Application of Perturbation Methods to the Theory of Nuclear Matter*†

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A generalized perturbation theory is developed in such a way that it can be applied to a many-body problem with strong forces between the particles. The Brueckner expression for the energy is shown to be the first-order term in a particular case of this expansion. Some of the higher-order terms in the expansion are studied, and the importance of self-consistency in the energy denominator of Brueckner's equation and of the use of the exclusion principle in intermediate states is assessed. A possible simplification of the methods used is suggested, which involves solving the Brueckner equation for the hard core, and using normal perturbation theory for the attractive part of the potential. The methods developed are used to analyze some details of previously published calculations.

The lack of equality between the Fermi energy and the binding energy in the nuclear matter calculations shows that there must be a rearrangement energy. A simple formula for the rearrangement energy is derived, and its importance for single-particle excited states, such as occur in the optical model, is shown. The relation between the rearrangement energy and the departure of the system from a degenerate Fermi-gas state is shown. The effect of the rearrangement energy on the ground-state energy is indirect, but it is as important as the self-consistency condition. The rearrangement energy seems to come mainly from the hard core, and simple numerical estimates of the rearrangement energy from a hard core potential show that it is somewhat less than 16 Mev at the Fermi surface. The ground-state energy is reduced by perhaps 1 Mev. There seems to be a discrepancy between the calculated and observed energy dependence of the real part of the optical model potential.

1. INTRODUCTION

THE object of the program of Brueckner and his collaborators[1–8] is to derive the properties of complex nuclei from the two-body forces which act between nucleons, as some of the properties of atoms have been calculated by Hartree and others[9] using the Coulomb interaction. It is only recently that high-energy scattering and polarization experiments have begun to show the details of nuclear forces, but it has been known for a long time that the forces have a short range, unlike the forces between electrons. For both the atom and the nucleus the particles have spin one half, and they obey the Pauli exclusion principle, but the nuclear particles have isotopic spin one half (two charge states) so that each state in coordinate space may be occupied by four particles, instead of only two. It is not certain that the forces between nucleons are simply the two-body forces observed in scattering experiments, but there is no definite evidence against this, and it is assumed in the above-mentioned calculations that they are.

The difficulties of making a calculation for particles interacting by strong short-range forces are great, and the simplest system to study is nuclear matter, since its translation invariance reduces the problem considerably. Only for the case of nuclear matter has much progress been made so far, but a detailed calculation has now been made by Brueckner and Gammel.[7,8] Ideally we wish to solve the Schrödinger equation for a system of A particles interacting through short-range two-body forces, and it is claimed that the method they use provides an approximate solution of the problem. A form of perturbation theory will be developed here which displays the degree of approximation involved in their calculations. It also shows how some simpler calculations might be made using similar methods.

There are two major difficulties in the application of perturbation theory to the nuclear matter problem. The first is that some forms of perturbation theory give rise to a spurious dependence on the total number of particles in the system, so that convergence problems arise which are absent from single-particle perturbation theory; the way to overcome this trouble has been shown by Goldstone[10] and by Hugenholtz.[11] The second difficulty is that the observed forces between free nucleons are strong, with a *repulsive core*, and perturbation theory will not converge for such forces; this is the trouble which is overcome by the use of Brueckner's *reaction matrix*. The Rayleigh-Schrödinger perturbation theory will be used to illustrate the first of these difficulties, and it will then be shown how it can be extended to include also the solution of the second difficulty. The method used was suggested by the paper of Tobocman[12] on the subject.

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2. RAYLEIGH-SCHRÖDINGER PERTURBATION THEORY

The difficulty about applying perturbation theory to a many-body problem is that the individual terms in the expansion may have an unsuitable dependence on A, the total number of particles in the system. Even if the expansion formally converges for all A, it may still be necessary to include A terms of the expansion to get approximately correct results. An example of this is given by the Brillouin-Wigner perturbation theory,[13] in which the Hamiltonian is $H=H_0+H_I$. Φ is an eigenstate of H_0, Ψ is an eigenstate of H with eigenvalue E, and P is the projection operator off Φ. The expansion is obtained by iterating the equation for Ψ,

$$\Psi=\Phi+P(E-H_0)^{-1}H_I\Psi. \tag{1}$$

This gives the energy as

$$E=(\Phi,H_0\Phi)+(\Phi,H_I\Phi)+\left(\Phi, H_I\frac{P}{E-H_0}H_I\Phi\right)+\cdots. \tag{2}$$

The first two terms in the expansion of E are proportional to A, but the third term is not, since the nondiagonal elements of H_I are proportional to A^{-1}: the number of momentum-conserving intermediate states is proportional to A^3, but the elements of $E-H_0$ are proportional to A for all excitations of just a few particles. The succeeding terms will also be constant, but there will be a number of order A which are of comparable importance, so there will be further corrections to E of order A. In the limit of an infinite number of particles, an infinite number of terms in the series must be taken into account.

The Rayleigh-Schrödinger expansion does not suffer from this trouble.[10,11] In the third term on the right of (2), $E-H_0$ is replaced by E_0-H_0, where E_0 is the eigenvalue of H_0 corresponding to Φ, and this is not proportional to A, so the whole term is proportional to A. This does not prove that there are not terms which depend on a lower power of A, for example transitions which do not conserve momentum, but whose cumulative effect is proportional to A. This seems unlikely when it is realized that the Rayleigh-Schrödinger expansion is a simple expansion in powers of a coupling constant g. If the Hamiltonian is written as

$$H(g)=H_0+gH_I, \tag{3}$$

an eigenstate, subject to some subsidiary conditions, can be found for all values of the coupling constant g between zero and one, and so the energy of the system can be expressed as a function of g in this range. If the density of the particles is constrained to be constant, it seems likely that the total energy will vary with g in a simple manner that does not depend on A in the limit of large A. If the energy is an analytic function of g, and if the power series converges uniformly for $|g|\leqslant 1$, then there is no trouble with the terms of very high order, since these are only important for values of g very close to the circle of convergence. For the same reasons, the expectation values of operators corresponding to physically observable quantities should be given in a satisfactory way by the Rayleigh-Schrödinger expansion. The observable is written as a power series in g, and its value can be obtained if the radius of convergence is greater than one.

The wave function is not a physical observable, so it is not surprising that its expansion in powers of g does not give satisfactory convergence. If the amplitude of the ground-state component of the wave function is chosen to be unity, then the amplitude of the first term in the expansion is of order A, the next of order A^2 and so on. In fact, as was pointed out by Bethe,[13] the probability of the system being in its ground state falls off like e^{-A}. This is what gives rise to what has been called the "unlinked-cluster problem," and it is analogous to the problem in classical statistical mechanics which is overcome by the use of Mayer's cluster integrals. In classical statistical mechanics it is also extremely unlikely that a system of interacting particles will be in the ideal gas state, even if the interactions are very weak. The wave function can still be used for the calculation of observables, since there is a cancellation of terms between the matrix element of the operator and the normalization factor. This has been treated in some detail by Goldstone[10] and Hugenholtz.[11]

The translation invariance of the nuclear matter problem dictates the choice of basis wave functions, so that Hartree-Fock self-consistency is no problem; if the unperturbed wave function is a determinant of single-particle states, then these states must be plane waves. It would be possible that the best choice of an unperturbed wave function were not the condensed Fermi gas, but it is likely to be. There are solutions which correspond to an unstable state in which the density is less than the equilibrium density, and in which condensation has not occurred (condensation of a dilute system would imply the existence of long-range correlations in the wave function). These are states of negative pressure, and are unstable, although there must be some perturbation to make such states condense.

In the generalization of this method that must be used here, (3) is replaced by

$$H=H(g), \tag{4}$$

where $H(0)=\mathrm{H}_0$ and $H(1)=H$, so that the simple Rayleigh-Schrödinger expansion is a special case of this expansion. We expand in terms of g, and, for the same reasons as before, we can expect that observables will be given satisfactorily by the power series, if the elements of $H(g)$ can be expressed as a power series in g. Even if the elements of $H(g)$ cannot be expressed as a power series in g, it is quite possible that the expansion of some

[13] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

other quantity will converge to the right answer. If $H(g)$ is analytic, and the expansion of an observable converges, its sum should be the actual value of the observable. It is of course hoped that only a very few terms in the expansion need be taken into account if the form of $H(g)$ is chosen correctly.

The form of the Schrödinger equation which is used is

$$|\Psi(g)\rangle=|\Phi\rangle+\frac{P}{E_0-H_0}[H(g)-H_0-E(g)+E_0]|\Psi(g)\rangle, \quad (5)$$

where the symbols have the meanings which were given earlier in this section. The energy is given by

$$E(g)=\langle\Phi|H(g)|\Psi(g)\rangle=E_0+\langle\Phi|H(g)-H_0|\Psi(g)\rangle, \quad (6)$$

if $|\Phi\rangle$ is normalized to unity. The notation

$$H_n=[\partial^n H(g)/\partial g^n]_{g=0}/n! \quad (7)$$

is used, and similarly

$$\Psi_n=[\partial^n\Psi(g)/\partial g^n]_{g=0}/n!. \quad (8)$$

A power series expansion of $E(g)$ can now be made by using (6) and (5). The solution is written as

$$E(g)=\sum_{n=0}^{\infty}E_n g^n. \quad (9)$$

The nth derivatives of (5) and (6) give

$$|\Psi_n\rangle=\frac{P}{E_0-H_0}\sum_{r=1}^{n}(H_r-E_r)|\Psi_{n-r}\rangle$$

$$=\frac{P}{E_0-H_0}\sum_{\substack{k_1+\cdots+k_s=n\\ k_i>0}}(H_{k_1}-E_{k_1})\times\frac{P}{E_0-H_0}\cdots\frac{P}{E_0-H_0}(H_{k_s}-E_{k_s})|\Phi\rangle, \quad (10)$$

$$E_n=\sum_{r=1}^{n}\langle\Phi|H_r|\Psi_{n-r}\rangle. \quad (11)$$

These can be combined to give the general formula

$$E_n=\sum_{\substack{l_1+\cdots+l_m=n\\ l_i>0}}\ \sum_{\substack{k_1+\cdots+k_{m-1}=m-1\\ k_i\geq 0}}(N_k+1)^{-1}\times\langle\Phi|H_{l_1}\mathcal{G}_{k_1}H_{l_2}\cdots\times\cdots\mathcal{G}_{k_{m-1}}H_{l_m}|\Phi\rangle, \quad (12)$$

where

$$\mathcal{G}_0=-|\Phi\rangle\langle\Phi|=P-1,\qquad \mathcal{G}_k=P(E_0-H_0)^{-k},\quad k>0, \quad (13)$$

and N_k is the number of times $\mathcal{G}_0$ occurs in the term.[14]

The advantage of this generalization of the method is that the elements of H do not have to be finite for the method to converge. It has not been assumed that the power series for $H(g)$ converges at $g=1$, or even that the elements of $H(g)$ are regular for $|g|<1$, but only that $E(g)$ is regular for $|g|<1$, which could be a weaker condition (it could also be a stronger condition).

3. EXPOSITION OF THE BRUECKNER METHOD

The actual Hamiltonian of the nuclear system is

$$H=\sum_{i=1}^{A}T_i+\sum_{ij}v_{ij}, \quad (14)$$

and is the sum of a single-particle part, the kinetic energy, and a two-particle part, the potential energy. The second summation on the right of (14) denotes a sum over all distinct pairs of particles, and this notation will be used throughout the work. To keep Φ as a determinant of single-particle wave functions, H_0 need not be chosen as a sum of single-particle operators, but the analysis is very much simpler if it is so chosen. It need not be a local operator, and it is not in the Hartree-Fock method. In fact, to preserve translation and rotation invariance in the nuclear matter problem, H_0 must be diagonal in momentum space, with elements equal to $\sum[T(k)+V(k)]$. The correct choice of H_0 is very important in order to get rapid convergence of the expansion, and it will be chosen much as the Hartree-Fock Hamiltonian is chosen.

It would be possible to choose $H(g)$ in such a way that it included many-particle operators to take account of many-particle correlations at any early stage of the expansion, but such a choice will not be considered here, and $H(g)$ will be a sum of one- and two-particle operators for all values of g. In order to study the Brueckner method, a *reaction matrix* G is defined which satisfies the equation

$$G_\alpha=v+vP_\alpha e_\alpha^{-1}G_\alpha, \quad (15)$$

where P_α is some projection operator, diagonal in the single-particle states, e_α is the sum of two single-particle operators with the dimensions of energy; all these operators are acting on antisymmetric two-particle wave functions. The subscript α denotes that the choice of P_α and e_α, and hence of G_α, may depend on the context in which G occurs; this will be explained in detail later. We want to express v in terms of G, and to do this we define the potential to be the function of g which satisfies

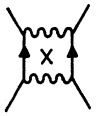
$$v_\alpha(g)=gG_\alpha-gv_\alpha(g)P_\alpha e_\alpha^{-1}G_\alpha. \quad (16)$$

G_α is kept constant, independent of g, and is defined by (15). $v(g)$ is then only equal to v, the real potential, for $g=1$, and is zero for $g=0$. The Hamiltonian is then written as

$$H(g)=\sum_i(T_i+V_i-gV_i)+\sum_{ij}v_{ij}(g), \quad (17)$$

[14] B. S. DeWitt, University of California Radiation Laboratory Report UCRL-2884, 1955 (unpublished), p. 115.

FIG. 1. Representation of the propagator $-P_\alpha e_\alpha^{-1}$.



which is in the same form as (4), so that the ideas developed in the previous section can now be applied. This is the basis of the treatment which will be given here.

In the calculations made by Brueckner and his collaborators,[1-8] the energy has always been worked out to first order in g, and the value of this is

$$E_0+E_1=\sum_{k<k_F} T(k)+\sum_{k_1,k_2<k_F}\langle \mathbf{k}_1\mathbf{k}_2|G_\alpha|\mathbf{k}_1\mathbf{k}_2\rangle, \quad (18)$$

where the ket $|\mathbf{k}_1\mathbf{k}_2\rangle$ is used to denote an antisymmetric two-particle wave function. Only if P_α and e_α are chosen suitably will this be a good approximation, and the choice of these depends on the choice of V, since we shall require some of the lower order terms in the expansion to be small; a variety of choices of P_α and e_α have been made in previous work, and some of these will be described later. The sums in (18) are over all momentum states with momentum less than the Fermi momentum k_F.

In analogy with the Hartree-Fock method, a single-particle energy, the *model energy*, is defined for a particle with momentum k as

$$\begin{aligned} M(k)&=T(k)+\sum_{k'<k_F}\langle \mathbf{k}\mathbf{k}'|G_\alpha|\mathbf{k}\mathbf{k}'\rangle \\ &=T(k)+V(k), \end{aligned} \quad (19)$$

and this defines H_0. It was suggested above that H_0 need not be a sum of single-particle operators, and an example of this would be for the definition of the G matrix used in (19) to depend on the state of excitation of the nucleus. Combining (17), (18), and (19), we get

$$\begin{aligned} E_0&=\sum_{k<k_F}[T(k)+V(k)], \\ E_1&=-\tfrac{1}{2}\sum_{k<k_F} V(k). \end{aligned} \quad (20)$$

Equation (12) as it stands is not in a convenient form for discussing the general terms of the perturbation expansion, for the many-body problem, but it can be shown that it is equivalent to a linked-cluster expansion like the one used by Goldstone.[10] The modification of his formalism for the Rayleigh-Schrödinger expansion which must be made is due to the terms H_l with $l>1$ which occur in (12). The H_l, defined by (7), are given term by term by the iteration of (16). At every place where there is a v interaction in the diagrams for the Rayleigh-Schrödinger expansion, there can be a series of G interactions, arising from Eq. (16), in the new expansion. The propagator $-P_\alpha e_\alpha^{-1}$ which must be used between these G interactions is represented by a cross, as shown in Fig. 1. It must be remembered that the operator P_α does not necessarily exclude states below the Fermi surface, and such states, behaving as particles and not as holes, will be shown by broken lines if there is any need to distinguish them. Φ is treated as the vacuum state in this formalism, so that a *hole* is created by removing a particle from the Fermi sea.

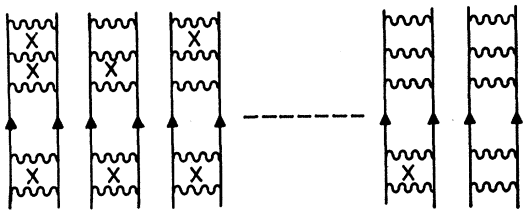


FIG. 2. A series of ladders of the same order.

An important concept for this theory is the "*ladder*" *part* of a diagram. It must be remembered that the vertices of these diagrams are ordered to understand this definition. A ladder consists of two *particle* lines (not hole lines) joined by a series of G interactions—the rungs of the ladder—so that no interaction occurs anywhere else in the diagram between the levels of the first and the last rung. Figure 1 can only occur as part or the whole of a ladder. The ladder may end in either particle lines or hole lines. The order of the ladder is one less than the number of rungs. A series of possible ladder diagrams of a certain order is shown in Fig. 2. The definition of P_α and e_α, and hence of G_α, will always be constant in a particular ladder, because we will wish the ladder diagrams to cancel with one another as nearly as possible, and each diagram of Fig. 2 will have the same set of G-matrix elements, but different propagators. These ladders are not identical with the ladders defined in Goldstone's paper.[10] Of particular interest is the "diagonal ladder" which begins and ends on the same pair of states.

Rules for computing the energy by means of diagrams in the Rayleigh-Schrödinger expansion have been given by Goldstone.[10] Certain modifications come from the use of a different expansion, so the rules are repeated here, with the necessary modifications. Particles are represented by lines going upwards, and holes by lines going downwards. One line must go into each vertex and one line out, and there are no external lines in the diagrams representing the ground-state energy, so that the particle and hole lines together form a system of closed loops. A vertex may be joined to itself or to another vertex at the same level only by a hole line, not by a particle line, and such an arrangement is called a *bubble*. G interactions are represented by wavy lines, and they always join two vertices through which fermion lines pass. A V interaction is represented by a dashed line ending in a cross --------×, and the other end goes to a vertex through which a fermion line passes. Both G and V lines are always drawn horizontally, so that a G

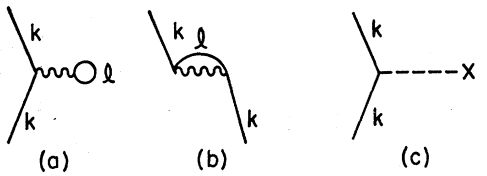


FIG. 3. The elements of graphs which will compensate one another by the definition of V.

interaction must join two vertices at the same level, called a *vertex pair;* otherwise the vertices must all be at different levels. Between the rungs of a ladder there may either be a cross or no cross. Both the G and V interactions conserve total momentum, spin, and charge in the nuclear matter problem. All distinct diagrams of any order, which are not composed of two or more parts not joined by any lines (*unlinked clusters*), must be drawn, and the fermion lines are given all possible combinations of momentum, charge, and spin. Two diagrams are distinct if the only difference between them is that the vertices occur in a different order. The *order* of a diagram is equal to the total number of G and V interactions which occur in it.

The contribution of a particular diagram is the product of the following factors. Each wavy line gives the matrix element of G and each dashed line gives the matrix element of V. The part of the diagram between two successive interactions gives the element of $(E_0-H_0)^{-1}$ unless it is part of a ladder with a cross to denote the factor $-P_\alpha e_\alpha^{-1}$; the (H_0-E_0) gives the sum of all the energies of the particle lines at that level, less the sum of all the energies of the hole lines at that level. Finally, each hole line gives a factor -1, and each closed fermion loop gives a factor of -1.

The scheme used by Goldstone[10] to describe the Brueckner theory is the following. $P_\alpha e_\alpha^{-1}$ is defined in such a way that all the ladder diagrams cancel exactly. For this to happen, P_α must be unity for states above the Fermi surface, and zero for states below the Fermi surface. $-e_\alpha$ must be equal to the unperturbed energy of the two-particle state on which it acts plus the excitation energy of the rest of the diagram at the same level as the ladder. Then $P(E_0-H_0)^{-1}$ will have the same value as $P_\alpha e_\alpha^{-1}$, and the sum of all the diagrams indicated in Fig. 2 will be zero. Therefore all diagrams which contain a ladder part can be dropped out. The G matrix depends on just one parameter, which is the excitation energy of the rest of the diagram at that level, $-W$, and so it will be written as $G(W)$. e_α then satisfies the equation

$$e_\alpha|\mathbf{m}_1\mathbf{m}_2\rangle=[W-M(m_1)-M(m_2)]|\mathbf{m}_1\mathbf{m}_2\rangle. \quad (21)$$

The difficulty we are now faced with is this: how is the self-consistent energy to be determined? The operator V which occurs as part of H_0 was originally defined by (19), but the G which occurs there is a function of W, and the value of W to be used has not been stated. The diagrams shown in Fig. 3 can only cancel exactly for the right value of the excitation energy of the rest of the diagram. 3(b) is just the exchange part of the bubble diagram. It can be seen that if any hole states enter or leave the vertices between which the G matrix acts, their energies must be subtracted from the energy of the rest of the diagram to get $-W$, since the hole lines overlap the ladder parts which were made to cancel by the definition of G, as is illustrated in Fig. 4. It is therefore tempting to define the energy of a hole by using $W=M(k)+M(l)$ to cancel the diagrams shown in Fig. 3. Only if this definition is used will (20) hold, but (18) is true anyway, so this definition is not necessary. The simplest corrections to (18) come from diagrams like those shown in Fig. 5, and 5(a) and 5(b) do not cancel with this definition, so it might seem better to define the hole energies to give some better cancellation of these terms. It will be shown in the next section, however, that there are further diagrams which do give exact cancellation when added to 5(a) and 5(b) if the hole energies are defined with $W=M(k)+M(l)$. The particle energies are defined to give some sort of average cancellation between 5(c) and 5(d). The G interaction in the middle of 5(c) is defined by the sum of certain ladders cancelling, and the rest of the diagram at that level (the *rest of the diagram* is everything at that level except the two particle lines in the ladder, as was explained above) consists of one excited pair, one hole line, and another hole line of momentum l from the bubble. V can be defined to cancel the G interactions only for a particular value of the energy of the pair and the hole.

The contributions of the type of diagrams shown in Fig. 5, and generally the difference between bubble and V interactions at the same place, can be written formally in a simple way. We suppose that a particular bubble diagram would have been cancelled by the corresponding part of the V matrix if the energy of the rest of the diagram had been $-W_0$, but in fact the energy of the rest of the diagram was $-W$. The remaining interaction therefore contains an element of $G(W)-G(W_0)$ as a factor. If the two-particle Hamiltonian is denoted by M, substitution of (21) in (15) gives

$$G(W)=v-vP(M-W)^{-1}G(W). \quad (22)$$

This can be written formally as

$$G^{-1}(W)=v^{-1}+P(M-W)^{-1}. \quad (23)$$

A combination of this with the same equation with W replaced by W_0 gives

$$\begin{aligned}G^{-1}(W_0)&=G^{-1}(W)+P(M-W_0)^{-1}-P(M-W)^{-1}\\&=G^{-1}(W)+P(W_0-W)\\&\quad\times(M-W_0)^{-1}(M-W)^{-1}, \quad (24)\end{aligned}$$

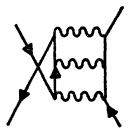


FIG. 4. A ladder part with one hole line and one particle line at each end.

and so

$$G(W)-G(W_0) = G(W_0)\frac{P(W_0-W)}{(M-W_0)(M-W)}G(W_0) \times\left[1-\frac{P(W_0-W)}{(M-W_0)(M-W)}G(W_0)\right]^{-1}. \quad (25)$$

This expression can be developed as a power series in $G(W_0)$, and if this is done a series of ladder-like diagrams is obtained which must be included in the complete diagram.

If complete cancellation of all bubble diagrams, as well as complete cancellation of all ladder diagrams, is required, then we must remove the condition that H_0 is a sum of single-particle operators, retaining, of course, the condition that it be diagonal in the representation which we have been using. We can still write it as the sum of operators V, but the V matrices will depend on the excitation of the rest of the diagram, since they are meant to cancel with the bubble diagrams. Since the energy of the rest of the diagram depends on the definition of V, there is a complicated self-consistency requirement. This problem of self-consistency is not the one which is solved in Appendix A of Brueckner and Gammel's paper,[8] as can be seen immediately from the fact that they do not modify the definition of hole energies on account of the excitation of the rest of the diagram, and so Figs. 5(a) and 5(b) do not cancel with one another.

4. CHOICE OF THE ENERGY DENOMINATORS

Since complete self-consistency, in the sense discussed in the last paragraph, seems very difficult to attain, all the calculations reported so far have used less stringent conditions to define the G and V matrices. In order to show how to assess the importance of self-consistency in the choice of the energy denominators, we will discuss a model in which the exclusion principle in intermediate states is treated exactly (in the defining equation of G), but the energy denominator is not made to satisfy any self-consistency principle. The perturbation expansion can be carried out as before, but there will be less cancellation of terms than there was in the previous section. A study of the low-order terms in the expansion will show how the energy denominators should be chosen without the requirement of exact self-consistency.

In the model which will now be discussed, an energy $e(k)$ is associated with each particle state and with each hole state; the form of this function $e(k)$ is as yet arbitrary. $G(W)$ is defined by (22), where M is just the sum of the $e(k)$ for the two-particle states on which it acts. If G occurs as part of a ladder, the W which is used to define G depends only on the four states with which the ladder begins and ends. If G does not occur as part of a ladder, the W which is used depends in just the same way on the four states which enter and leave the vertices at which it acts. A bubble counts as one hole line entering and one hole line leaving the vertex. We will only be concerned with diagonal ladders at present, so there are only two different lines at the vertex. If both lines are particle lines, as in Fig. 2, we put $W=-2\Delta$, where Δ is a constant. If one line is a particle line, and the other a hole line of momentum l, as in Fig. 4, we put $W=-\Delta+e(l)$. If both the lines are hole lines, we put $W=e(l_1)+e(l_2)$. This definition of the G matrices will be justified and a suitable value of Δ chosen later, when we try to make an approximate cancellation of certain low-order diagrams.

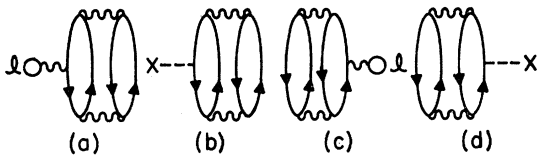


FIG. 5. The simplest ground-state diagrams which compensate one another by the definition of V.

The first-order energy is given by (18), and there is no need to define H_0 in order to evaluate that. The G which occurs there acts between two bubbles of momentum k_1 and k_2, and so, in accordance with the rules given in the previous paragraph, $W=e(k_1)+e(k_2)$. V will now be defined so that it exactly cancels all bubble diagrams as shown in Fig. 3. The rules of the previous paragraph give

$$V(l)=\sum_{l'<k_F}\langle \mathbf{ll'}|G[e(l)+e(l')]|\mathbf{ll'}\rangle, \quad (26)$$

$$V(m)=\sum_{l<k_F}\langle \mathbf{lm}|G[e(l)-\Delta]|\mathbf{lm}\rangle, \quad (27)$$

where $\mathbf{l}$ is a hole state and $\mathbf{m}$ is a particle state. The propagators to be used in the perturbation expansion are now defined, since (19) gives the single-particle energies in the unperturbed system.

The first thing to do is to combine the ladder diagrams of the same order, as shown in Fig. 2, into one term. This is quite simply done by using as a propagator the sum of the two propagators which occur, which is

$$P(E_0-H_0+M-W)(E_0-H_0)^{-1}(M-W)^{-1}. \quad (28)$$

An asterisk is placed between the rungs of a ladder to show that this has been done. This was made possible by keeping the definition of G constant within a ladder.

In Fig. 6 are shown a number of types of diagram which give corrections to the energy. Figures 6(a), (b), (c), and (d) give all the second- and third-order diagrams, since (e) is an exchange part of (d). It must be noticed that only (a), (b), (f), (g), and (n) would be zero in the completely self-consistent scheme, and all the other diagrams would give contributions. (c), the scattering of a hole from a hole, is a kind of four-particle cluster, and (d), (e), and (k) are three-particle clusters,

as are some of the other diagrams shown.[5,13] (o) is an example of a case in which there is no cancellation of what would otherwise be a ladder part, because of an interaction which occurs in another part of the diagram at the same level. This is an unsatisfying feature of the theory as it has so far been developed, and will be discussed in a forthcoming paper. It must be observed however, that if there were no asterisk in the ladder part of (n), it would give a larger contribution than (o), since the extra energy denominator which occurs in it is larger than the one which occurs in (o); this gives some justification for the scheme in which a ladder is only called a ladder if no other vertex occurs at the same level. Figures 6(g) and 6(m) would cancel exactly if $e(k)=M(k)$, which is the justification of the definition of the hole energies by (26). 6(h) and 6(i) might be important, and these will be discussed in Sec. 10.

The impression given by a study of Fig. 6 is that the cancellation of the ladder diagrams becomes less and less important as the excitation of the rest of the diagram becomes higher, since there will be more and more similar diagrams which are not cancelled even in the completely self-consistent scheme. If we can cancel out (a), (b), and higher order ladders of that kind, and (f) and higher order ladders of that kind, and also (g) and (n), then we will have got the theory as refined as is reasonable in its present form.

The propagator in a ladder diagram like 6(a) or 6(b) is obtained by substituting the rules for W in (28), and the result of this is

$$\frac{M(l_1)+M(l_2)-M(m_1)-M(m_2)+e(m_1)+e(m_2)-e(l_1)-e(l_2)}{[M(l_1)+M(l_2)-M(m_1)-M(m_2)][e(m_1)+e(m_2)-e(l_1)-e(l_2)]}, \tag{29}$$

where l_1, l_2 are the hole states and m_1, m_2 are the particle states. This expression shows at once the importance of self-consistency in the definition of the energies, since, if $e(k)$ is exactly the same function as $M(k)$, then (29) is exactly zero. It is often more convenient not to attempt exact equality of these functions, but to take $e(k)$ as a quadratic function which fits $M(k)$ near $k=k_F$; this is known as the *effective mass approximation*, and has been used by several authors.[4,6,13,15] Expression (29) should then be used to check whether the contribution of diagrams 6(a), 6(b), and so on is small. If 6(a) gives a contribution less than the three-particle cluster terms, then there is no point in improving the self-consistency.

If we denote the G matrix defined by the use of a function $e(k)$ by $G\{e\}$, then we can get the not very surprising result that the sum of diagrams 6(a), 6(b), etc., is just $G\{M\}-G\{e\}$. This can be proved by using (23) to get an equation similar to (25).

Next we try and reduce the contributions from diagrams like 6(f) by a suitable choice of Δ. It will be assumed that $e(k)$ is the same as $M(k)$ for this discussion, since, if this were not true, the modification would be much less than it was for 6(a) and 6(b). If the hole and particle lines are labeled 1, 2, 3 from left to right, and the intermediate state of the second particle is m_2', then the propagator is, by use of (28),

$$\frac{M(l_1)-M(m_1)+M(l_2)+\Delta}{\{M(l_1)-M(m_1)+M(l_2)-M(m_2')+M(l_3)-M(m_3)\}\{M(m_2')+M(m_3)-M(l_3)+\Delta\}}. \tag{30}$$

Δ should be chosen so as to make this expression small on the average, so it should be equal to the excitation energy of a typical pair less the energy of a typical hole, and $-M(0)$ should be a good order of magnitude for it.

Now let us make a small change in Δ. This will alter the particle energies and hence the G matrices, and so produce a change in the first-order energy. This change in the first-order energy must be exactly compensated by a change in the higher-order energies, since the total energy is not changed merely by a change in the perturbation procedure. The most direct effect it could have, which is also likely to be the most important higher-order effect, is to alter (30) by an amount determined by taking its partial derivative with respect to Δ, equal to $-[M(m_2')+M(m_3)-M(l_3)+\Delta]^{-2}$, and so to make a change in the contribution from 6(f). If these assumptions are correct, it can be seen that an increase in Δ should lead to an increase in E_0+E_1. Moreover, if we multiply the derivative of (30) by $M(k_F)-M(0)$, we should get a quantity greater in magnitude than (30). Therefore, if we vary Δ about a reasonable value by an amount $M(k_F)-M(0)$, the change in the first-order energy provides an outside estimate of the contribution from 6(f). Something very like this has been done by Brueckner and Gammel,[8] and they do indeed find that the effect is very small.

It can be shown quite simply that if we sum over all the ladder parts which can occur in a diagram, we get back to the expansion proposed by Goldstone, which was explained in Sec. 3. We suppose that each of the series of elements shown in Fig. 7 is part of a larger diagram, that the energy of the rest of the diagram at that level is W, and that the G matrices were defined with $W=W_0$; the two-particle Hamiltonian will be understood for H_0, and M will denote the two-particle operator used in the definition of G by (22). If K is written for the sum of the diagrams in Fig. 7, and is

[15] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1957).

regarded as an operator, it obeys the equation

$$K=G-GP(H_0-W)^{-1}K+GP(M-W_0)^{-1}K. \quad (31)$$

This reduces to

$$G^{-1}=K^{-1}+P(M-W_0)^{-1}-P(M_0-W)^{-1}, \quad (32)$$

and comparison with (24) shows that K is just the G matrix used by Goldstone.[10] The two ways of estimating the corrections to the first-order energy are therefore equivalent. In the first, the ladder diagrams were made to cancel exactly by making G depend on the energy of the rest of the diagram; this made us unable to cancel all the bubble diagrams. In the second, all the bubble diagrams were eliminated by making G independent of the rest of the diagram, but the ladder diagrams would not all cancel.

5. THE EXCLUSION PRINCIPLE

The operator P_α which is used in (15) has so far been defined so that it excludes all states below the Fermi surface. The methods used previously can be extended to treat the case in which P_α is some other single-particle operator, with eigenvalues between zero and one, and the only difference will be in the propagators which must be used in the ladder diagrams. To illustrate how such a case is treated, we will take P_α to be just the principal-value operator, which is unity unless the denominator is zero. Particles may now propagate in a ladder part below the Fermi surface, and such "particles" are denoted by broken lines in the diagrams. Figure 8 shows some of the new diagrams which now contribute to the energy. Figures 8(a) and 8(b) seem to be the most important corrections, and their total contribution is

$$\sum_{l_1,l_2,l_3<k_F}\sum_{m_1>k_F}\frac{|\langle \mathbf{l}_1\mathbf{l}_2|G|\mathbf{l}_3\mathbf{m}_1\rangle|^2}{M(l_3)+M(m_1)-M(l_1)-M(l_2)+\frac{1}{2}D}+\sum_{l_1,l_2,l_3,l_4<k_F}\frac{|\langle \mathbf{l}_1\mathbf{l}_2|G|\mathbf{l}_3\mathbf{l}_4\rangle|^2}{M(l_3)+M(l_4)-M(l_1)-M(l_2)}, \quad (33)$$

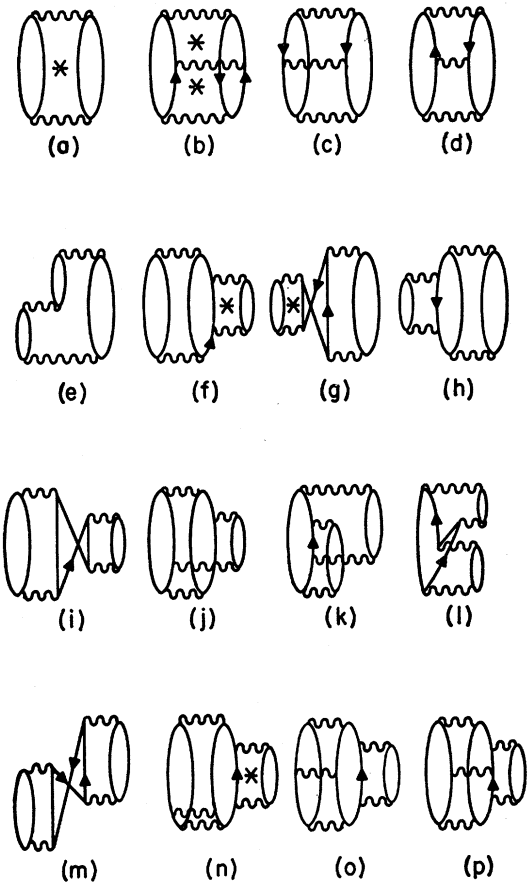


FIG. 6. Some of the simpler ground-state diagrams which remain after the bubble parts have been cancelled by use of (26) and (27).

where $\frac{1}{2}D$ is a constant which is added to the particle energies in the energy denominator which defines G, for reasons which will become clear. Owing to the presence of this constant D, the contribution of Fig. 6(a) is now

$$-\sum_{l_1,l_2<k_F}\sum_{m_1,m_2>k_F}\frac{|\langle \mathbf{l}_1\mathbf{l}_2|G|\mathbf{m}_1\mathbf{m}_2\rangle|^2 D}{[M(m_1)+M(m_2)-M(l_1)-M(l_2)][M(m_1)+M(m_2)-M(l_1)-M(l_2)+D]}. \quad (34)$$

If D is zero, then the second term of (33) vanishes,[6] the first term is positive, and (34) is zero. As D is increased, the first term of (33) gets smaller, going to zero when D becomes large, and (34) goes steadily to a negative limit. The approximation will therefore be improved if D is made equal to some positive constant, probably less than the Fermi kinetic energy, but it is only by trial that a suitable value can be found.

For $D=0$ the terms represented by 8(c) and 8(d) are also positive, since, even if some of the particles in the intermediate state are below the Fermi surface, the total energy is positive, and the rest of the diagram just gives the square modulus of some matrix element. Therefore they cannot cancel with the first term of (33), and the convergence would be better if 6(f) were made to help to cancel them. This means that the convergence would be better if the model energies were the ones proposed by Bethe[13] and Goldstone,[10] with the exclusion

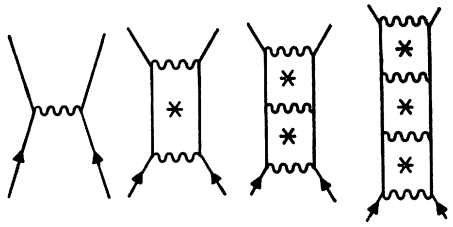


FIG. 7. A series of ladder parts of successive orders.

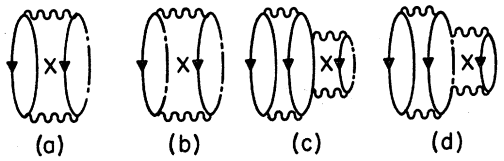


FIG. 8. Some simple ground-state diagrams when the exclusion principle is neglected in intermediate states. The broken lines represent particles propagating below the Fermi surface.

principle used in intermediate states. This appears to contradict the remark of Brueckner and Wada[6] that the change in self-consistent energy largely compensates for the neglect of the exclusion principle. The reason for this will be discussed below, but it is essentially just because the lower effective mass has a similar effect to D in (33) and (34) that they got this result.

The choice of P_α as the principal-value operator has been made in a number of different calculations, since (15) can then be transformed into a differential equation, and it has been discussed in some detail here to show what effect the choice of P_α has on the validity of the approximation. It is also convenient and much more accurate to choose P_α so that it is a function of the relative momenta of the two particles only, and it will then be chosen from geometrical considerations as some function varying smoothly between zero and one. Such a scheme can be treated in the same way, and there will be a tendency for Figs. 6(a) and 8(a) and 8(b) to cancel.

6. THE SUM OF TWO POTENTIALS

Equation (15) is not a linear equation, so it is not solved by separating the potential into the sum of two parts and solving for each part separately. However, such a separation might be used as the starting point for a perturbation calculation. The solution of (15) for a pure repulsive core is well known,[6,15,16] and it is also known that the potential itself is a good solution of (15) below the Fermi surface for a nonsingular potential.[13,17] If the actual potential v is the sum of a potential w for which (15) can be solved to give the solution G and a nonsingular potential u, then we can replace (16) by

$$w(g)=gG-gw(g)Pe^{-1}G, \tag{35}$$

$$v(g)=w(g)+gu, \tag{36}$$

and use (36) as the basis of the expansion. If we still denote the G interaction by a wavy line, and the u interaction by a dashed line, then the diagonal elements of u must be added to (18), and, in addition to the diagrams of Fig. 6, there will be diagrams like those of Fig. 9 as corrections to the energy.

The single-particle energy $M(k)$ which is used in the perturbation expansion will be defined so as to cancel the bubble diagrams with both G and u interactions, so the correct energy denominator to be used is not given by the self-consistent solution of (15), but by the self-consistent solution of the whole problem. The most important class of corrections to the energy is likely to be the one shown by Figs. 9(a), (b), and (c), and the higher-order diagrams consisting of just a single ladder. Figure 9(a) is known to be unimportant for a well-behaved potential; the addition to the energy of 9(b) and 9(c) is equivalent to replacing the expectation value of u in the model state Φ by its expectation value in the state $(1+Pe^{-1}G)\Phi$. Since this is the solution of the Schrödinger equation which is solved to find the G matrix, this correction would be quite convenient to make.

It is also possible to use the solution of (15) for the potential w as a starting point for an iterative solution with the potential v; the solution with w will be called F and the solution with $v=u+w$ will be called G. The equation for F gives

$$w=v-u=(1+FPe^{-1})^{-1}F, \tag{37}$$

and substitution of this in the equation for G gives

$$\begin{aligned} G=&[1-uPe^{-1}-(1+FPe^{-1})^{-1}FPe^{-1}]^{-1} \\ &\times[u+(1+FPe^{-1})^{-1}F] \\ =&(1-uPe^{-1}-FPe^{-1}uPe^{-1})^{-1}(u+F+FPe^{-1}u), \end{aligned} \tag{38}$$

which can be expanded as an iterative equation for G.

7. DISCUSSION OF PUBLISHED CALCULATIONS

A number of calculations of the energy of nuclear matter at various densities have been made on the basis of Eqs. (15) and (18). They have used various definitions of P_α and e_α in order to simplify the completely self-consistent scheme. The validity of all these approximations can be discussed in terms of the theory which has been presented here, and a brief discussion of several of them will be given.

(*a*) *Brueckner and Gammel.*[7,8]—The calculations by these authors come closer than any others to satisfying the self-consistent energy condition, and use the potentials deduced by Gammel, Christian, and Thaler[18] from nucleon-nucleon scattering data as their starting point. The exclusion principle was used in intermediate states, although, for a given center-of-mass momentum and relative momentum, its effect was averaged over angles; this should indeed be a very good approximation. The energy denominators were defined in a way very similar to the scheme described in Sec. 4, with the self-consistency condition $e(k)=M(k)$ obtained by iteration. The main difference is that they treated the particle energies more precisely, so that their energies depend on the hole which was created with them. In diagram 6(f) the second hole energy has been allowed for exactly, and the only reason that this diagram is not exactly zero is that the excitation energy of the pair on the left has been allowed for in the definition of G only

[16] D. J. Thouless, Proc. Roy. Soc. (London) **A239**, 108 (1957).
[17] D. J. Thouless, Phys. Rev. **107**, 559 (1957).
[18] Gammel, Christian, and Thaler, Phys. Rev. **105**, 311 (1957).

by an average value. That this is unimportant was shown by the fact that the total energy was very insensitive to the choice of this average excitation energy. The sign of the energy change was the one derived in Sec. 4, and its magnitude shows that the contribution of Fig. 6(f) cannot be more than 1 Mev per particle.

(*b*) *Bethe and Goldstone.*[15]—The calculation for a repulsive core carried out by these authors used $P_\alpha=0$ for relative momenta of the two particles less than k_F and $P_\alpha=1$ for relative momenta greater than k_F. This was only done for center-of-mass momentum zero, and for other values of the total momentum corrections would have to be taken into account. They used the effective mass approximation for the energy denominator.

(*c*) *Brueckner and Wada.*[6]—This paper describes calculations done with a square well potential outside a repulsive core. The exclusion principle is neglected for the calculation of G, and P_α is taken to be the principal-value operator. The exclusion principle is then taken into account by using the value of G found initially in an expression which is stationary to first order in G. Although the procedure is not equivalent, it is as good as evaluating the contributions of Figs. 8(a) and 8(b). The remark that the effect of the exclusion principle is small must be treated with caution, as was pointed out in Sec. 5. Almost the same value for the total energy was obtained with and without the exclusion principle, but a different effective mass was used in each case, on grounds of self-consistency. It was pointed out in Sec. 5 that, even if the exclusion principle is neglected in intermediate states, the same energy denominators ought to be used. Brueckner and Wada used the variation of the hole energies from momentum zero to the Fermi momentum to find the self-consistent energy, although what is actually wanted is the variation in the region of the Fermi surface. When the exclusion principle is not used, the second term of (33) makes an important contribution to the hole energy, being positive for zero momentum and negative for the Fermi momentum, although averaging out strictly to zero. If this term is neglected, as it is if only first-order terms are considered, the energy is lowered at zero momentum and raised at high momenta, so the "self-consistent" effective mass is lowered. Although this choice of effective mass cannot be justified on grounds of self-consistency as better than the higher effective mass obtained when the exclusion principle is used, it does have much the same effect as adding a constant to the particle energies, which certainly should improve the approximation. Such a treatment of the exclusion principle has been used in Appendix B of Brueckner and Gammel.[8]

8. EQUATION FOR THE SEPARATION ENERGY ON THE FERMI SURFACE

It can be shown quite simply that the condition for matter to be in equilibrium at zero pressure is that the average energy of the particles is equal to the energy required to remove one particle at constant volume[13]; this is just a thermodynamic relation. The average energy of a particle in nuclear matter is the coefficient of A in the semiempirical mass formula. The energy required to remove one particle at constant volume is the separation energy of a particle on the Fermi surface, since changing the number of particles at constant volume has the effect of changing the Fermi momentum.

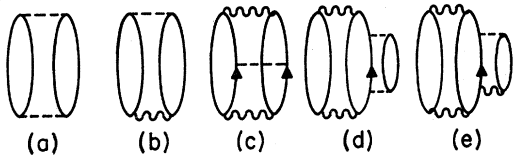


FIG. 9. Some simple ground-state diagrams when (36) is used as the basis of the expansion.

The results of Brueckner and Gammel[8] do not give equality of the average energy, E_B, and the model energy at the Fermi surface, $M(k_F)$, but the two differ by about 15 Mev at equilibrium density, with $M(k_F)$ less than E_B. This means that the model energy cannot be interpreted as the energy lost by the removal of the particle, but there is also a *rearrangement energy*. This rearrangement energy must correspond to the change in the correlation structure of the nucleus when a particle is removed, and it is manifested by a change in the effective interaction between the particles, given by the operator G.

The model energy of a particle with momentum $\mathbf{k}$, $M(k)$, is defined by (19) as the kinetic energy $k^2/2m$ plus the sum of the expectation values of the operator G over all pairs of particles which contain the one particle. G is defined by

$$G=v+vPe^{-1}G, \tag{39}$$

where v is the potential energy operator, P is the projection operator which excludes pair states with one or more particles in the Fermi sea, and e is a diagonal matrix equal to the energy of the initial state minus the energy of the intermediate state of the two particles. We are interested in the diagonal element of G for a pair of particles both in the Fermi sea, since the total energy is the total kinetic energy plus the sum of the diagonal elements of G for all distinct pairs of particles in the Fermi sea. This definition of G and its applications were discussed at length in Secs. 3, 4, and 5.

If G were kept constant after a particle on the Fermi sphere, of momentum $\mathbf{k}_F$, had been removed from the medium, with the volume kept constant, then the energy loss computed would be the kinetic energy plus the interaction energy of the particle with all the other particles, and this is just the model energy $M(k_F)$ as it is in the Hartree-Fock theory. To be consistent it is necessary to recalculate the G matrix, since both P and e in Eq. (39) are changed by removing the particle. Only if this is done will the equality of separation energy and

average energy be obtained, since the optimum density was found by Brueckner and Gammel[8] calculating the form of G appropriate to each density. Since the change in G made by removing one particle is of relative order A^{-1} (A is the total number of particles), only first-order changes need be considered, and the separation energy is

$$M(k_F)-\sum_{l_1,\ l_2<k_F}\langle \mathbf{l}_1\mathbf{l}_2|\delta G|\mathbf{l}_1\mathbf{l}_2\rangle+O(A^{-1})$$
$$=M(k_F)-R(k_F)+O(A^{-1}), \quad (40)$$

where $R(k_F)$ is the *rearrangement energy.*

Differentiation of (39) gives

$$\delta G=v\underset{e}{\text{—}}G-v\underset{e}{\text{—}}\delta e\underset{e}{\text{—}}G+v\underset{e}{\text{—}}\delta G, \quad (41)$$

and the solution of this is

$$\delta G=G\underset{e}{\text{—}}G-G\underset{e}{\text{—}}\delta e\underset{e}{\text{—}}G. \quad (42)$$

The first term on the right of this expression represents the effect of the change in the exclusion principle, and, if G were defined without the exclusion principle in intermediate states, as it is in part of the paper by Brueckner and Wada,[6] this term would be absent. δP is equal to 1 for pair states in which one member of the pair has momentum $\mathbf{k}_F$ and the other is outside the Fermi sea, since such an intermediate state is now allowed, but it was forbidden before the particle of momentum $\mathbf{k}_F$ was removed. δP is zero for all other states. Explicitly this gives a contribution to the diagonal part of δG equal to

$$|\langle \mathbf{l}_1\mathbf{l}_2|G|\mathbf{m}\mathbf{k}_F\rangle|^2$$
$$\times[M(l_1)+M(l_2)-M(m)-M(k_F)]^{-1} \quad (43)$$

for $m>k_F$, where $\mathbf{l}_1+\mathbf{l}_2=\mathbf{m}+\mathbf{k}_F$. If $m<k_F$, the change in the matrix element is zero. The contribution to the rearrangement energy is obtained by substituting this in (40). It is important to notice that (43) cannot be positive, and so the energy required to remove a particle is reduced by this term. In Goldstone's notation,[10] it can be represented by a diagram like Fig. 10, together with some diagrams obtained from this by interchanging the labels of the lines.

The second term in (42) gives the effect due to the change in energy of the initial and intermediate states which results from the removal of one particle. To a first approximation the energy change comes from the loss of the interaction with the particle $\mathbf{k}_F$, which is represented by diagrams like Fig. 11. There are also higher order terms which come from the change in G producing a change in the particle and hole energies in intermediate states, and these are represented by diagrams like those in Fig. 12. If the rearrangement energy is small compared to the potential energy, these terms will be small compared to the rearrangement energy, and can be neglected. This seems to be the case in Brueckner and Gammel's calculations.

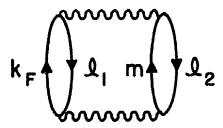


FIG. 10. Graph representing the contribution to the rearrangement energy due to the exclusion principle in intermediate states.

9. EXCITED STATES

We know that states of nuclear matter with a finite number of particles excited above the Fermi surface are not stable, and must always decay in such a way that more particles are excited. The statistical weight of states with an extra particle excited increases indefinitely with A, so that the only stable excited states are those with a number of particles of order A excited; this is true unless there is an energy gap at the Fermi surface, so that an excited particle cannot lose enough energy to excite another without going back to the Fermi sea. There is some evidence that such an energy gap exists in nuclear matter,[19] and it is an essential feature of theories of superconductivity.[20,21] We will not consider the existence of an energy gap here, but will assume that all these excited states are unstable. If there is an energy gap, it will have a serious effect on particles whose excitations are comparable with the magnitude of the energy gap, but the situation will not be changed by the gap for more highly excited particles. If these states decay slowly, their energy will be of interest, and can be calculated by Brueckner's methods. The *excitation energy* $E(m)$ of a state $\mathbf{m}$ outside the Fermi sea is defined as the energy of a system which has one particle of momentum $\mathbf{m}$ in addition to the degenerate Fermi gas, less the energy of the undisturbed Fermi gas. The excitation energy $E(l)$ of a state inside the Fermi sea is the energy of the undisturbed Fermi gas, less the energy of a system with a particle missing from the state $\mathbf{l}$. The rearrangement energy of a state with an extra particle is the excitation energy less the model energy, and the rearrangement energy of a state with a particle missing is the model energy less the

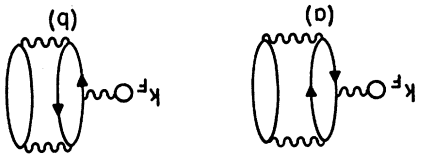


FIG. 11. Graphs representing the lowest-order contribution to the rearrangement energy due to the change in the energy denominator.

[19] C. De Dominicis and P. C. Martin, Bull. Am. Phys. Soc. Ser. II, **3**, 224 (1958).
[20] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).
[21] N. N. Bogoljubov, Nuovo cimento **7**, 794 (1958).

excitation energy:

$$E(l)=M(l)-R(l),$$
$$E(m)=M(m)+R(m). \tag{44}$$

If only a few particles are excited, the excitation energies can be added, since their mutual interactions and their effect on the medium are both of order A^{-1}, and so the interference between them is negligible, in general. The energy of an excited state is calculated by adding together the excitation energies of the particles and then subtracting the excitation energies of the holes. This is equivalent to adding together the model energies of the particles, subtracting the model energies of the holes, and then adding the rearrangement energies of the holes and particles. A change in the volume of the system can be simulated by removing particles from or adding particles to the Fermi surface, and, for excited states as well as for the ground state, this gives no first-order change in the energy at optimum density. This remark implies that the density of the nucleus cannot adjust itself when an excited particle is added to give a lower energy. There can, however, exist excitations for which the energies are not additive, as, for example, when a hole and a particle are coupled together to give a state

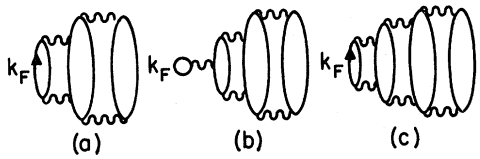


FIG. 12. Higher order contributions to the rearrangement energy.

of lower energy than the individual energies of the hole and particle. Such an excitation is called "zero sound".[22]

The rearrangement energy can be obtained just as it was obtained in Sec. 8, and it is equal to

$$R(k)=\sum_{l_1,l_2<k_F}\langle \mathbf{l}_1\mathbf{l}_2|\,\delta G\,|\mathbf{l}_1\mathbf{l}_2\rangle. \tag{45}$$

If a particle of momentum $\mathbf{l}$ is removed from the Fermi sea, δG is given by (42), where δP is 1 for pair states with one particle of momentum $\mathbf{l}$ and the other outside the Fermi sea, and is zero otherwise. δe is the interaction energy of the missing particle with the intermediate state less its interaction with the initial state (the sign comes from the definition of e as the energy of the initial state less the energy of the intermediate state). If a particle of momentum $\mathbf{m}$ is added outside the Fermi sea, δG is still given by (42), where δP is -1 for pair states with one particle of momentum $\mathbf{m}$ and the other particle somewhere outside the Fermi sea, and is zero otherwise. δe is the interaction energy of the added particle with the initial state less its interaction with the intermediate state. The diagrams which give the rearrangement energy for these excited states are Figs. 10, 11, and 12 with $\mathbf{k}_F$ replaced by $\mathbf{l}$ or $\mathbf{m}$. It must be noticed that, in the calculation of $R(k)$, the state $\mathbf{k}$ comes into the diagrams as a bubble or as a particle line, never as a hole line, whether the state was initially occupied or unoccupied.

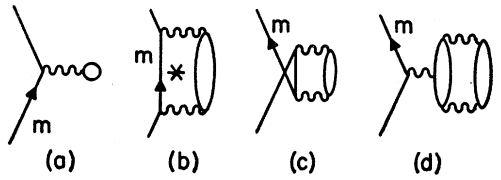


FIG. 13. Simplest contributions to the excitation energy of a particle with momentum $\mathbf{m}$.

There is another way of deriving the equation for the rearrangement energy which sheds light on the reason for the difference between model energy and excitation energy. In Sec. 8 it was treated as if it came from a change in the interaction between the other particles, firstly because of the alteration of the exclusion principle in intermediate states, secondly because of the change in the energy spectrum which in turn changed the G matrices. The method developed in Sec. 3, however, allows a perturbation expansion to be made with a variety of differently defined G matrices, so it is also possible to make the perturbation expansion for the system of degenerate Fermi gas plus one particle using the G defined for the undisturbed degenerate Fermi gas.

Since the G matrix is the same as it was for the ground-state calculations, there is now no change in the interactions between the particles in the Fermi sea, and we just have to consider the interaction between the additional particle and the rest of the nucleus. Graphs of this can be made by representing the extra particle as an external line, just as the scattering of a real particle is represented in field theory. The simplest diagrams which contribute to the excitation energy are shown in Fig. 13. Figure 13(a) represents the simple interaction of a particle with the rest of the nucleus, and is the major part of the potential energy. Figure 13(b) is a ladder diagram; it would give zero if the Δ of (27) were zero, and this shows that the off-energy-shell propagation which was allowed for in the definition of $M(m)$ no longer occurs. The model energy defined on the energy shell is the right energy to use here. Figure 13(c) is equivalent to Fig. 10; it represents the fact that we had allowed for a ground-state excitation, in the definition of the G matrix, which is now forbidden because one of the intermediate states is occupied by the extra particle.

In order to calculate the expectation value of a single-particle operator O (for example, the magnitude of the momentum), which is $\langle\Psi|O|\Psi\rangle/\langle\Psi|\Psi\rangle$, it would be necessary to add to its expectation value in the model

[22] L. D. Landau, J. Exptl. Theoret. Phys. U.S.S.R. **32**, 59 (1957) [translation: Soviet Phys. JETP **5**, 101 (1957)]; V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 139 (1958) [translation: Soviet Phys. JETP **7**, 96 (1958)].

FIG. 14. Simplest graphs representing the ground-state expectation value of a single particle operator O.

ground-state wave function (the degenerate Fermi gas) a sum of linked graphs, each of which contains the operator O once. The two most important graphs are shown in Fig. 14, where a square on a line means that the expectation value of O for the single-particle state represented by that line must be multiplied by the usual contributions from the rest of the diagram; the square counts as a vertex, and so the energy denominator comes twice in each of the diagrams of Fig. 14. The sum of diagrams like 13(a) and 13(d) then gives approximately the expectation value of the potential energy of the additional particle not in the model ground state, but in the true ground state. This gives a more directly physical interpretation of Fig. 11.

Figure 13(b) has one very important feature which must be observed. The intermediate state can have either less or more energy than the initial state, and so the propagator has a singularity. If the singularity is removed, following the example of scattering theory, by adding a small imaginary quantity to the denominator, an imaginary contribution to the energy is obtained. In fact, if the model energy of the particle state is defined on the energy shell, so that 13(b) gives zero, the G matrix itself has an anti-Hermitian part, and the model energy has an imaginary part. If the anti-Hermitian part of G is small, it is equal to $-i\pi G\delta(e)G$, where $\delta(e)$ implies that the intermediate states have the same energy as the initial state. This imaginary part of the energy comes naturally as a result of the possible transitions to the other states of the same energy, and gives the inverse lifetime of the state. The expression has been used by Bethe and Shaw[23] to calculate the imaginary part of the optical-model potential for nuclear matter.

The configurations in which one state in the Fermi sea is unoccupied can be treated in the same way. The empty state is represented by an external hole line, and some of the diagrams which contribute to the excitation energy are shown in Fig. 15. Figures 15(a) and (d) together give the expectation value of the potential energy in the true ground state. Figure 15(b) gives zero, since the hole-state energies were defined on the energy shell by (26). Figure 15(c) represents the hole scattering into another hole state by creating a pair (exciting one other particle), and then scattering back into its original state; this is a process not compensated by the definition of G. Again the propagator has a singularity, which contributes an imaginary part of the energy, and a hole state is not stable.

[23] H. Bethe and G. Shaw (private communication), also reported by H. Bethe at the Pittsburgh Conference on Nuclear Structure, 1957 (unpublished), p. 137.

It is interesting to notice that there is no gap in the excitation energy spectrum at the Fermi surface, since the same sort of terms arise in the evaluation of excitation energy for both holes and particles near the Fermi surface. There must be no gap if the average energy is a continuous function of density, since the excitation energy at the Fermi surface is equal to the energy gained by adding one more particle, or the energy lost by removing one particle. An energy gap can only exist if the excitation energies are not simply additive.

10. GROUND-STATE ENERGY

The ground-state energy in the Brueckner theory is the sum of the kinetic energies plus half the sum of the potential energies. For this to be true, the model energies must be used, and there is no justification for using the excitation energies. In the Hartree-Fock theory there does exist this relation between the ground-state energy and the excitation energies of hole states, so that the extent of its failure in the Brueckner theory represents the deviation from a true independent-particle system. The hole excitation energy is, in principle, a measurable quantity, so that there exists a

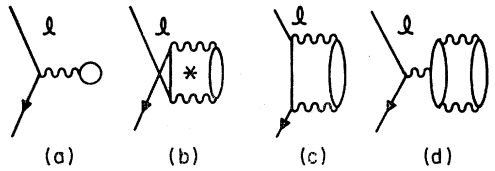


FIG. 15. Simplest contributions to the excitation energy of a hole with momentum $\mathbf{l}$.

physically measurable rearrangement energy of the nucleus equal to twice the total energy less the kinetic energy of the Fermi gas less the sum of the hole excitation energies. This would be zero if the Hartree-Fock theory were exact. Some interesting results have been obtained by several authors who use the assumption that it is zero.[24–26] There is a simpler way of expressing the deviation of the system from an independent-particle model, which is the density matrix in momentum space (probability distribution of a momentum state being occupied), which would be a step function if there were no deviation. The arguments of the previous section show that there is some indirect connection between the two. Experimental evidence for the form of the density matrix in momentum space has been discussed by a number of authors,[27,28] and there are several lines of evidence which suggest that the deviations from the Fermi distribution are considerable, although none

[24] V. F. Weisskopf, Nuclear Phys. **3**, 423 (1957).
[25] R. B. Hall and R. J. Eden, Nuclear Phys. **6**, 157 (1958).
[26] G. R. Satchler, Phys. Rev. **109**, 429 (1958).
[27] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955).
[28] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950); J. S. Levinger, Phys. Rev. **84**, 43 (1951); E. M. Henley, Phys. Rev. **85**, 204 (1952); P. A. Wolff, Phys. Rev. **87**, 434 (1952).

of them gives much evidence about the very high momentum components which are of particular importance for the rearrangement energy. A theoretical argument by Migdal[29] shows that there is a discontinuity in the momentum distribution at the energy at which the excitation energy equals the average energy E_B.

The effects we have discussed in this chapter do not alter the ground-state energy in the lowest orders, but they do have some relation to fourth- and higher-order terms in the perturbation series. For example, Figs. 13(c) and (d) can occur as insertions into a particle line just as 13(a) can. The simplest types of ground-state diagrams in which these insertions occur are shown in Fig. 16. Figures 16(a) and (e) are the same as Figs. 6(i) and (h) which are mentioned in Sec. 4. Any of these insertions can be made any number of times in the same line, and so they can be taken into account by a redefinition of the self-consistent particle energies which are used in the definition of G, just as the insertion of

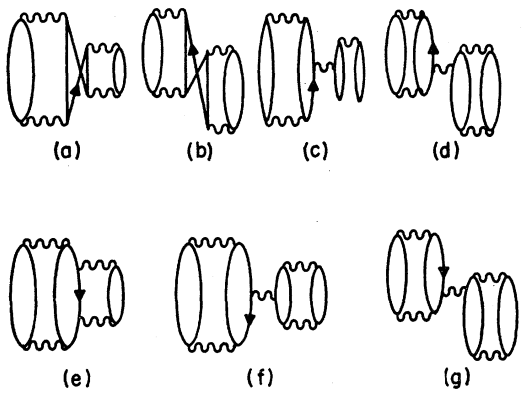


FIG. 16. Some ground-state diagrams involving Figs. 13 and 15 as insertions.

Fig. 13(a) was originally taken into account. Figures 16(a) and (b) are each different ways of inserting Fig. 13(c), but the energy denominator is larger for the central region of 16(a) than it is for 13(c), so the energy change due to this is equal to the energy change produced by 13(c) evaluated off the energy shell. Figures 16(a) and (b) can be added together, however, and then they give a change of the self-consistent particle energy equal to 13(c) evaluated on the energy shell. Figures 16(a) and (b) contain the same matrix elements as 13(c), but the propagators add together to give this simple result, just as Figs. 6(g) and 6(m) exactly cancelled in Sec. 4. Figures 16(c) and (d) represent ways of inserting 13(d) into a ground-state diagram, where again 13(d) must be evaluated off the energy shell. It is not in general possible to add together combinations of such diagrams so that the off-energy-shell effect disappears, but it is possible in the limit of a small contribution from 13(d), so that only the lowest-order diagrams, shown in Fig. 16, are important. If we add to 16(c) and (d) the two diagrams obtained by interchanging the order of the top two vertex pairs, the off-energy-shell effect cancels out. Since the diagrams are symmetrical, this process includes every diagram twice, so we should divide by two. The lowest-order effect of 13(d) on the ground-state energy can then be obtained by adding to the self-consistent particle energies half its effect on the excitation energies. This factor of a half shows that the best self-consistent energy is *not* equal to the excitation energy of a particle, although this is probably better than the model energy.

Figure 16(e) shows the only way of inserting 15(c) into the lowest order ground-state diagram. The insertion is defined off the energy shell, and therefore raises the self-consistent hole energy less than 15(c) on the energy shell would. This means that the hole self-consistent energies are lower than the excitation energies, and so an extra energy gap is introduced by the off-energy-shell effect. This makes the theory more symmetrical, since in Sec. 4 the model energies were defined on the energy shell for holes and off the energy shell for particles, but now there is a correction added which must be calculated off the energy shell for holes and on the energy shell for particles. Figures 16(f) and (g) are similar to 16(c) and (d), and again can be taken into account to lowest order by adding half the effect of 15(d) to the self-consistent energies.

11. ESTIMATION OF THE REARRANGEMENT ENERGIES

From the calculations that have already been made by Brueckner and his collaborators,[8,30] it is possible to make some estimate of the rearrangement energy defined by (44) and (45). The most direct evidence is the difference between the Fermi energy and the average energy which was mentioned earlier. Table I gives, for

TABLE I. The results used by Brueckner, Gammel, and Weitzner[30] (private communication from K. A. Brueckner) for the energy as a function of momentum at three different densities, and for the binding energy. The Fermi energy is extrapolated from these figures, and then the difference between this and the binding energy is given. The rate of change of binding energy with density is calculated on the basis of a quadratic fit. $R(k_F)$ is given by Eq. (47). Momenta are in inverse fermis (1 fermi$\equiv 10^{-13}$ cm), energies in Mev.

k/k_F	$k_F=1.551$	$k_F=1.420$	$k_F=1.357$
0.1	−107.9	−94.1	−86.8
0.3	−100.8	−88.5	−81.4
0.5	−86.8	−77.1	−71.3
0.7	−66.9	−60.8	−56.7
0.9	−41.9	−40.0	−37.9
E_B	−14.68	−14.43	−13.78
$M(k_F)$	−28.4	−28.2	−27.1
$E_B-M(k_F)$	13.7	13.8	13.3
$\rho dE_B/d\rho$	2.1	−3.6	−5.9
$R(k_F)$	15.8	10.2	7.4

[29] A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. **32**, 399 (1957) [translation: Soviet Physics JETP **5**, 333 (1957)].

[30] Brueckner, Gammel, and Weitzner, Phys. Rev. **110**, 431 (1958).

three different densities near the optimum density, the energy for five momenta less than the Fermi momentum, the binding energy, and some other figures deduced from these. The extrapolated value of the Fermi energy is compared with the binding energy; the difference between the two is steadily just below 14 Mev, and its nearly constant value is quite a surprising result. The fact that the difference is about 14 Mev at equilibrium density means that the value of $R(k_F)$ is about 14 Mev at that density. To interpret the difference between Fermi energy and average energy away from equilibrium density, we notice that the addition of one particle at constant volume on the Fermi surface can also be regarded as a change in density. This leads to the equation

$$AE_B(\rho)+E(k_F)=(A+1)E_B(\rho+\rho/A), \quad (46)$$

and substitution of (44) in this gives

$$R(k_F)=E_B(\rho)-M(k_F)+\rho dE_B/d\rho. \quad (47)$$

The value of $\rho dE_B/d\rho$ in Table I is found by making a quadratic fit to get the binding energy as a function of k_F. These values of $\rho dE_B/d\rho$ probably vary too much, since Brueckner, Gammel, and Weitzner[30] give the compressibility modulus as 172 Mev, and this quadratic fit gives the compressibility as 200 Mev. However, the rearrangement energy is clearly a very rapidly varying function of the Fermi momentum, and it seems to vary approximately as $k_F{}^5$.

Since a nonsingular potential such as a Yukawa potential does not cause much higher momentum excitation,[13,31] and Figs. 10 and 11 both depend on the high-momentum components of the wave function, we expect a large part of the rearrangement energy to come from the repulsive core. At equilibrium density the product of the core radius, $r_c=0.4$ fermi, and the Fermi momentum is 0.6, so that the method of Bethe and Goldstone[15] should give a good qualitative picture of what happens. We will consider only an S-state pure repulsive core, and pairs of particles with total momentum zero. The formulas derived by the author[16] for general angular momentum in the limit of small $k_F r_c$ will be used here, although it would not be very laborious to make a better approximation than this. These formulas give

$$(\mathbf{k}|G|\mathbf{k}_0)=4\pi(\sin kr_c)/m^*k \quad (48)$$

before averaging over spin and isotopic spin states, and this will be taken as the form of the G matrix. $\mathbf{k}_0$ is the initial relative momentum, less than k_F, and $\mathbf{k}$ is the intermediate relative momentum, which will be greater than k_F. (48) could be written more symmetrically, but it has been assumed that $k_0 r_c$ is small, whereas it may be necessary to take kr_c large. The derivation of this expression is not valid for k_0 near to k_F.

We now use (48) to evaluate (43). The matrix elements involved are all approximately equal to $16k_F{}^3r_c/3m^*\pi A$ for singlet-triplet and triplet-singlet (spin and isotopic spin) states, and zero otherwise, because of the antisymmetry requirement and the S-state potential used. The contribution to $R(k_F)$ is then

$$(4k_F{}^6r_c{}^2/3\pi^2m^{*2}) \times\langle[M(m)+M(k_F)-M(l_1)-M(l_2)]^{-1}\rangle_{\mathrm{Av}}, \quad (49)$$

where the average is taken over energy denominators which satisfy the exclusion principle and momentum conservation. In the effective-mass approximation this is a quantity which can be evaluated explicitly. The general expression which gives this average not only for the contribution to $R(k_F)$, but for $R(l)$ where $\mathbf{l}$ is anywhere in the Fermi sea, is

$$\begin{aligned}
&\frac{36m^*}{k_F{}^6}\Bigg\{\int_{\frac{1}{2}(k_F-l)}^{\frac{1}{2}(k_F+l)}p^2dp\int_{-1}^{(4p^2+l^2-k_F{}^2)/4pl}dz \\
&\qquad+\int_{\frac{1}{2}(k_F+l)}^{k_F}p^2dp\int_{-1}^{1}dz\Bigg\} \\
&\times\Bigg\{\int_0^{k_F-p}\frac{q^2dq}{p^2+l^2-2plz-q^2} \\
&\qquad+\int_{k_F-p}^{\sqrt{(k_F{}^2-p^2)}}\frac{q(k_F{}^2-p^2-q^2)dq}{2p(p^2+l^2-2plz-q^2)}\Bigg\} \\
&=(3m^*/80k_F{}^6)\{l^{-1}(-52k_F{}^5+60l^2k_F{}^3-16l^5)\log(k_F-l) \\
&\quad+l^{-1}(52k_F{}^5-60l^2k_F{}^3-16l^5)\log(k_F+l)+32l^4\log l \\
&\quad+(15k_F{}^4-30l^2k_F{}^2+31l^4)\log 2 \\
&\quad+l^{-1}(16k_F{}^4-16l^2k_F{}^2+4l^4)\sqrt{(2k_F{}^2-l^2)} \\
&\quad\times\log([\tfrac{1}{2}k_F{}^2-\tfrac{1}{2}l\sqrt{(2k_F{}^2-l^2)}]/ \\
&\quad[\tfrac{1}{2}k_F{}^2+\tfrac{1}{2}l\sqrt{(2k_F{}^2-l^2)}])-10k_F{}^4+12lk_F{}^3 \\
&\qquad+60l^2k_F{}^2-34l^4\}. \quad (50)
\end{aligned}$$

In the integral, $2p$ is the total momentum of a pair, $\cos^{-1}z$ is the angle between the total momentum and $\mathbf{l}$, and q is the initial relative momentum. This expression is equal to $1.52m^*/k_F{}^2$ for $l=0$ and $0.84m^*/k_F{}^2$ for $l=k_F$. The contributions to $R(l)$ are 9.8 Mev at $l=0$ and 5.5 Mev at $l=k_F$, for $k_F=1.5$ fermi^{-1} and $m^*=0.7m$.

This method probably gives an overestimate of the contribution to the rearrangement energy, since the attractive part of the potential will interfere destructively with the hard core for momentum transfer less than the reciprocal of the range of the potential. Brueckner has calculated the corresponding contribution from a pure Yukawa force, chosen to fit low-energy scattering data, and finds 7.5 Mev at zero momentum and 1.4 Mev on the Fermi surface.[32] This confirms that the repulsive core does dominate above the Fermi surface, although the attractive part of the potential seems

[31] W. J. Swiatecki, Phys. Rev. **103**, 265 (1956).

[32] K. A. Brueckner (private communication).

to be important for the evaluation of (43) below the Fermi surface. The dependence of (49) on k_F is a fourth power dependence, which is in qualitative agreement with the strong dependence of the calculated rearrangement energy on the Fermi momentum. The attractive part of the potential will not give a contribution which behaves in this way, since the matrix elements decrease as the momentum transfer required to excite a pair above the Fermi surface increases.

The type of contribution shown in Fig. 11 depends even more on the high-momentum components of the ground state, and so the use of a pure repulsive-core potential to determine their values should be rather better than in the foregoing evaluation of Fig. 10. The deviation of the ground state from a Fermi gas is given to first order by the wave function $Pe^{-1}G|\Phi\rangle$. The probability of a particular particle lying in the momentum range between k and $k+\delta k$ is, for $k>k_F$,

$$\frac{3k^2}{k_F{}^3}N(k)\delta k=\tfrac{3}{8}A\left(\frac{8k_F{}^3}{3\pi m^*A}\right)^2\frac{3A}{2k_F{}^3}\times\int_k^{k+\delta k}\frac{m^{*2}}{k^4}\frac{\sin^2kr_c}{k^2}k^2dk, \quad (51)$$

$N(k)$ is the probability of occupation of a particular state. The factor $3Ak^2\delta k/k_F{}^3$ is the total number of states in the range considered. This gives

$$N(k)=(4k_F{}^6/3\pi^2k^6)\sin^2kr_c, \quad k>k_F, \quad (52)$$

where the initial energy of the particle has been neglected. Integration of (52) shows that the probability of a particle being in an excited state, outside the Fermi sea, is approximately

$$1-N(k)=4k_F{}^2r_c{}^2/\pi^2, \quad k<k_F. \quad (53)$$

If we assume that a particle, once it has been excited, no longer interacts with a particle on the Fermi surface, we find that the loss of potential energy of a particle on the Fermi surface due to the excitation of the other particles is

$$(4k_F{}^2r_c{}^2/\pi^2)V(k_F). \quad (54)$$

Although the particle on the Fermi surface will still interact with the excited particles, the interaction will be less, since the angular average of their relative momenta is larger. The value of (53) for $k_F=1.5$ fermi^{-1} is 0.146, and so, since Table I gives $V(k_F)$ as -75 Mev, the contribution to $R(k_F)$ is 10.9 Mev. This is, of course, too large, since we have neglected the interaction of the particle on the Fermi surface with the excited particles, but it indicates that the contribution to $R(k_F)$ from Fig. 11 is of the same order of magnitude as the contribution from Fig. 10, and possibly somewhat larger. The order of magnitude agrees with the value of $R(k_F)$ which was calculated.[8,30] Both terms are much smaller than the potential energy of a particle, so it is correct to neglect the contributions from Fig. 12. This term also is strongly dependent on k_F in the correct way, since $-V(k_F)$ increases rapidly with k_F.

The evaluation of the left-hand side of (53) can be confirmed by a very different method; if we define

$$G(W)=v+vP(W+e)^{-1}G(W), \quad (55)$$

and differentiate this, we get

$$G'(W)=-G(W)P(W+e)^{-2}G(W). \quad (56)$$

The expectation values of the matrix elements of $G'(0)$ summed over all pairs in the Fermi sea which contain the one particle just give the quantity on the left of (53). Brueckner and Gammel[8] have calculated G for different values of W, and the raising of the particle energy curve by about 24 Mev when W (which they call Σ) is increased by 140 Mev indicates that this quantity is about $\frac{1}{6}$ for particle states, and it shows no signs of getting less for hole states.

These estimates are too crude to give a quantitative estimate of the change in the ground-state energy due to the effects discussed in Sec. 4. The contribution to $R(0)$ from Fig. 11 would be given by (54) as about 14 Mev. Only half of this energy goes into the self-consistent energy (Sec. 10), so that the hole-state energies are raised by perhaps 10 Mev with respect to an average particle-state energy. If the elements of $G'(0)$ summed over all pairs containing one particular particle are about $-\frac{1}{6}$, the ground-state energy may be reduced by about 1 Mev. A better estimate could be made if the ratio of the contributions from Figs. 10 and 11 were known.

12. APPLICATIONS

The optical model provides the most direct application of the theory of excited states. This was treated by Brueckner, Eden, and Francis[33]; and by Bethe and Shaw[23] in the following way. When the nucleon has entered the nucleus, the state of the system is the metastable state with one particle outside the filled Fermi sea. The energy of the system less the energy of the undisturbed nuclear matter must equal the energy of the incident nucleon, but this is just how we defined the excitation energy of the state. The real part V of the optical-model potential is the energy of the excited state less the kinetic energy in the medium,

$$E(k)=V+k^2/2m. \quad (57)$$

Measurement of the real part of the optical-model potential gives k as a function of E for $E>0$. Some corrections must be made for the imaginary part of the potential, but these are small at low energies.

In the effective-mass approximation to the excitation energy,

$$E(k)=E_B+(k^2-k_F{}^2)/2m^{**}, \quad (58)$$

where E_B is the average nuclear binding. Thus

$$-V(E)=-E_Bm^{**}/m+k_F{}^2/2m-E(1-m^{**}/m). \quad (59)$$

[33] Brueckner, Eden, and Francis, Phys. Rev. **100**, 891 (1955).

This agrees quite well with the potential at zero energy, but the potential falls off at higher energies at a rate which suggests that $1-m^{**}/m$ is $\frac{1}{3}$,[34] whereas the model-energy effective mass is $0.7m$, and the rearrangement energy tends to raise this even more. The critical way in which (59) depends on the deviation of the effective mass from the real mass makes it a useful way of measuring the effective mass.

Another piece of evidence for a small effective mass is the giant resonance in the nuclear photoeffect.[24] The variation of the position of the giant resonance with atomic weight has been explained by using an effective mass of half the real mass of a nucleon. It is clear that the excitation energy, not the model energy, should be used in Wilkinson's theory, since it involves the excitation of a single particle to a higher state.[35]

The symmetry energy[36,8] involves this effective mass also, but, since it also depends on the details of the isotopic spin dependence of the reaction matrix, it does not provide good evidence for the value of the effective mass.

13. CONCLUSIONS

The perturbation theory developed in Secs. 2 and 3 has been used to study various features of the nuclear many-body problem. It was shown how the self-consistency problem was connected with the cancellation of certain terms in the perturbation expansion, and so corrections can be made for a failure of self-consistency. The neglect of the exclusion principle in intermediate states of the Brueckner equation can also be corrected by taking account of additional terms, but, in practice, it seems that this is a serious approximation, and the correcting terms which can be calculated easily are quite large.

It seems that the validity of the Brueckner approximation depends on the magnitude of the three- and four-body cluster terms, as has often been suggested. The departure of the true ground-state wave function from the model wave function has already largely been taken into account when the energy is calculated by use of the Brueckner equation, but it does have important effects on other quantities. The expectation value of some operator in the first-order correction to the wave function has occurred at several points in this discussion. The lowest-order effect of a departure from self-consistency of the energy denominators, the effect of off-energy-shell propagation, the effect of adding a constant to the energy denominators, and the rearrangement energy all involve such an expectation value.

The importance of the exclusion principle in intermediate states is determined by a different set of terms, also of second order in the reaction matrix, but involving less momentum transfer in the interactions. Since the attractive part of the nuclear potential dominates for low momentum transfer and the repulsive part for high momentum transfer, the exclusion principle's importance depends to a greater extent on the attractive potential. Part of the rearrangement energy comes from the use of the exclusion principle, and this could be large even for a purely attractive potential.

The departure of the ground-state wave function from the model wave function involves interactions with a large momentum transfer, so the repulsive part of the potential will play an important role here. At intermediate momenta the attractive and repulsive parts interfere, but at high momenta the repulsive part dominates. For this reason the repulsive core has several important effects which cannot easily be obtained by standard perturbation theory, even if the hard core should affect the ground-state energy only slightly, but which can be derived simply by using the Brueckner theory. The rearrangement energy and the existence of high-momentum components of the wave function are its most obvious and important effects. Too much departure of the ground-state wave function from the model wave function would make even the Brueckner theory inapplicable, partly because of the difficulty of dealing with the off-energy-shell propagation, but the calculated departure is not alarmingly large.

The rearrangement energy is as important as the rest of the potential energy for the setting up of a self-consistency condition, except that it is smaller in magnitude. In calculations made so far it has been neglected, but there is no doubt that it should be taken into account. It could either be allowed for in the definition of the single-particle energies, or it could be corrected for by calculating the higher order terms in the perturbation series. Either of these methods involves a calculation whose difficulty depends on the size of the rearrangement energy, since this determines the accuracy required. A method of including the rearrangement energy in the self-consistent calculation will be treated in a further paper.

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[34] See, for example, A. E. Glassgold, Revs. Modern Phys. **30**, 419 (1958).
[35] D. H. Wilkinson, Physica **22**, 1039 (1956).
[36] A. E. S. Green, Phys. Rev. **95**, 1006 (1954); A. E. S. Green, Revs. Modern Phys. **30**, 569 (1958).