p_-^2+i\eta)-\sum\nolimits_+(p_+^2-i\eta), \quad (47)$$

where ν_c is the number (0, 1, or 2) which specifies whether the initial and/or final state lines are cut.

In the summations in (46), the momenta p_+ are restricted to be above the Fermi surface, the p_- to be below. This has several interesting consequences. In the first place, there are a large number of diagrams in which momentum conservation requires a p_+ to be equal to a p_-. All these then vanish. Such is the case for the contributions of Figs. 2 (g)-(j). Here the simplest interaction with the medium as represented by the simplest closed loop or its associated exchange interaction can result only in forward scattering of the interacting particle and therefore cannot reverse the sense of its propagation line. Of greater moment is the fact that the structure of the D_i provides us with information concerning the analytic properties of $M(\mathbf{p},p_0)$. This will be discussed in the next section, in connection with a brief consideration of excited-state energies.

We now turn to the formulation of the rules for calculating $E(N,0)$ from time-independent formulas.

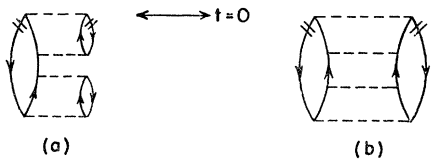

FIG. 6. Selected contributions to the fourth-order energy. Each of them corresponds to one or two distinct diagrams of the self-energy kernel $\mathfrak{M}_R$ obtained by removing the downward-directed lines indicated.

These rules are closely those given for $M(\mathbf{p},p_0)$ and follow directly from consideration of Eq. (34). Thus to every contribution to $\mathfrak{M}_R$ there is a contribution to $E(N,0)$. It is also easy to see that at most two different diagrams of $\mathfrak{M}_R$ give the same diagram for $E(N,0)$ when the single-particle propagator is closed. This may be concluded by considering the alternative definition of $E(N,0)-E_0(N,0)$ as one-half the sum of all distinct, connected, closed-loop diagrams which disappear at time $t=0$. According to Eq. (34), the diagram of Fig. 6(a), for example, could have been formed from those contributions to $\mathfrak{M}_R$ in which either of the two downward directed lines marked with $\diagdown\!\!\diagdown$ or $//$ is opened. In such cases the factor of $\frac{1}{2}$ in (34) is removed. The only other type of possibility, illustrated in Fig. 6 (b), arises from but a single distinct term of $\mathfrak{M}_R$, and the factor of $\frac{1}{2}$ is thus retained in calculating the contribution of this diagram.

The time integrations may now be performed and the spatial integrals transformed to sums over momenta as for $M(\mathbf{p},p_0)$, with the even simpler result that a typical contribution, exclusive of a possible factor $\frac{1}{2}$, has the form

$$\lambda^n(-1)^{l+m}\frac{1}{\Omega^n}\sum_{\mathbf{q}_1\cdots\mathbf{q}_n}\sum_{\mathbf{p}_1\cdots\mathbf{p}_{2n}}\prod_{j=1}^{2n-1}\delta_j \times v(|\mathbf{q}_1|)\cdots v(|\mathbf{q}_n|)[\bar{D}_{n-1}\cdots\bar{D}_1]^{-1}, \quad (48)$$

where

$$\bar{D}_i^{-1}=\sum\nolimits_-(p_-^2+i\eta)-\sum\nolimits_+(p_+^2-i\eta), \quad (49)$$

the right-hand side having already been defined in connection with Eq. (47). In the present case, since the number of p_+ equals the number of p_-, $\bar{D}_i^{-1}$ never vanishes and the factors $i\eta$ may be dropped. Though there are $2n$ vertices, in (48) we have already taken cognizance of the fact that momentum is automatically conserved at one of them if it is at the $2n-1$ others. There are $(n-1)!$ terms of type (48) for every original diagram.

VI. ANALYTIC PROPERTIES OF THE SELF-ENERGY OPERATOR AND OF THE ONE-PARTICLE GREEN'S FUNCTION

We turn in this section to a further study of the properties of the self-energy operator and of the one-particle Green's function. The reason for this is that though the various perturbation expansions may well converge for potentials that are sufficiently regular and sufficiently weak, the attempt to infer the analytic properties of the functions so defined from the individual terms of the series can, of course, be quite misleading. But, as we shall see, both the occurrence and position of "one-particle" excitations, for example, depend intimately on just such analytic properties, in this case of the self-energy operator.

We consider first the general form of the full one-

particle Green's function. We have

$$t>t': G(x;x')$$
$$=i\sum_{\mathbf{p},\alpha}\langle N,0|\psi(x)|N+1,\mathbf{p},\alpha\rangle\langle N+1,\mathbf{p},\alpha|\psi^\dagger(x')|N,0\rangle$$
$$=i\sum_{\mathbf{p},\alpha}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-i\epsilon_+(\mathbf{p},\alpha)(t-t')]$$
$$\times\langle N,0|\psi(0)|N+1,\mathbf{p},\alpha\rangle\langle N+1,\alpha|\psi^\dagger(0)|N,0\rangle, \quad (50)$$

where

$$\epsilon_+(\mathbf{p},\alpha)=E(N+1,\mathbf{p},\alpha)-E(N,0), \quad (51)$$

and we have used only completeness and translational invariance. In (50) and (51), α is a parameter of degeneracy distinguishing states of given momentum. It is in fact clear from a consideration of the unperturbed situation that α will take on an infiite number of values. Introducing the "wave function"

$$\Omega^{-\frac{1}{2}}\varphi_+(\mathbf{p},\alpha)=\langle N,0|\psi(0)|N+1,\mathbf{p},\alpha\rangle, \quad (52)$$

which can be interpreted as the amplitude for finding the ground state of N particles if we remove a particle from the state $|N+1,\mathbf{p},\alpha\rangle$, we thus have

$$t>t': G(x;x')=\frac{i}{\Omega}\sum_{\mathbf{p},\alpha}|\varphi_+(\mathbf{p},\alpha)|^2$$
$$\times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-i\epsilon_+(\mathbf{p},\alpha)(t-t')]. \quad (53)$$

For the unperturbed Green's function we have $\epsilon_+(\mathbf{p},\alpha)=p^2$, and $|\varphi_+(\mathbf{p},\alpha)|^2=\theta_+(p^2-p_F^2)$ is a projection operator onto that state of momentum $\mathbf{p}$ which is of single-particle character. In the general case the range of summation over $\mathbf{p}$ need bear no simple relationship to the corresponding unperturbed sum, where the $(N+1)$-particle states must carry at least momentum p_F. For the sake of simplicity we shall however ignore the possible occurrence in this first discussion of "anomalous" bound states and suppose that the $\epsilon_+(\mathbf{p},\alpha)$ has a minimum for $|\mathbf{p}|=p_F$, though states with $|\mathbf{p}|<p_F$ certainly occur. More precisely, there will be a spectrum $\epsilon_+(\mathbf{p},\alpha)$ for every p, with some minimum. We are now supposing that the minimum among the minima occurs for $|\mathbf{p}|=p_F$. That such a minimum may occur for some systems is reasonable physically since the unperturbed state consisting of the Fermi-gas state of N particles plus one particle, of momentum p_F, just above the Fermi sea is the $(N+1)$-particle state of least energy. Unperturbed states with $|\mathbf{p}|<p_F$, for instance, must contain at least two of the $N+1$ particles above the Fermi sea.

Similarly we find

$$t<t': G(x;x')=-\frac{i}{\Omega}\sum_{p,\alpha}|\varphi_-(\mathbf{p},\alpha)|^2$$
$$\times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-i\epsilon_-(\mathbf{p},\alpha)(t-t')], \quad (54)$$

where

$$\epsilon_-(\mathbf{p},\alpha)=E(N,0)-E(N-1,-\mathbf{p},\alpha), \quad (55)$$
$$\Omega^{-\frac{1}{2}}\varphi_-(\mathbf{p},\alpha)=\langle N-1,-\mathbf{p},\alpha|\psi(0)|N,0\rangle. \quad (56)$$

Arguing as above, we suppose the maximum value of $\epsilon_-(\mathbf{p},\alpha)$ to occur for $|\mathbf{p}|=p_F$ and to be the same as the above minimum. Their common value defines the perturbed Fermi energy ϵ_F.

We may note that $\sum_\alpha|\varphi_-(\mathbf{p},\alpha)|^2$ is the momentum distribution in the ground state,

$$N=\int d^3x\langle N,0|\psi^\dagger(x)\psi(x)|N,0\rangle=\sum_{\mathbf{p},\alpha}|\varphi_-(\mathbf{p},\alpha)|^2. \quad (57)$$

We have further a connection between $|\varphi_-(p,\alpha)|^2$ and $|\varphi_+(p,\alpha)|^2$,

$$1=\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\langle N,0|\{\psi^\dagger(x),\psi(0)\}|N,0\rangle$$
$$=\sum_\alpha[|\varphi_+(\mathbf{p},\alpha)|^2+|\varphi_-(\mathbf{p},\alpha)|^2]. \quad (58)$$

It is also worth remarking that the approximation of $|\varphi_-(\mathbf{p},\alpha)|^2$ and $|\varphi_+(\mathbf{p},\alpha)|^2$ by their Fermi-gas values can lead to serious errors. As an example of this, let us consider the exact Eq. (5) for the ground-state energy of N particles. The kinetic-energy term here is

$$\sum_{\mathbf{p},\alpha}p^2|\varphi_-(\mathbf{p},\alpha)|^2. \quad (59)$$

A measure of the difference between the value of (59) and the kinetic energy of the Fermi gas can be achieved by considering the perturbation form of the second term of (5). Comparing this with the corresponding term of (9), we see that in jth order it is j times larger than the latter.

In general then we must expect $|\varphi_\pm(\mathbf{p},\alpha)|^2$ to be smooth functions, subject only to conditions like (57) and (58), and the corresponding summations in (53) and (54) to involve no sharp cutoff. With these introductory remarks, we renew our discussion of the self-energy operator.

We have so far defined the irreducible self-energy operator as a power series. A considerable summation of terms in this series [consider the time-dependent form, $M(x;x')$] can be achieved by use of the exact propagator[15] in place of $G^{(0)}$. We need merely write

$$M[G^{(0)}]=M'[G], \quad (60)$$

and mean by M' that subclass of diagrams of M obtained by omitting all diagrams that can be obtained from $M'[G^{(0)}]$ by replacing $G^{(0)}$ by $G^{(0)}+G^{(0)}M_RG^{(0)}=G$.

It is now a simple matter to record the time-ordered contributions to M' by reference to the corresponding

[15] Further summations, which can be carried out by the introduction of "vertex operators" and "polarization operators," might well alter some of our conclusions. To this extent our discussion, based on the partial summation introduced here, is subject to some of the same reservations voiced at the beginning of this section. We plan to return to these questions in the future.

term in M, Eq. (46). From Eqs. (59) and (60), it is clear that the following replacements are necessary: (i) Summations are no longer restricted to be above or below the Fermi sea. In fact we have

$$\sum_{|\mathbf{p}|>p_F} \rightarrow \sum_{\mathbf{p},\alpha} |\varphi_+(\mathbf{p},\alpha)|^2,$$
$$\sum_{|\mathbf{p}|<p_F} \rightarrow \sum_{\mathbf{p},\alpha} |\varphi_-(\mathbf{p},\alpha)|^2. \tag{61}$$

(ii) In the energy denominators we set

$$p_+^2 \rightarrow \epsilon_+(\mathbf{p},\alpha),$$
$$p_-^2 \rightarrow \epsilon_-(\mathbf{p},\alpha). \tag{62}$$

We now apply the above rules together with the assumed nature of the excitation spectrum to draw conclusions about the analytic properties of the self-energy operator. These follow from the general structure of the energy denominator of $M'(\mathbf{p},p_0)$ which is now

$$p_0-v_c p_0+\sum[\epsilon_-(\mathbf{p},\alpha)+i\eta]-\sum[\epsilon_+(\mathbf{p},\alpha)-i\eta]. \tag{63}$$

We use only the assumption that $\epsilon_+(\mathbf{p},\alpha)$ has its minimum value for $|\mathbf{p}|=p_F$, and that at the same point $\epsilon_-(\mathbf{p},\alpha)$ has its maximum value, both referring to the common point $\epsilon_\pm=\epsilon_F$. It follows from (63) that for every $\mathbf{p}$, $M'(\mathbf{p},p_0)$ is real for $p_0=\epsilon_F$ (and only this value) which is a branch point of the operator. We may well consider that branch lines run from ϵ_F to $+\infty$ slightly below the real axis and from ϵ_F to $-\infty$ slightly above the real axis corresponding, respectively, to the continuum of values of ϵ_+ and ϵ_-.

We remark finally that summed formulas for the energy for Eq. (29), for instance, may be recorded, but no use of such formulas will be made in the remainder of this paper, and we relegate this exercise to the interested reader.

VII. DISCUSSION OF SINGLE-PARTICLE EXCITATIONS

The discussion above bears directly on our ability to define real single-particle excitation energies. The wave function of a single-particle state will satisfy the equation

$$[-i\partial_t+p^2]\langle N,0|\psi(x)|N+1,\mathbf{p}\rangle - \int d^4x'\, M(x;x')\langle N,0|\psi(x')|N+1,\mathbf{p}\rangle=0, \tag{64}$$

which is immediately plausible[16] as one of the homogeneous equations associated with the equation for $G(x;x')$. Since we have

$$\langle N,0|\psi(x)|N+1,\mathbf{p}\rangle=\Omega^{-\frac{1}{2}}\varphi_+(\mathbf{p}) \times\exp[i\mathbf{p}\cdot\mathbf{x}-i\epsilon_+(\mathbf{p})t], \tag{65}$$

we obtain from (64) and (65) the equation

$$[\epsilon_+(\mathbf{p})-p^2+M'(\mathbf{p},\epsilon_+(\mathbf{p}))]\varphi_+(\mathbf{p})=0. \tag{66}$$

We are thus constrained to find the real solutions of the eigenvalue equation

$$\epsilon_+(\mathbf{p})=p^2-M'(\mathbf{p},\epsilon_+(\mathbf{p})). \tag{67}$$

It is essential to realize that the solutions of (67) by no means constitute the entire spectrum, but merely those eigenfunctions of $N+1$ particles which can be connected adiabatically (in the sense of perturbation theory) to an eigenstate of the unperturbed Hamiltonian consisting of the Fermi-gas state of N particles plus one additional particle above the Fermi sea. In other words they are the true bound states of single-particle character, and the corresponding ϵ_+ may be considered as the excitation energy of a "real" particle in the medium. From the considerations of the previous section, there can be at most one such state, namely for $\epsilon_+=\epsilon_F$. For this reason we have suppressed the degeneracy index α, since $\epsilon_+(\mathbf{p},\alpha)$ does not obey (67) in general.

In addition to the real root, there may of course be complex pseudo-roots which describe metastable single-particle states of the system (which decay into collective states).

That the equation

$$\epsilon_F=p_F^2-M'(p_F,\epsilon_F) \tag{68}$$

is indeed satisfied hardly requires formal proof. For if we write (68) as

$$E(N+1,\mathbf{p}_F)=E(N,0)+p_F^2-M'(\mathbf{p}_F,\epsilon_F), \tag{69}$$

we can view the result as an equation for the ground state of $N+1$ particles for the given volume Ω. From (69) we can at least aver the existence of a real power series for $E(N+1,\mathbf{p}_F)$. To generate this power series, we first replace $M'(\mathbf{p}_F,\epsilon_F)$ by its formal equivalent $M(\mathbf{p}_F,\epsilon_F)$ (i.e., the power series for M). Then from (68) or (69) we carry out a further expansion,

$$\epsilon_F=p_F^2-M(\mathbf{p}_F,p_F^2)-\frac{\partial M}{\partial p_F^2}(\epsilon_F-p_F^2)-\cdots, \tag{70}$$

of M about $\epsilon_F=p_F^2$. It follows from the structure of *its power series* that $M(\mathbf{p}_F,p_F^2)$ is real. Indeed the Taylor series (70) is term by term real and when solved for ϵ_F by successive iteration yields the desired power series for that quantity.

It is now essential to remark that in (68) or any of its equivalent expressions, we have an alternative formula for computing the binding energy per particle in the limit of large N and Ω. From its very definition, ϵ_F is the binding energy of an additional particle at constant volume. Expanding this finite change in a power series in the change in the number of particles,

[16] Equation (64) can also be derived by use of the limiting procedures of M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

we have

$$\epsilon_F=\frac{\partial E(N,0)}{\partial N}\bigg|_\Omega+\frac{1}{2!}\frac{\partial^2 E(N,0)}{\partial N^2}\bigg|_\Omega+\cdots. \tag{71}$$

In the limit of large N and Ω, we may neglect the second and higher derivatives, for remembering that $E(N,0)=N\epsilon(\rho)$, where $\epsilon(\rho)$, the binding energy per particle, is a function only of the density, we have for a saturating system

$$\frac{\partial E(N,0)}{\partial N}\bigg|_\Omega=\epsilon(\rho)+N\frac{\partial\epsilon(\rho)}{\partial N}=\epsilon(\rho), \tag{72}$$

and

$$\frac{\partial^2 E(N,0)}{\partial N^2}\bigg|_\Omega=\frac{1}{\Omega}\frac{d\epsilon(\rho)}{d\rho}\to 0 \tag{73}$$

as $\Omega\to\infty$, ρ constant.

Having noted the identity of the Fermi energy to the binding energy per particle, it is possible to demonstrate directly that $\epsilon(\rho)$ satisfies (68), through the proof is somewhat lengthy. We proceed from the identities

$$\epsilon(\rho)=\frac{\partial E(N,0)}{\partial N}\bigg|_\Omega=\frac{d}{d\rho}\left[\frac{E(N,0)}{\Omega}\right]=\frac{2\pi}{p_F{}^2}\frac{d}{dp_F}\left[\frac{E(N,0)}{\Omega}\right]. \tag{74}$$

We compute the last form of Eq. (74) directly from the power series for $E(N,0)$ in which, according to Eq. (10), we make the replacements

$$\sum_{\mathbf{p}_i}\to\Omega(2\pi)^{-3}\int_0^{p_F}d\mathbf{p}_i \quad\text{or}\quad \Omega(2\pi)^{-3}\int_{p_F}^{\infty}d\mathbf{p}_i, \tag{75}$$

as appropriate. We then find

$$\frac{2\pi}{p_F{}^2}\frac{d}{dp_F}\left[\frac{E(N,0)}{\Omega}\right]=p_F{}^2-M_R(\mathbf{p}_F,p_F{}^2). \tag{76}$$

To establish Eq. (76), we first note that it holds trivially for the kinetic-energy term. To see that its validity extends throughout, we study Eq. (48), a typical term of $E(N,0)$. The contribution of (48) to the left-hand side of (76) is then a sum of $2n$ terms, one for each particle or hole line, each one a time-ordered contribution to $M_R(\mathbf{p},p_0)$, evaluated for $|\mathbf{p}|\to p_F$, $p_0\to p_F{}^2$. Conversely any contribution to $M_R(\mathbf{p},p_0)$ is related uniquely to some parent term (48) in the present sense. For those terms of (48) which bear a factor $\frac{1}{2}$, the process of differentiation will yield the corresponding terms of M_R exactly twice.

A cautionary word or two is required before proceeding. In order to claim the formal identity (76), it is necessary to retain and differentiate a large class of contributions to $E(N,0)$ which actually vanish. These comprise all diagrams in which at least two of the particles, one constrained to move above the Fermi sea, the other below it, are nevertheless required to carry momenta of the same magnitude. The resulting kernel $M_R(\mathbf{p}_F,p_F{}^2)$ is then highly singular, owing to the presence of terms in which there is at least one intermediate state containing but a single particle and a single hole both at the Fermi surface. To complete the demonstration of the assertion preceeding Eq. (74), we must finally show that a careful evaluation of $M_R(\mathbf{p}_F,p_F{}^2)$ will establish the identity[17]

$$M_R(\mathbf{p}_F,p_F{}^2)=M(\mathbf{p}_F,\epsilon_F). \tag{77}$$

We refer for this last point of the proof to the Appendix.

We complete our account of the positive results of this section by the statement without proof, that by methods similar to those which lead to Eq. (66), we can show that $\varphi_-(\mathbf{p})$ satisfies the equation

$$[\epsilon_-(\mathbf{p})-p^2+M'(\mathbf{p},\epsilon_-(\mathbf{p}))]\varphi_-(\mathbf{p})=0. \tag{78}$$

Thus the energy of a "hole," analogous to that of a particle, is real only at the perturbed Fermi energy ϵ_F. On the other hand, for $|\mathbf{p}|\neq p_F$ the solutions of Eqs. (66) and (78) are complex. Where the imaginary part of $\epsilon_+(\mathbf{p})$, for example, is "small," we can associate with those values of $\mathbf{p}$ (near the Fermi surface) approximate single-particle excitations in a manner familiar from the theory of atomic spectra, with the qualitative difference that we have a continuum of such states. Similar remarks obtain for holes near the Fermi surface.

VIII. GENERALIZATION OF THE THEORY TO FINITE TEMPERATURES

Though the main purpose of this paper has been to study the zero-temperature problem, we cannot forbear from pointing out the relative ease with which the theory can be generalized to finite temperatures by the method of Green's functions.[18] Let us study the internal energy of a system of N particles at reciprocal temperature $\beta=1/kT$. This quantity, reducing to the ground-state energy as $\beta\to\infty$, is given by the well-known canonical average

$$E_{N'}=\frac{\mathrm{tr}[H_{N'}\exp(-\beta H_{N'})]}{\mathrm{tr}[\exp(-\beta H_{N'})]}, \tag{79}$$

where $H_{N'}$ is specifically the N'-particle Hamiltonian, and therefore "tr" means the trace with respect to states of a definite number, N', of particles *and appropriate symmetry*. In accordance with our program we shall deal here with Fermi statistics alone.

As is usual in such problems—because of the great formal simplification resulting therefrom—we replace

[17] Equations (76) and (77) together constitute the theorem of van Hove and Hugenholtz, reference 10.

[18] The authors wish to point out that they did not arrive at this generalization before seeing the work of P. C. Martin and J. Schwinger, reference 6, where the general Green's function of the type illustrated by Eqs. (82) and (83) is defined.

the averaging process (79) by the grand canonical average

$$\langle E\rangle=\frac{\sum_{N'=0}^{\infty}\mathrm{tr}\{H_{N'}\exp[-\beta H_{N'}+\mu N']\}}{\sum_{N'=0}^{\infty}\mathrm{tr}\{\exp[-\beta H_{N'}+\mu N']\}}$$
$$=\frac{\mathrm{Tr}\{H\exp[-\beta H+\mu N]\}}{\mathrm{Tr}\{\exp[-\beta H+\mu N]\}},\quad(80)$$

where H and N are now the second-quantized operators of Eqs. (1) and (3), respectively, and the symbol "Tr" denotes trace restricted only with respect to symmetry. The parameter μ is chosen so as to yield the correct number of particles,

$$\langle N\rangle=\frac{\mathrm{Tr}\{N\exp[-\beta H+\mu N]\}}{\mathrm{Tr}\{\exp[-\beta H+\mu N]\}}.\quad(81)$$

The essential new step in the development is the recognition that if we introduce the explicit form of H into the numerator of (80), the resulting expression can be made formally identical to Eq. (5) for the ground-state energy if only we introduce a new set of temperature-dependent Green's functions, of which the relevant members are

$$G_\beta(x;x')=\frac{i\,\mathrm{Tr}\{T(\psi(x)\psi^\dagger(x'))\exp[-\beta H+\mu N]\}}{\mathrm{Tr}\{\exp[-\beta H+\mu N]\}},\quad(82)$$

and

$$G_\beta(x_1,x_2;x_1',x_2')=\frac{(i)^2\,\mathrm{Tr}\{T(\psi(x_1)\psi(x_2)\psi^\dagger(x_2')\psi^\dagger(x_1'))\exp[-\beta H+\mu N]\}}{\mathrm{Tr}\{\exp[-\beta H+\mu N]\}}.\quad(83)$$

These functions satisfy formally the same equations as developed from Sec. III onwards. In particular if we confine ourselves here to the perturbation series for $\langle E\rangle$, all that is required completely to define that series is to obtain the form of the noninteracting one-particle Green's functions $G_\beta^{(0)}$.

Toward this end we carry out the discussion analogous to that developed for $G^{(0)}=G_\infty^{(0)}$ at the beginning of Sec. V. We first state the result,

$$G_\beta^{(0)}(x;x')=\frac{1}{\Omega}\frac{1}{2\pi}\sum_{\mathbf{p}}\int_C dp_0$$
$$\times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-p_0(t-t')][-p_0+p^2]^{-1},\quad(84)$$

with the contour C chosen as follows: It is the weighted average of two contours C_1 and C_2. C_1 runs from $-\infty$ to ∞ slightly above the real axis and has its contribution weighted by the factor $[1+\exp(-\beta p^2+\mu)]^{-1}$, the probability that at temperature β^{-1} the state of momentum $\mathbf{p}$ of the noninteracting gas is empty; C_2 runs from $-\infty$ to ∞ slightly below the real axis, its contribution being weighted by the factor $\exp(-\beta p^2+\mu)\times[1+\exp(-\beta p^2+\mu)]^{-1}$, the probability that under the given conditions the said momentum state is occupied. The same result is achieved by writing in place of (84)

$$G_\beta^{(0)}(x;x')=\frac{1}{\Omega}\frac{1}{2\pi}\sum_{\mathbf{p}}\int_{-\infty}^{\infty}dp_0$$
$$\times\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip_0(t-t')]G_\beta^{(0)}(\mathbf{p},p_0),\quad(85)$$

with

$$G_\beta^{(0)}(\mathbf{p},p_0)=[1+\exp(-\beta p^2+\mu)]^{-1}[-p_0+p^2-i\eta]^{-1}$$
$$+\exp(-\beta p^2+\mu)[1+\exp(-\beta p^2+\mu)]^{-1}$$
$$\times[-p_0+p^2+i\eta]^{-1}.\quad(86)$$

Performing the integral with respect to p_0, we then obtain

$$t>t':\ G_\beta^{(0)}(x;x')=\frac{i}{\Omega}\sum_{\mathbf{p}}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip^2(t-t')]$$
$$\times[1+\exp(-\beta p^2+\mu)]^{-1},\quad(87)$$

$$t<t':\ G_\beta^{(0)}(x;x')=\frac{-i}{\Omega}\sum_{\mathbf{p}}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-ip^2(t-t')]$$
$$\times\exp(-\beta p^2+\mu)[1+\exp(-\beta p^2+\mu)]^{-1}.\quad(88)$$

The correctness of the results (87), (88) may now be verified directly from the definition (82), employing the methods of Sec. V.

It is furthermore clear from (87) and (88) that also the general structure (with its associated diagrams) of the time-independent form of the perturbation series is maintained.[19] The designation of "particle" and "hole" now refers to the contributions from (87) and (88), respectively. Instead of sharp cutoffs in the momentum sums, as occurs in the limit of zero temperature, the corresponding sums here range throughout all possible values, weighted by the probabilities for non-occupation and occupation, respectively, of the given momentum state. Further development of this and related approaches to statistical mechanics will be referred to future publications. We do remark, however, that to define completely the procedure described here, the value of μ must be obtained from a simultaneous study of the expansion for $\langle N\rangle$.

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[19] This result has been achieved by a different method by C. Bloch and C. de Dominics, reference 6.

APPENDIX

We study here the validity of the formula

$$M_R(\mathbf{p}_F,p_F^2)=M(\mathbf{p}_F,\epsilon_F), \tag{A.1}$$

Eq. (77) of the text. We shall show that these functions, with a proper interpretation of M_R are represented by identical power series. The series in question can be readily constructed by solving Eq. (70) by iteration. Let

$$M_0=M(\mathbf{p}_F,p_F^2),\quad \dot{M}_0=\frac{\partial}{\partial p_0}M(\mathbf{p}_F,p_0)\,|_{P F^2},\ \cdots. \tag{A.2}$$

We then find that

$$\begin{aligned}M(\mathbf{p}_F,\epsilon_F)&=M_0+M_0\dot{M}_0+\left[M_0\dot{M}_0^2+\frac{1}{2!}M_0\ddot{M}_0\right]\\&\quad+\left[M_0\dot{M}_0^3+\tfrac{3}{2}M_0^2\dot{M}_0\ddot{M}_0+\frac{1}{3!}M_0^3\dddot{M}_0\right]+\cdots\\&=M_0+\frac{1}{2!}\left(\frac{\partial}{\partial p_0}M^2\right)_0+\frac{1}{3!}\left(\frac{\partial^2}{\partial p_0^2}M^3\right)_0\\&\quad+\frac{1}{4!}\left(\frac{\partial^3}{\partial p_0^3}M^4\right)_0+\cdots,\end{aligned} \tag{A.3}$$

the last a highly suggestive form.

We next turn to the study of $M_R(\mathbf{p},p_0)$. The rules for constructing this kernel are, of course, the same as those for $M(\mathbf{p},p_0)$ except that there are additional diagrams. Thus if we consider the Feynman diagram corresponding to Fig. 6(a), this gives rise as usual to 4! time-ordered contributions to $M_R(\mathbf{p},p_0)$ To deduce this series (A.3) from the limiting form of the complete time-ordered series is cumbersome. Essential simplification is achieved if we realize the availability of an alternative form of $M_R(\mathbf{p},p_0)$: It follows directly from Eq. (22) that

$$\begin{aligned}M_R(\mathbf{p},p_0)&=M(\mathbf{p},p_0)+M(\mathbf{p},p_0)G^{(0)}(\mathbf{p},p_0)M(\mathbf{p},p_0)\\&\quad+MG^{(0)}MG^{(0)}M+\cdots,\end{aligned} \tag{A.4}$$

with

$$G^{(0)}=\lim_{\eta\to 0}\left[-p_0+p^2-i\eta\epsilon(p^2-p_F^2)\right]. \tag{A.5}$$

We are now concerned with the limit of (A.4) as $|\mathbf{p}|\to p_F$, $p_0\to p_F^2$. Because of the multiplicity of limiting procedures and the singularity of $M_R(\mathbf{p},p_0)$ at the limiting value, a correct order of the limits must be inferred from the original expression for $E(N,0)$, from which $M_R(\mathbf{p}_F,p_F^2)$ is obtained by differentiation. Since in the former all energy denominators are non-singular, the limit $\eta\to 0$ may be taken first. A further study of the structure of the terms of $E(N,0)$, Eq. (48), shows that the limiting procedure is correctly represented if we replace $G^{(0)}$ by $\mathcal{P}[-p_0+p^2]^{-1}$, the principal value, and study the function

$$\int dp_0\,\delta(p_0-p_F^2)\lim_{|\mathbf{p}|\to p_F}M_R(\mathbf{p},p_0). \tag{A.6}$$

We have therefore in general to study the "function"

$$\delta(x)\mathcal{P}x^{-n}=(-1)^n[(n+1)!]^{-1}\delta^{(n)}(x). \tag{A.7}$$

If we adopt Eq. (A.7) provisionally, then the evaluation of (A.6) leads immediately to (A.3).

Finally the validity of (A.7) may be concluded by means of the representations[20]

$$\mathcal{P}x^{-1}=\lim_{\sigma\to 0}\frac{x}{x^2+\sigma^2},\quad \delta(x)=\frac{1}{\pi}\lim_{\sigma\to 0}\frac{\sigma}{x^2+\sigma^2}. \tag{A.8}$$

For example,

$$\begin{aligned}\delta'(x)&=\frac{1}{\pi}\lim\frac{1}{dx}\left(\frac{\sigma}{x^2+\sigma^2}\right)=-\frac{1}{\pi}\lim\frac{2\sigma x}{(x^2+\sigma^2)^2}\\&=-2!\delta(x)\mathcal{P}x^{-1},\end{aligned} \tag{A.9}$$

$$\begin{aligned}\delta''(x)&=\frac{1}{\pi}\lim\frac{d^2}{dx^2}\frac{\sigma}{x^2+\sigma^2}=\frac{1}{\pi}\lim\left\{\frac{-2\sigma}{(x^2+\sigma^2)^2}+\frac{8x^2\sigma}{(x^2+\sigma^2)^3}\right\}\\&=\frac{1}{\pi}\lim 3!\frac{\sigma}{(x^2+\sigma^2)^2}=3!\delta(x)\mathcal{P}x^{-2},\end{aligned} \tag{A.10}$$

etc.

[20] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, Cambridge, 1954), third edition, p. 70.