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Electron Self-Energy Approach to Correlation in a Degenerate Electron Gas*†

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A new method of computing the correlation energy of a degenerate electron gas is presented in which the interactions are studied by considering the self-energy of a lone particle impurity in the system. The self-energy results as in quantum electrodynamics from the action of the proper field set up by the charged particle back on itself; the Feynman space-time formulation of quantum mechanics is employed in the self-energy calculation, which is carried out along lines already laid out by Lindhard. The Feynman propagator, which takes the particle from one point in space-time to another, is derived. A slight but essential change in the particle propagator is needed to allow for exchange effects when the particle impurity is an additional electron in the degenerate electron gas. This gives the electron gas a dual role: it acts as a dielectric medium which can be polarized and also as a vacuum from which electron-hole pairs can be created and undergo exchange with incident electrons. The polarization propagator for the effective potential set up by the impurity in the electron gas, considered as a dielectric medium, is derived heuristically in the text from Lindhard's dynamic dielectric constant and more rigorously in an Appendix from the momentum-exciton model. The electron self-energy is a Feynman integral involving the particle and polarization propagators and defines an optical potential which is found to have both real and imaginary parts. For momenta less than the Fermi momentum, it is shown in a second Appendix that the optical potential is simply the negative of the self-energy of a hole in the Fermi sea. The imaginary part of the optical potential for an electron of momentum p is proportional to $(p/p_0-1)^2$ (where p_0 is the Fermi momentum), and gives rise to damping. Thus the concept of a one-electron state is only valid for small excitation energies and breaks down when the electron is appreciably far removed from the Fermi surface. The mean free path for high electron density is given (in units of $\hbar/p_0$) by $3.98r_s^{-\frac{1}{2}}$ times the above function of momentum. (r_s is the unit-sphere radius in Bohr radii.) The derivative of the real part of the optical potential with respect to momentum, evaluated at the Fermi surface, gives a correction to the specific heat in agreement with Gell-Mann. The value of the optical potential itself is related by Seitz's theorem to the derivative of the correlation energy with respect to density. Integration over density yields an expression for the ground state energy which agrees with the results of other investigators. Finally a brief discussion is given of Bethe's theorem, which directly relates the optical potential to the ground state correlation energy per particle. Although Bethe's theorem is not valid for the idealized electron gas with uniform positive background, it does apply to actual metals in equilibrium.

I. INTRODUCTION

THE correlation energy of a degenerate electron gas has been computed by Wigner,[1] Macke,[2] Pines,[3] Gell-Mann and Brueckner,[4] and many other investigators.[5] All of these investigations quite naturally concern themselves, from the outset, with all the myriad interactions which simultaneously take place between the infinitely many pairs of electrons in the gas. The present paper presents what is believed to be a somewhat simpler approach to the correlation problem, in which one individual electron at the surface of the Fermi sea is singled out for close attention. The polarization of the gas around the electron is studied in detail. The action of the polarization cloud back on the electron, resulting in a self-energy, is computed. This self-energy is related, by Seitz's[6] theorem, to the derivative of the correlation energy with respect to density. An integration with respect to density then gives an expression for the correlation energy per electron in agreement with results of the above authors. The self-energy is calculated by means of perturbation theory, using Feynman's[7] time-dependent formulation of quantum mechanics. The propagator for the electron, both in "positive" and "negative" energy (relative to the Fermi energy) intermediate states, is exhibited in Sec. II.

The present work forms a continuation of an earlier investigation into the collective interactions in a degenerate electron gas by means of the "momentum-exciton model."[8] In this earlier work the strength of the transition matrix element between the ground state and the excited states of an interacting electron gas was studied by considering the inelastic scattering of a fast incident electron. This approach is used in Appendix I of the present paper to evaluate the Green's function or "polarization propagator" of the electron gas. Section III contains an alternative derivation of the polarization propagator, in which use is made of Lindhard's[9] frequency and wave-number dependent dielectric constant. The propagator describes the Coulomb field set up by the electron gas as a result of its being excited into the spectrum of higher energy states by an incident electron. The self-energy of the

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[1] E. P. Wigner, Phys. Rev. **46**, 1002 (1934).

[2] W. Macke, Z. Naturforsch. **5a**, 192 (1950).

[3] D. Pines, Phys. Rev. **92**, 626 (1954); D. Pines in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1955), Vol. 1, p. 367.

[4] M. Gell-Mann and K. Brueckner, Phys. Rev. **106**, 364 (1957).

[5] See, for example J. Hubbard, Proc. Roy. Soc. (London) **A240**, 539 (1957); **A243**, 336 (1958); E. W. Montroll and J. C. Ward, Phys. Fluids **1**, 55 (1958); P. Nozières and D. Pines, Phys. Rev. **109**, 1009 (1958).

[6] F. Seitz, *Modern Theory of Solids* (McGraw-Hill Book Company, Inc., New York, 1940), p. 343.

[7] R. P. Feynman, Phys. Rev. **76**, 749; 769 (1949).

[8] R. A. Ferrell and J. J. Quinn, Phys. Rev. **108**, 570 (1957).

[9] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 8 (1954).

incident electron is evaluated as a function of its momentum in Sec. IV, along lines already laid out by Lindhard.[9] Use is then made of the derivative of this result with respect to momentum in Sec. V to compute the correlation correction to the specific heat of the electron gas. The result is identical to that already obtained by Gell-Mann[10] in a somewhat different way.

In Sec. VI it is noted that the self-energy contains a negative imaginary part, which leads to damping of the one-electron state for momentum vectors appreciably far removed from the Fermi sea. The mean free path is evaluated at the end of Sec. VI. So far as the present authors are aware, this is a new result, although the idea of damping by electron collisions is certainly not new and has been investigated quantitatively in various ways by Landsberg[11] (for the Auger broadening of x-ray band emission spectra), Lee-Whiting[12] (for the moderation of positrons), and many others.[13]

Section VII contains a discussion of the self-energy in terms of an optical potential, such as is familiar in nuclear physics. The properties of the optical potential are further discussed in Appendix II. The calculation of the correlation energy from the self-energy of an electron at the Fermi surface is also carried out in Sec. VII. It is further noted that if one is dealing with real metals at equilibrium then Bethe's[14] theorem applies, and the correlation energy is exactly equal to the self-energy at the Fermi surface, with no integration over density necessary. A short summary constitutes Sec. VIII.

II. ELECTRON-HOLE THEORY

In calculating the self-energy of a point charge moving through a degenerate electron gas, we can think of the quiescent unperturbed electron gas as a kind of vacuum. It is actually very similar to the vacuum state in quantum electrodynamics, in which all the negative energy states are filled, if the Fermi energy is considered as the zero level of energy. We can picture the self-energy as arising from the following process: the approach of the point charge is followed by a virtual excitation and subsequent de-excitation of the vacuum, that is by virtual creation and reabsorption of a quantum of excitation in the electron gas. The propagation of the disturbance in the electron gas, which can be treated as a dielectric medium with specific dynamical properties, is studied in Sec. III below. In the present section we concentrate on the propagation of the external point charge itself, from the time it excites the medium and recoils into some intermediate state to the time the medium acts back on the point charge and gives up its excitation. For the calculation we will use Feynman's[7] time-dependent formulation of quantum mechanics.

The propagator for the passage of a particle which can be distinguished from the electrons in the degenerate Fermi gas from the space-time point 1 to the space-time point 2 is

$$K(2,1)=\int\frac{d^3k d\omega}{(2\pi)^4}\frac{i}{\omega-k^2/2m+i\delta}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}, \qquad (1)$$

where δ is an infinitesimally small positive quantity. (Throughout this paper we use units in which $\hbar$, the reduced Planck's constant, equals unity.) There is a slight but essential change in this propagator if the point charge, of mass m, is not a distinguishable particle, but rather an additional electron. Because of the exclusion principle, no electron of energy less than the Fermi energy can propagate through the vacuum state of the electron gas. On the other hand, a vacancy in the Fermi sea can propagate through the gas as a positive energy hole. The modification in the propagator which is necessary to take this into account is analogous to that discussed by Feynman[7] for the theory of the positron and yields

$$K_+(2,1)=\int\frac{d^3k d\omega}{(2\pi)^4}\frac{i}{\omega-E(\mathbf{k})(1-i\delta)}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}, \qquad (2)$$

where $E(\mathbf{k})\equiv(k^2-p_0^2)/2m$ is the energy measured relative to the Fermi energy and takes on both positive and negative values. p_0 is the Fermi momentum. δ is again an infinitesimally small positive quantity, but now we have written it so that the product of $E(\mathbf{k})$ times δ replaces the δ of Eq. (1). Thus the imaginary part of the denominator changes its sign when the energy itself does. This modification in the propagator has the important consequence that only positive frequencies are propagated forward in time and only negative frequencies backwards. The phase factor representing the change in the wave function of the system when it passes from an earlier time to a later time, regardless of the temporal sequence of points 1 and 2, is consequently always of the form $e^{-i|\omega||t_{21}|}$. This is required by quantum mechanics and by the fact that the vacuum is the lowest energy state of the system (taken for convenience here to have zero energy). Thus the intermediate state has always positive energy and consists of an electron for $t_{21}>0$ and a hole for $t_{21}<0$. These two cases are illustrated by the Feynman graphs of Figs. 1(a) and 1(b), respectively. The direction of time is taken as upwards, so that the

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[10] M. Gell-Mann, Phys. Rev. **106**, 369 (1957).

[11] P. T. Landsberg, Proc. Phys. Soc. (London) **A62**, 806 (1949).

[12] G. E. Lee-Whiting, Phys. Rev. **97**, 1557 (1955).

[13] See for example L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956) [translation: Soviet Phys. JETP **3**, 920 (1957)]. In this paper Landau shows how many of the properties of a Fermi liquid depend only upon the behavior of the fermions in the immediate vicinity of the Fermi surface. Applied to the electron gas, his formulas give very simply the correlation corrections to both the specific heat and the paramagnetic spin susceptibility (details to be published).

[14] H. Bethe, Phys. Rev. **103**, 1353 (1956). This theorem is also discussed by V. F. Weisskopf [Nuclear Phys. **3**, 423 (1957)] and N. M. Hugenholtz and L. Van Hove [Physica **24**, 363 (1958)]. For papers on the general theory of an interacting Fermi gas see J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957), and N. M. Hugenholtz, Physica **23**, 481 (1957); **23**, 533 (1957).

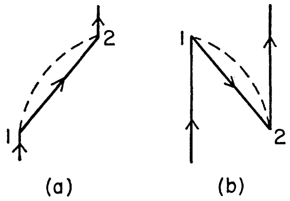

FIG. 1. Feynman graphs for the self-energy process. In graphs of type (a) the particle in the intermediate state propagates forward in time. (Flow of time is upward.) Inclusion of graphs of type (b) when the particle considered is an additional electron and can propagate backward in time accounts for exchange effects. The process (b) represents the simultaneous creation at a space-time point 2 of a disturbance and an electron-hole pair. The hole propagates forward in time to 1 (represented by backward electron propagation) and annihilates with the incident electron absorbing the disturbance.

intermediate state in Fig. 1(a) is an electron while in Fig. 1(b) it is a hole. The dashed lines represent the propagation of the polarization through the gas, and will be discussed further in the next section.

The explicit form of $K_+(2,1)$ is quite simple, and is easily obtained by carrying out the integration over the frequency variable ω in Eq. (2). As illustrated in Fig. 2, the contour is along the real axis in the complex ω plane. For $\mathbf{k}$ such that $E(\mathbf{k})$ is positive the integrand has a pole in the fourth quadrant, while for $E(\mathbf{k})<0$ the pole is in the second quadrant [Figs. 2(a) and 2(c), respectively]. Let us first consider $t_{21}>0$, in which case the factor $e^{-i\omega t_{21}}$ damps out in the direction of the negative imaginary axis. The contour can therefore be displaced downwards, according to Cauchy's theorem, and will leave only the residue at the pole as shown in Fig. 2(b):

$$\frac{i}{2\pi}\int\frac{d\omega}{\omega-E(\mathbf{k})(1-i\delta)}e^{-i\omega t_{21}}=e^{-iE(\mathbf{k})t_{21}}. \tag{3}$$

This result holds only for both t_{21} and $E(\mathbf{k})$ positive. For $t_{21}>0$ but $E(\mathbf{k})<0$ the downward displacement of the contour yields zero. On the other hand, for $t_{21}<0$, the integration is effected by displacing the contour upwards which yields zero for positive $E(\mathbf{k})$, as seen from Fig. 2(a). For negative $E(\mathbf{k})$, Fig. 2(c) is transformed into Fig. 2(d) and yields the residue

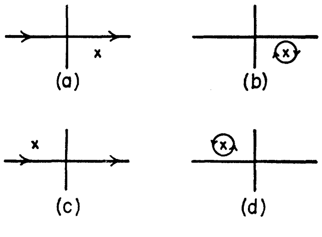

FIG. 2. Contours of integration in the complex ω plane for the particle propagator $K_+(2,1)$. The propagator is defined as an integration along the real axis as shown in (a) and (c). For positive particle energy, $E(\mathbf{k})$, the integrand has a pole in the fourth quadrant as indicated in (a), while for negative $E(\mathbf{k})$ this pole occurs in the second quadrant as shown in (c). For $t_{21}>0$ the factor $e^{-i\omega t_{21}}$ in the propagator damps out in the direction of the negative imaginary axis, and the contour can be displaced indefinitely downward leaving only the residue from the pole indicated by (b). On the other hand, for $t_{21}<0$, the contour can be displaced upward leaving only the residue from the pole indicated by (d). Consequently only positive frequencies are propagated forward in time, negative frequencies backward.

$$\frac{i}{2\pi}\int\frac{d\omega}{\omega-E(\mathbf{k})(1-i\delta)}e^{-i\omega t_{21}}=-e^{-iE(\mathbf{k})t_{21}}. \tag{4}$$

The negative sign which appears in this case is required by the fact that the hole, which propagates in the intermediate state of the system, eventually annihilates with a different electron than was associated with its creation. Before the probability amplitude for this process can be added coherently to that for the processes involving intermediate states with $E(\mathbf{k})>0$, it is necessary to interchange the two electrons, resulting in the minus sign in Eq. (4).

Substitution of the right-hand members of Eq. (3) and (4) into Eq. (2) gives

$$\begin{aligned}K_+(2,1)&=\frac{1}{(2\pi)^3}\int_{k>p_0}d^3k\; e^{i\mathbf{k}\cdot\mathbf{x}_{21}-iE(\mathbf{k})t_{21}},\quad t_{21}>0\\&=\frac{-1}{(2\pi)^3}\int_{k<p_0}d^3k\; e^{i\mathbf{k}\cdot\mathbf{x}_{21}-iE(\mathbf{k})t_{21}},\quad t_{21}<0.\end{aligned} \tag{5}$$

These expressions can be further reduced by carrying out the indicated integrations over momentum space, yielding

$$\begin{aligned}K_+(2,1)=-\left(\frac{-im}{8\pi^3t}\right)^{\frac{1}{2}}\exp\left(i\frac{p_0^2t}{2m}\right)\frac{1}{x}\frac{\partial}{\partial x}\exp\left(i\frac{mx^2}{2t}\right)\\\times\left\{\theta(t)-(\tfrac{1}{2}i)^{\frac{1}{2}}\mathfrak{C}\left(\left[\frac{p_0^2t}{2m}\right]^{\frac{1}{2}}-\left[\frac{mx^2}{2t}\right]^{\frac{1}{2}}\right)\right.\\\left.-(\tfrac{1}{2}i)^{\frac{1}{2}}\mathfrak{C}\left(\left[\frac{p_0^2t}{2m}\right]^{\frac{1}{2}}+\left[\frac{mx^2}{2t}\right]^{\frac{1}{2}}\right)\right\},\end{aligned} \tag{6}$$

where

$$\begin{aligned}\theta(t)&=1,\quad t>0\\&=0,\quad t<0,\end{aligned}$$

$$t\equiv t_{21},\quad x\equiv|\mathbf{x}_{21}|,$$

and

$$\mathfrak{C}(v)=(2/\pi)^{\frac{1}{2}}\int_0^v\exp(-iu^2)du=\frac{v}{|v|}[C(v^2)-iS(v^2)].$$

The C and S functions are the well-known Fresnel integrals[15] and the $\mathfrak{C}$ function is represented by the familiar Cornu spiral[15] when the latter is interpreted as an Argand diagram. Because of the inversion symmetry of the Cornu spiral, the second and third terms in the bracket of Eq. (6) cancel one another in the

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[15] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), fourth edition, pp. 35–37.

limit of $p_0 \rightarrow 0$, and one obtains the familiar[16] propagator for the single-particle Schrödinger equation:

$$K(2,1)=\left(\frac{-im}{2\pi t_{21}}\right)^{\frac{3}{2}} \exp(imx_{21}^2/2t_{21}). \tag{7}$$

This expression can, of course, be found directly by carrying out the integrations in the right-hand member of Eq. (1). These integrated, explicit expressions for the propagators have been presented only for the sake of definiteness. In actual application the integral forms given by Eqs. (1) and (2) are more useful, and no further use will be made in this paper of Eqs. (6) and (7). We now proceed in the next section to describe the propagation of the polarization disturbance through the electron gas, considered as a dielectric medium. The degenerate gas thus plays a dual role: (1) it serves as a sort of reference state, or vacuum, from which electron-hole pairs can be created; (2) in addition it has particular electrical properties which will be described by a dielectric constant dependent on frequency and wave number.

III. POLARIZATION PROPAGATOR

To find the self-energy corresponding to the Feynman diagrams shown in Fig. 1, we must know the effective potential acting on the electron at space-time point 2 as a result of the disturbance in the medium which it causes at 1. This is a problem which has been studied thoroughly by Lindhard.[9] Considering the electron at 1 as setting up the external source function $\rho(\mathbf{x}_1,t_1)$, and designating the electric scalar potential at 2 by $\varphi(\mathbf{x}_2,t_2)$, we have the following simple relationship between the Fourier transforms:

$$\varphi(\mathbf{k},\omega)=\frac{4\pi}{k^2}\frac{\rho(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}. \tag{8}$$

This equation can be considered a definition of the dielectric constant[17] $\epsilon(\mathbf{k},\omega)$ as a function of the wave number $\mathbf{k}$ and the frequency ω. This function was computed by Lindhard[9] for a degenerate electron gas and found to be represented by

$$\epsilon(\mathbf{k},\omega)=1-\frac{3}{64\gamma^2 z^2}\left[\frac{(u-z)^2-1}{2z}\ln\left(\frac{u-1-z}{u+1-z}\right)\right.$$
$$\left.+\frac{(u+z)^2-1}{2z}\ln\left(\frac{u+1+z}{u-1+z}\right)-2\right], \tag{9}$$

where $u=\omega/kv_0$ and $z=k/2p_0$. $v_0=p_0/m$ is the Fermi velocity and $\gamma=p_0^2/2m\omega_p$ is the ratio of the Fermi energy to the classical plasma frequency. In terms of r_s, the unit-sphere radius in units of the Bohr radius of the hydrogen atom, γ is given by

$$\gamma = 1.061 r_s^{-\frac{1}{2}}. \tag{10}$$

The medium is thus characterized by the single parameter γ, which is a measure of the density of the gas. In the limit of infinite density ($\gamma\rightarrow\infty$) the gas becomes rigid or unpolarizable, and $\epsilon\rightarrow 1$. Returning now to Eq. (8) and re-expressing it in terms of the space-time functions gives

$$\varphi(\mathbf{x}_2,t_2)=\int d^3x_1 dt_1\, G(2,1)\rho(\mathbf{x}_1,t_1), \tag{11}$$

where the polarization propagator, or Green's function, is

$$G(2,1)=\int\frac{d^3k d\omega}{(2\pi)^4}\frac{4\pi}{k^2\epsilon(\mathbf{k},\omega)}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}. \tag{12}$$

There are some interesting properties of the propagator which it will be worthwhile to exhibit. First of all, in the limit of infinite density, or alternatively, in the complete absence of a dielectric medium, $\epsilon\rightarrow 1$ and the integration can be carried out explicitly, giving

$$G(2,1)=x_{21}^{-1}\delta(t_{21}), \tag{13}$$

where δ is the Dirac delta function. This is just the instantaneous Coulomb interaction. Another simple limiting case is that of long wavelengths, for which the dielectric constant reduces to

$$\epsilon\approx 1-\omega_p^2/\omega^2, \tag{14}$$

where ω_p is the classical plasma frequency. This expression vanishes at $\omega=\pm\omega_p$, leading to poles in the denominator. The Green's function is consequently an improper integral and is not completely defined without a prescription for dealing with the poles. This question is discussed further below. For the moment, the following rule suffices: ϵ is considered to have an infinitesimally small imaginary part, of the same sign as ω itself. The integration then yields

$$G(2,1)\approx x_{21}^{-1}[\delta(t_{21})-\theta(t_{21})\omega_p \sin\omega_p t_{21}], \tag{15}$$

where the θ function has already been defined in Sec. II. The first term in the brackets represents, again, the instantaneous Coulomb interaction while the second expresses the dynamic screening of the medium. The inertial nature of the gas is clearly portrayed by the behavior of this term, which represents a lagging response to the external forcing charge density. If $\rho(1)$ is applied as a pulse, the resulting $\varphi(2)$ contains in addition to the instantaneous direct interaction, a term proportional to $\sin\omega_p t_{21}$, which attains an appreciable magnitude only after enough time has elapsed for the electron gas to respond. On the other hand, when $\rho(1)$ is very slow and does not vary appreciably over a time ω_p^{-1}, we can set the second term in brackets also

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[16] W. Pauli, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. 24, Part 1, Chap. 2, p. 104.

[17] For a recent discussion of the general theory of the dielectric constant see P. Nozières and D. Pines, Phys. Rev. **109**, 762 (1958).

equal to $\delta(t_{21})$, since

$$\int \theta(t_{21})\omega_p \sin\omega_p t_{21} dt_{21} = 1.$$

Then $G(2,1)\approx 0$, corresponding to complete screening. This result holds only for functions $\rho(1)$ which also do not vary appreciably over distances equal to the screening length, since Eq. (14) is valid only in the long-wave length approximation.

A further feature of Eq. (14) appears if $\rho(1)$ oscillates at the frequency ω_p. This is the case of resonance, and the integral

$$\varphi(2) = \int d^3x_1 dt_1\, G(2,1)\rho(1)$$

diverges as the integration is carried out over the time. The amplitude of the scalar potential field produced in the medium builds up to larger and larger values as the external charge density continues to act. This is, of course, just the condition for a natural oscillation of the medium itself, and showed up already in the fact that the integrand in Eq. (12) had a pole at this frequency. In general, for any wavelength, the condition

$$\epsilon(\mathbf{k},\omega) = 0 \tag{16}$$

yields the Bohm-Pines dispersion relation[18] and determines the frequency of plasma oscillation as a function of the wave number. In this manner, furthermore, one can also fix the cutoff, or minimum-wavelength oscillation which can be sustained by the gas.[19]

By means of Eq. (16) it is now possible to give a more complete discussion of the poles in the integrand of the Green's function. The normal modes of oscillation to which they correspond must eventually be damped. Let the mean life be designated by λ^{-1}. Then the electric potential must contain terms with the time dependences $e^{-i(\pm\omega_r - i\lambda/2)t}$, where ω_r is a real number, and $\pm\omega_r - i\lambda/2$ are the roots of Eq. (16). Thus the poles are shifted below the real axis in the complex ω plane, which is easily seen to be equivalent to the prescription which led to Eq. (14). In the ideal limiting case of infinitely long lifetime the poles can be regarded as lying on the real axis and the prescription can alternatively be taken to consist of a path of integration parallel to the real axis but displaced an infinitesimal amount upward. The situation is illustrated in Fig. 3(a), where it will be noted that a cut along the real axis has also been introduced. This is required by the logarithmic nature of $\epsilon(\mathbf{k},\omega)$. The branch points occur at $\omega = \pm(kv_0 + k^2/2m)$. The quantity in parenthesis is the maximum energy which can be absorbed by the electron gas when a single electron is given the additional momentum $\mathbf{k}$. For real values of ω of magnitude smaller than this limit, $\epsilon(\mathbf{k},\omega)$ contains an imaginary part, representing the absorptive behavior of the medium. It is easily seen that Imϵ changes sign if one (a) crosses the cut, or (b) stays on the same side of the cut but passes the origin (i.e., if one changes the sign of the real variable ω). Thus, as established by Lindhard, the dielectric constant satisfies the reality condition

$$\epsilon^*(\mathbf{k},\omega) = \epsilon(-\mathbf{k}, -\omega) = \epsilon(\mathbf{k}, -\omega), \tag{17}$$

where the asterisk signifies the complex conjugate. The last result follows from the isotropy of the gas.

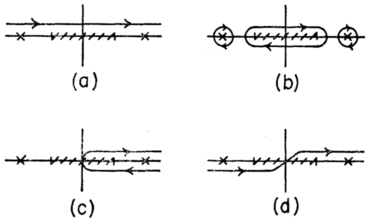

FIG. 3. Contours of integration in the complex ω plane for the polarization propagator. The propagator $G(2,1)$ is defined as an integration along the upper side of the real axis as shown in (a). The cut along the real axis is required by the logarithmic nature of the dielectric constant and corresponds to the continuum of single-electron excitations. For $t_{21}<0$ the factor $e^{-i\omega t_{21}}$ in the propagator damps out in the direction of the positive imaginary axis, and the contour can be displaced indefinitely upward to give $G(2,1)=0$. For $t_{21}>0$ the contour can be displaced downward leaving the residues and the loop around the cut shown in (b). Consequently $G(2,1)$ does not satisfy the quantum mechanical requirement that only positive frequencies propagate forward in time. The contour for the true quantum-mechanical Green's function $G_+(2,1)$, is given by the modification shown in (c) for the case $t_{21}>0$. The contour (d) is equivalent to (c) for $t_{21}>0$ and in addition correctly describes the case $t_{21}<0$.

Just as with the poles, there is clearly some additional ambiguity in the definition of the Green's function, depending upon which side of the cut the path of integration passes. This was already settled by Lindhard, however, who specified that Eq. (9) correctly yields the imaginary part of ϵ, with the right sign, if ω is considered to have an infinitesimally small positive imaginary part. Thus, as with the poles, the path of integration should be taken along the upper side of the cut [Fig. 3(a)]. It may further be noted that this choice is consistent with the requirement of causality. Returning to Eq. (12), we see that for $t_{21}<0$ the factor $e^{-i\omega t_{21}}$ damps out the integrand for large positive imaginary values of ω. Therefore the contour of Fig. 3(a) can be displaced indefinitely upward, giving $G(2,1)=0$ for the case of 2 preceding 1. Thus the Green's function is causal, and does not give any disturbance in the medium until after the external charge density has been applied. For the case of positive t_{21}, on the other hand, the contour can be closed below and yields the residues and the integral around the cut shown in Fig. 3(b). Since the left-hand half of this diagram represents negative values for the real part of ω, it is clear that the Green's function $G(2,1)$ does not satisfy the requirement discussed in Sec. II in relation to the electron-hole propagator, *viz.*, that only positive frequencies should

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[18] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).

[19] R. A. Ferrell, Phys. Rev. **107**, 450 (1957).

propagate forward in time. $G(2,1)$ is thus not the actual quantum-mechanical propagator for polarization in the electron gas, but bears a close relationship to it as can be seen as follows: The electron gas is excited out of its ground, or "vacuum," state Ψ_0 by the disturbance at 1 and into the state $\Psi_0+\Psi'$, where Ψ' represents the small perturbation. This perturbed part of the wave function disappears at 2 by virtue of the interaction of the electrons of the gas with the additional single electron. If this total interaction is designated by the Hermitian operator H_I, the true quantum-mechanical propagator, which we denote by G_+, is proportional to $(\Psi_0,H_I\Psi')$. On the other hand, the G function determined above refers to a *measurement* of the scalar potential, and is hence proportional to the expectation value

$$\begin{aligned}\langle H_I\rangle &= \big((\Psi_0+\Psi'),\, H_I(\Psi_0+\Psi')\big)\\ &= (\Psi_0,H_I\Psi')+\text{c.c.},\end{aligned} \quad (18)$$

where the complex conjugated term results from the Hermiticity of H_I. (The vacuum expectation value of H_I is zero.) Thus for $t_{21}>0$, we can write

$$G(2,1)=G_+(2,1)+G_+{}^*(2,1), \quad (19)$$

where, as previously, the asterisk signifies the complex conjugate. Because of the conjugated term, G contains both positive and negative frequencies, even though G_+ has only positive frequencies. We can solve for G_+ simply by selecting the positive frequency part of G. This is accomplished as shown in Fig. 3(c) by choosing a loop along the positive real axis which passes through the cut at the origin. For $t_{21}>0$ an equivalent contour, designated henceforth as C, is that shown in Fig. 3(d), where the path follows the lower side of the real axis for negative frequency and the upper side for positive frequency. An advantage of C over the path of Fig. 3(c) is that it also correctly reproduces the behavior of $G_+(2,1)$ for the reversed time sequence $t_{21}<0$. Here we are now including the additional effect, previously not considered, of the disturbance originating at 2 and being received at the later time 1. Thus, rather than requiring $G_+(2,1)$ to vanish in this case, we impose the condition

$$G_+(2,1)=G_+(1,2), \quad (20)$$

which is clearly satisfied by the integral

$$G_+(2,1)=\int_C \frac{d^3k d\omega}{(2\pi)^4}\frac{4\pi}{k^2\epsilon(\mathbf{k},\omega)}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}. \quad (21)$$

It should further be noted that, with the present convention that $-\omega$ is on the opposite side of the cut as $+\omega$, the relation

$$\epsilon(\mathbf{k},-\omega)=\epsilon(\mathbf{k},\omega) \quad (22)$$

replaces the previous reality condition [Eq. (17)], which required $\epsilon^*(\mathbf{k},\omega)$ as the right-hand member. Thus, by changing to the contour C we have effectively changed functions. Lindhard has, as a matter of fact, distinguished this new function by a special subscript, and in the Appendix of his paper[9] has given what is essentially a derivation of Eq. (21)—in, however, a quite different way. An alternative derivation of the polarization propagator G_+ can be constructed from the momentum-exciton model[8] and is presented in Appendix I. This method simply involves computing the excitation matrix elements for the positive energy continuum states encircled by the loop in Fig. 3(c). It may have some advantage since it avoids the restriction inherent in the concept of the dielectric constant and what has been called the "strong random phase approximation."[19] It is for example, quite straightforward to compute the exchange correction to the Bohm-Pines dispersion relation in the exciton-model,[20] while the dielectric constant approach does not lend itself very well to this problem. We are also aware that the polarization propagator can be derived[21] in the manner of Gell-Mann and Brueckner[4] by summing graphs in perturbation theory. Although this method is, of course, quite effective, we feel that it is not as satisfactory as the others discussed here because the series which is summed is actually divergent for some values of the variables.[22]

IV. SELF-ENERGY

From the work of Feynman[7] it is easily seen that the self-energy corresponding to Fig. 1 for an electron of momentum $\mathbf{p}$ and zero-order energy $E(\mathbf{p})$ is

$$\begin{aligned}\text{S.E.}(\mathbf{p})=e^2\int d^3x_1dt_1\, e^{-i\mathbf{p}\cdot\mathbf{x}_2+iE(\mathbf{p})t_2}\\ \times K_+(2,1)G_+(2,1)e^{i\mathbf{p}\cdot\mathbf{x}_1-iE(\mathbf{p})t_1}.\end{aligned} \quad (23)$$

Substituting from Eqs. (2) and (21) and carrying out the space-time integration reduces Eq. (23) to

$$\begin{aligned}\text{S.E.}(\mathbf{p})=e^2\int_C \frac{d^3kd\omega}{(2\pi)^4}\frac{4\pi}{k^2\epsilon(\mathbf{k},\omega)}\\ \times\frac{i}{E(\mathbf{p})-\omega-E(\mathbf{p}-\mathbf{k})(1-i\delta)}.\end{aligned} \quad (24)$$

On the other hand, the self-energy of the electron when it is moving in a true physical vacuum, in the complete

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[20] R. A. Ferrell (to be published). The result agrees with the time-dependent Hartree-Fock treatment of reference 19 when the latter has been corrected for an error of omission. Terms which were inadvertently omitted considerably reduce the magnitude of the effect of exchange. H. Kanazawa and S. Tani [Progr. Theoret. Phys. Japan **19**, 153 (1958)], seem also not to have included all of the important terms.

[21] D. Dubois and M. Gell-Mann (private communication).

[22] For an alternative approach which is free of this difficulty see K. Sawada, Phys. Rev. **106**, 372 (1957); Sawada, Brueckner, Fukuda, and Brout, Phys. Rev. **108**, 507 (1957); and G. Wentzel, Phys. Rev. **108**, 1593 (1957). This approach is, however, inherently restricted to the strong random phase approximation and, consequently, seems not to lend itself to further refinement of the theory

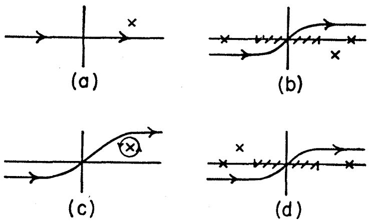

FIG. 4. Contours of integration in the complex ω plane for the self-energy. The self-energies of an electron in vacuum and in the electron gas are given by the path integrals illustrated in (a) and (b), respectively. The difference which exists in the contours for negative-energy intermediate states is removed by shifting the path as shown in (c). The resulting residue, summed over the intermediate states, constitutes the exchange energy. On the other hand, for positive-energy intermediate states the pole of the electron propagator falls above the real axis and the contours for the self-energies in the gas [now illustrated by (d)] and in vacuum agree without further modification.

absence of any dielectric medium, is

$$[\text{S.E.}(\mathbf{p})]_0=e^2\int\frac{d^3kd\omega}{(2\pi)^4}\frac{4\pi}{k^2}\times\frac{i}{E(\mathbf{p})-\omega-E(\mathbf{p}-\mathbf{k})+i\delta}. \tag{25}$$

This result has been obtained by replacing K_+ in Eq. (23) by K [from Eq. (1)] and by setting $\epsilon=1$ in G_+. What we are actually interested in here is the difference in these self-energies, or the change

$$\Delta E(\mathbf{p})=\text{S.E.}(\mathbf{k})-[\text{S.E.}(\mathbf{k})]_0 \tag{26}$$

in the energy of the electron when it is "transplanted" from the actual vacuum into the electron gas. The subtraction indicated in Eq. (26) corresponds to the "mass renormalization" of quantum electrodynamics and is most easily carried out by taking the difference within the integral sign. Before doing this, however, it is desirable to make the integrands as nearly alike as possible by shifting the pole at $\omega=E(\mathbf{p})-E(\mathbf{p}-\mathbf{k})+i\delta$ in the vacuum self-energy so that it agrees with the position $\omega=E(\mathbf{p})-E(\mathbf{p}-\mathbf{k})(1-i\delta)$ of the pole in S.E. An equivalent procedure is to shift the path of integration in Eq. (25) up from the real axis past the pole so as to leave the pole below the new contour. This is necessary only for values of $\mathbf{k}$ such that $E(\mathbf{p}-\mathbf{k})$ is negative. Figures 4(a) and 4(b) compare the path of integration in Eq. (25) with that in Eq. (24), while Fig. 4(c) shows the residue which results when the contour of Fig. 4(a) is deformed in the desired way. Figure 4(d), on the other hand, shows the S.E. integration for a typical positive value of $E(\mathbf{p}-\mathbf{k})$. Here the pole occurs above the contour, as in Fig. 4(a), and the electron-propagation factors in the integrands of Eqs. (24) and (25) agree without any modification.

The contribution of the residue illustrated in Fig. 4(c) to the integral over frequency in Eq. (25) is

$$\oint\frac{d\omega}{2\pi}\frac{i}{E(\mathbf{p})-\omega-E(\mathbf{p}-\mathbf{k})+i\delta}=1. \tag{27}$$

As is clear from the discussion of Sec. II, the residues can be attributed to the exchange of the additional electron with the indistinguishable electrons of the medium. Their total contribution to the energy shift will therefore be denoted by the subscript "ex," and amounts to

$$E_{\text{ex}}(\mathbf{p})=-e^2\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}=-e^2\int_{|\mathbf{p}'|\leq p_0}^{|\mathbf{p}-\mathbf{k}|\leq p_0}\frac{d^3p'}{(2\pi)^3}\frac{4\pi}{|\mathbf{p}-\mathbf{p}'|^2}, \tag{28}$$

where we have put $\mathbf{p}'=\mathbf{p}-\mathbf{k}$. Equation (28) is the familiar equation for the exchange correction to the single electron energy. The integration is quite straightforward[23] and yields

$$E_{\text{ex}}(\mathbf{p})=-\frac{e^2p_0}{2\pi}\left(\frac{p^2-p_0^2}{pp_0}\ln\left|\frac{p-p_0}{p+p_0}\right|+2\right), \tag{29}$$

an expression whose logarithmically singular behavior at the Fermi surface has received a great deal of attention. It is of course, precisely because of the infinite slope of this function at $p=p_0$ that the effect of correlation is so essential in computing the specific heat. Without the correlation correction the density of states at the Fermi surface is drastically reduced, resulting in a specific heat very much smaller than the Sommerfeld value, and with a temperature dependence quite different from the experimentally observed simple linear behavior of the latter. The contribution of correlation, which will be designated by the subscript "corr," is defined by the equation

$$\Delta E(\mathbf{p})=E_{\text{ex}}(\mathbf{p})+E_{\text{corr}}(\mathbf{p}) \tag{30}$$

as the part of the energy shift which is left over after the exchange contribution is separated out. According to Eqs. (24), (25), and (26),

$$E_{\text{corr}}(\mathbf{p})=-e^2\int_C\frac{d^3kd\omega}{(2\pi)^4}\frac{4\pi}{k^2}\times\left[\frac{1}{\epsilon(\mathbf{k},\omega)}-1\right]\frac{i}{\omega-(E-E'+iE'\delta)}, \tag{31}$$

where we have introduced the abbreviations $E\equiv E(\mathbf{p})$ and $E'\equiv E(\mathbf{p}-\mathbf{k})$. Equation (31) exhibits the inherent dependence of the correlation correction on the polar-

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[23] See, for example, F. Seitz, reference 6, p. 340 or the appendix of Pines' paper (reference 3).

izability of the gas since, if ϵ is replaced by unity (nonpolarizable medium), E_{corr} vanishes identically.

Before proceding to carry out the integration in the right-hand member of Eq. (31) it is well to note that the electron propagator has a pole at $\omega=E-E'+iE'\delta$. This pole occurs in the second and fourth quadrants except for special values of E' falling in the interval $0\leq E'\leq E$ (except in Sec. VII we deal only with positive E). This complication can be eliminated by again pushing the contour up, or, alternatively, by bringing the pole down to the position $(E-E')(1-i\delta)$. In either case, a further residue contribution results of the form

$$e^2\int_{0\leq E'\leq E}\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}\left[\frac{1}{\epsilon(\mathbf{k},E-E'+i\delta)}-1\right],$$

where $E'=E(\mathbf{p}-\mathbf{k})$ and δ is a positive infinitesimal. The restriction on the integration is equivalent to $p_0\leq|\mathbf{p}-\mathbf{k}|\leq p$. The second term in brackets is most easily handled by combining it with the integrand of Eq. (28). This yields

$$E_{\mathrm{ex}}'(\mathbf{p})=-e^2\int_{p'\leq p}\frac{d^3p'}{(2\pi)^3}\frac{4\pi}{|\mathbf{p}-\mathbf{p}'|^2}=-\frac{e^2p}{\pi}. \qquad (32)$$

The remainder can be split up into the imaginary part

$$E_{\mathrm{corr}}{}^{\mathrm{im}}(\mathbf{p})=e^2\int_{0\leq E'\leq E}\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}\,\mathrm{Im}\frac{1}{\epsilon(\mathbf{k},E-E'+i\delta)}, \qquad (33)$$

and the real part

$$E_{\mathrm{corr}}{}^{\mathrm{res}}(\mathbf{p})\cong e^2\int_{p_0\leq|\mathbf{p}-\mathbf{k}|\leq p}\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}\frac{1}{\epsilon(\mathbf{k},0)}. \qquad (34)$$

Since the imaginary part of ϵ vanishes for zero frequency we have inserted into Eq. (34) the static dielectric constant. This approximation is valid for small excitation energies E. In the remaining line integral, which we designate by $E_{\mathrm{corr}}{}^{\mathrm{line}}(\mathbf{p})$, we may now set $\delta=0$ since the path of integration is already completely prescribed by the contour C. Thus $E_{\mathrm{corr}}{}^{\mathrm{line}}(\mathbf{p})$ is identical to the right hand member of Eq. (31), except for the absence of the $iE'\delta$ term in the electron propagator. A further simplification in this line integral is suggested by the work of Gell-Mann and Brueckner.[4] The contour C can be deformed into the imaginary axis of the ω plane by the substitution $\omega=iw$, where w is a real number. This transformation is illustrated by Fig. 5(b), which also shows the residue. Figure 5(a) shows the original contour integral with the pole above the positive real axis. Because $\epsilon(\mathbf{k},iw)^*=\epsilon(\mathbf{k},-iw)=\epsilon(\mathbf{k},iw)$, [see Eq. (22)], it is clear that the line integral is real and is given by

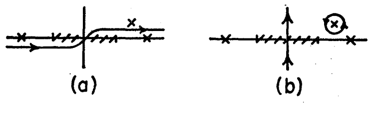

FIG. 5. Contours of integration in the complex ω plane for the self-energy of an electron outside the Fermi sea. For special intermediate-state energies the pole in the integrand occurs in the first quadrant as shown in (a). (b) illustrates the deformation of the contour into a line integral along the imaginary axis. The line integral has simple analytic properties, while the residue arising from the change in the contour has a singularity in its slope at the Fermi surface which cancels the undesirable singularity in the exchange energy.

$$E_{\mathrm{corr}}{}^{\mathrm{line}}(\mathbf{p})=-4\pi e^2\int\frac{d^3k\,dw}{k^2(2\pi)^4}\times\left[\frac{1}{\epsilon(\mathbf{k},iw)}-1\right]\frac{E-E'}{w^2+(E-E')^2}, \qquad (35)$$

where the w-integration runs from $-\infty$ to $+\infty$.

Collecting terms, we find that the sole contributor to the imaginary part of the correlation energy is $E_{\mathrm{corr}}{}^{\mathrm{im}}(\mathbf{k})$, while the total real part is

$$\mathrm{Re}E_{\mathrm{corr}}=(E_{\mathrm{ex}}'-E_{\mathrm{ex}})+E_{\mathrm{corr}}{}^{\mathrm{res}}+E_{\mathrm{corr}}{}^{\mathrm{line}}. \qquad (36)$$

It should be noted that the term $E_{\mathrm{ex}}'-E_{\mathrm{ex}}$ has, except for sign, the same singular behavior as E_{ex}. But adding E_{ex} to the correlation energy gives the smoothly varying function E_{ex}'. Thus the customary break-up of the total energy into exchange and correlation energy is quite artificial. The total energy has no pathological behavior as a function of the electron momentum. It is also of interest that in the present method of treating the interaction of the electrons the exchange energy is automatically included in a very natural way, and does not have to be included "as an afterthought,"[24] as in the calculation of Gell-Mann and Brueckner.

V. SPECIFIC HEAT

The low-temperature specific heat of a degenerate Fermi gas is proportional to the density of single-particle excited states per unit energy interval. Consequently if the zero-order energy E is shifted by ΔE as a result of the interaction the specific heat per electron c_V is changed from the Sommerfeld value for non-interacting fermions[25]

$$c_S=\frac{\pi^2KT}{2E_0}K \qquad (37)$$

by the factor

$$c_V/c_S=\left(1+\left.\frac{d\Delta E}{dE}\right|_{E=0}\right)^{-1}\approx1-\frac{1}{v_0}\left.\frac{d\Delta E}{dp}\right|_{p=p_0}. \qquad (38)$$

K is the Boltzmann constant and T the temperature. The right-hand member of Eq. (38) can either be greater than unity, representing an enhancement of the specific heat, or less than unity, corresponding to an inhibition. These cases arise when the slope of the

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[24] This feature of the Gell-Mann-Brueckner calculation has been remarked by L. Onsager [Bull. Am. Phys. Soc. Ser. II, 3, 146 (1958)].

[25] See, for example, F. Seitz, reference 6, p. 150.

curve $\Delta E(\mathbf{p})$ *vs* p is negative or positive, respectively. As seen below, the contribution to the slope arising from the Coulomb interactions of the electrons is positive, at least at high density. But since contributions to the self-energy are additive, a further term must be included for real metals because of the motion of the ions and their response to the polarizing field of the electron. This electron-phonon self-energy has generally a stronger slope than that resulting from the electron-electron interactions. It has been computed for sodium and found to give about a 20% enhancement to the electron specific heat.[26] Since it is planned to publish separately the details of the electron-phonon calculation, we shall limit the present discussion to the electron-electron interactions. Although the result of this work is identical to that of Gell-Mann,[10] it is presented as an example of the method of the self-energy. Because of the considerable simplicity of this method it should be possible to improve the accuracy of the result with a reasonable expenditure of additional labor.

In Sec. VI it will be shown that the derivative of the imaginary part of ΔE vanishes at the surface of the Fermi sea. Therefore for ΔE in Eq. (38) we can substitute $E_{\text{ex}}+\text{Re}E_{\text{corr}}$, where the second term is given by Eq. (36). If we neglect correlation and keep only the E_{ex} term, we find an infinite slope for $dE_{\text{ex}}/dp|_{p=p_0}$ and, as already mentioned, the specific heat is reduced to zero. But already including the correction term $E_{\text{ex}}'-E_{\text{ex}}$ gives the finite slope [see Eq. (32)] $dE_{\text{ex}}'/dp=-e^2/\pi$. Of the other correction terms, it will be shown in Sec. VII that the line integral contributes a slope of higher order in the density parameter r_s. With the customary simplification of restricting the calculation to high density, this contribution can be neglected. The only remaining term in Eq. (36) is

$$E_{\text{corr}}{}^{\text{res}}=(p-p_0)\frac{e^2}{\pi}\int_0^{2p_0}\frac{dk}{k\epsilon(\mathbf{k},0)}. \tag{39}$$

Here we have already integrated normal to the Fermi surface over a spherical shell of thickness $(p-p_0)$ and have also performed the angular integration about the direction of $\mathbf{p}$. From Eq. (9)

$$\epsilon(\mathbf{k},0)=1+\frac{3}{16\gamma^2z^2}F(z), \tag{40}$$

where

$$F(z)=\frac{1}{2}+\frac{1-z^2}{4z}\ln\left|\frac{1+z}{1-z}\right|. \tag{41}$$

For small values of z, $F(z)\approx 1$ and $\epsilon\approx 1+3/16\gamma^2z^2=1+q^2/k^2$, where $q=(4k_0/\pi a_0)^{\frac{1}{2}}=0.815(r_s)^{\frac{1}{2}}k_0$ is the reciprocal of the screening length. This expression can be derived from the Fermi-Thomas statistical model and yields a Yukawa-type potential for the screened Coulomb field about a fixed point charge impurity. Setting $F(z)$ equal to unity is not permitted when large momentum transfers, or short wavelengths, are important.[27] This is not the case here, however, and the approximation is justified. Substituting into the integral of Eq. (39) and using Eq. (10) gives

$$\int^1\frac{zdz}{z^2+3/16\gamma^2}=\frac{1}{2}\ln\left(\frac{1+3/16\gamma^2}{3/16\gamma^2}\right)$$
$$\approx-\tfrac{1}{2}\ln r_s+0.898, \tag{42}$$

where consistent with the high-density approximation, the term of order r_s has been dropped. Thus we have

$$E_{\text{corr}}{}^{\text{res}}=-(p-p_0)\frac{e^2}{2\pi}(\ln r_s-1.97), \tag{43}$$

giving

$$\left.\frac{d\Delta E}{dp}\right|_{p=p_0}=-\frac{e^2}{2\pi}(\ln r_s+0.203), \tag{44}$$

and a specific heat inhibition of

$$c_V/c_S=1+0.083r_s(\ln r_s+0.203), \tag{45}$$

in complete agreement with Gell-Mann.[10] Although this expression gives a reduction for small values of r_s, the correlation correction vanishes for $r_s=0.816$ and changes to an enhancement for values larger than this. Since terms of higher order in r_s must also be included, it is clear that not even the sign of the effect can presently be predicted for the electron densities actually occurring in metals ($r_s\gtrsim 2$). Equation (45) is thus of no practical value without the higher order corrections. Its interest is mainly theoretical, in connection with the limit $r_s\to 0$. This limit can be effected either by increasing the density to infinity or by keeping the density fixed and reducing the electron charge to zero. Thus, by including correlation, one has a theory which continuously approaches the Sommerfeld theory of free electrons as the interaction is turned off. This is not the case when only the effect of the interaction through exchange is included and correlation is neglected.

VI. DAMPING OF ONE-ELECTRON STATES

It has been noted in Sec. IV that the self-energy of an electron of momentum greater than the Fermi momentum has an imaginary part as well as a real part. In addition to an energy shift there is consequently an attenuation of the one-electron state, with a mean life of minus one-half of the reciprocal of the imaginary part of the self-energy. After this time the one-electron state will have passed most of its original excitation energy to the electron gas. This damping is a consequence of the fact that a single-electron excitation is not a stationary state of the combined system of

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[26] R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, **1**, 217 (1956); **3**, 203 (1958).

[27] J. J. Quinn and R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, **3**, 53 (1958).

electron and degenerate gas, and results from inelastic collisions of the original electron with the electrons of the gas. The damping is, however, greatly retarded by the collective screening of the Coulomb field of the electron by the gas, acting as a dielectric medium. Without the screening the effect of the collisions would be so drastic that the mean life would be vanishingly small, and the concept of the single-electron excitations, which is at the basis of the specific heat calculation of the preceding section, would have no validity.

The imaginary part of the self-energy is to be calculated by substituting into Eq. (33). For the sake of simplicity we shall consider only small excitation energies E. Consequently we are near the static limit for the dielectric constant, where the imaginary part vanishes, and can make the approximation

$$\begin{aligned}\operatorname{Im}\epsilon(\mathbf{k},\, E-E'+i\delta)^{-1}\\ \approx -\epsilon(\mathbf{k},0)^{-2}\operatorname{Im}\epsilon(\mathbf{k},\, E-E'+i\delta).\end{aligned} \tag{46}$$

From Eq. (9), for $0\leq u\leq <1-z$ we have

$$\operatorname{Im}\epsilon(\mathbf{k},\, \omega+i\delta)=\frac{3\pi u}{32\gamma^2z^2}=\frac{3\pi\omega}{128E_0\gamma^2z^3}, \tag{47}$$

where $E_0=p_0^2/2m$ is the Fermi energy. Substituting into Eq. (33) and carrying out the integration normal to the Fermi sphere gives

$$\int_{0\leq E'\leq E} dk_n(E-E')=\frac{E^2}{2v_0}=2\frac{E_0^2}{v_0}\left(\frac{p}{p_0}-1\right)^2, \tag{48}$$

which leads to

$$E_{\mathrm{corr}}{}^{\mathrm{im}}=-\frac{3p_0e^2}{128\gamma^2}\left(\frac{p}{p_0}-1\right)^2\int_0^1\frac{dz}{z^4\epsilon(\mathbf{k},0)^2}. \tag{49}$$

Without the screening represented by the dielectric constant the integral would be severely divergent and result in an infinite damping rate, as mentioned above. But with $\epsilon(\mathbf{k},0)\approx 1+3/(16\gamma^2z^2)$, [Eq. (40)], the integral takes on the finite value $16\pi\gamma^3/3\sqrt{3}$, so that

$$E_{\mathrm{corr}}{}^{\mathrm{im}}=-\frac{\pi p_0e^2\gamma}{8\sqrt{3}}\left(\frac{p}{p_0}-1\right)^2. \tag{50}$$

Multiplying by -2 and taking the reciprocal gives the mean life. Including an additional factor of the Fermi velocity gives the mean free path λ. In units of the reduced de Broglie wavelength p_0^{-1} (essentially the interelectron spacing), we have

$$\begin{aligned}\lambda p_0=4\sqrt{3}\pi^{-1}\gamma^{-1}a_0p_0(p/p_0-1)^{-2}\\ =3.98r_s^{-\frac{1}{2}}(p/p_0-1)^{-2}.\end{aligned} \tag{51}$$

As pointed out in the preceding section on the specific heat, by including correlation the present theory passes continuously to the Sommerfeld case of no interaction as the interaction is turned off. This behavior is exhibited also by Eq. (51), which yields an infinite mean free path (in units of the deBroglie wavelength) in the limit $r_s\rightarrow 0$. The mean free path also becomes infinite as the excitation energy is reduced and $p\rightarrow p_0$. The damping approaches zero sufficiently rapidly that $dE_{\mathrm{corr}}{}^{\mathrm{im}}/dE|_{E=0}=0$. This would seem to be an *a priori* requirement on any reasonable theory of the self-energy, since without it the specific heat correction of Eq. (38) has no meaning.

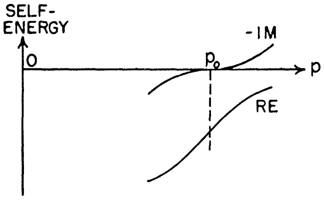

FIG. 6. Optical potential of an electron of momentum p moving through a degenerate electron gas of Fermi momentum p_0. The real part is represented by the curve labeled RE and has a finite value and slope at the Fermi surface. The curve labeled $-$IM is the negative of the imaginary part and gives rise to a damping of single-electron states. It has zero value and slope at the Fermi surface. The optical potential for $p<p_0$ is defined as the negative of the self-energy of a hole.

VII. OPTICAL POTENTIAL AND GROUND-STATE CORRELATION ENERGY

Figure 6 is a plot of both the real part and the negative of the imaginary part of the renormalized self-energy of an electron in a degenerate electron gas, $\Delta E(\mathbf{p})$, as a function of the electron momentum $\mathbf{p}$. This self-energy constitutes an effective potential similar to the optical potentials used in nuclear physics, where the real part gives rise to elastic scattering and the imaginary part to absorption. As emphasized above, the electron gas has a dual role: it acts as a dielectric medium which can be polarized and also as a vacuum from which electron-hole pairs can be created and undergo exchange with incident electrons.

In Sec. VI the imaginary part of the self-energy of an electron whose momentum is slightly greater than the Fermi momentum was determined and interpreted in terms of a damping of single-electron states. The curve labeled $-$IM appearing in Fig. 6 demonstrates the behavior of the negative of the imaginary part of the optical potential, which we designate by the function $\mathrm{IM}(p-p_0)\equiv\operatorname{Im}\Delta E(\mathbf{p})$. So far we have discussed only the self-energy associated with momenta $p>p_0$, corresponding to a particle outside the Fermi sea. As shown in Fig. 6, the optical potential can be extended to momenta $p<p_0$, inside the Fermi sea. Since the state of momentum $\mathbf{p}$ is normally occupied when $|\mathbf{p}|<p_0$ the optical potential in this case is defined as the negative of the self-energy of a hole of momentum $-\mathbf{p}$. In Appendix II it is proved that the renormalized hole self-energy is indeed correctly represented by the negative of the function $\Delta E(\mathbf{p})$ and furthermore that the imaginary part of this function has the following

property for small values of $p-p_0$:

$$\mathrm{IM}(-|p-p_0|)=-\mathrm{IM}(|p-p_0|). \tag{52a}$$

This change of sign at the Fermi surface is illustrated by the upper curve in Fig. 6 and has the consequence that holes inside the Fermi sea damp out in the same way as electrons outside. The mean free path of holes is thus also given by Eq. (51).

The extension of the real part of the optical potential, $\mathrm{RE}(p-p_0)\equiv\mathrm{Re}\Delta E(\mathbf{p})$ for momenta $p<p_0$ is given by the curve labeled RE in Fig. 6. It is established in Appendix II that $\mathrm{RE}(p-p_0)$ is continuous and has continuous slope at the Fermi surface. Therefore, for small values of $p-p_0$,

$$\begin{aligned}\mathrm{RE}(-|p-p_0|)-\mathrm{RE}(0)&\\ &=-[\mathrm{RE}(|p-p_0|)-\mathrm{RE}(0)].\end{aligned} \tag{52b}$$

Since the real part of the self-energy of a hole of momentum p is the negative of $\mathrm{RE}(p-p_0)$, it follows from Eq. (52) that the correlation correction to the density of states for holes is the same as for electrons. This was tacitly assumed when writing down Eq. (38) of Sec. V since holes and electrons contribute equally to the specific heat.

The sum of the real part of the potential at the Fermi surface together with the kinetic energy $E_0=p_0^2/2m$, represents the amount of energy needed to introduce into the system one additional electron of momentum p_0 while keeping the volume constant. The negative of this quantity, which is just the energy required to remove a particle of momentum p_0, is often used in nuclear physics and is called the separation energy. The value of the separation energy, which we designate by $-E_S$, is given by

$$E_S=(p_0^2/2m)+E_{\mathrm{ex}}(p_0)+E_{\mathrm{corr}}{}^{\mathrm{line}}(p_0). \tag{53}$$

From Eq. (29), $E_{\mathrm{ex}}(p_0)=-e^2p_0/\pi$. The residue contributions have been omitted; they vanish at the Fermi surface. Only $E_{\mathrm{corr}}{}^{\mathrm{line}}(p_0)$ remains to be evaluated. Introducing $v=w/(2k_0v_0z)$ and $P=p/p_0$ into Eq. (35) and simplifying gives

$$E_{\mathrm{corr}}{}^{\mathrm{line}}(\mathbf{p})=\frac{e^2p}{\pi^2}\int_{-\infty}^{\infty}dv\int_0^{\infty}dz\left(\frac{1}{\epsilon}-1\right)\times\int_{-1}^{1}d\mu\frac{P\mu-z}{(P\mu-z)^2+v^2}. \tag{54}$$

Performing the integrations over μ and z exactly gives an expression of the form $F\ln r_s+G$, where the functions F and G may be expanded as power series in r_s with coefficients which depend only on v and P. Integrating over z exactly is lengthy, but to obtain only the leading term in the expansion simplifying approximations can be made. Since the main contribution to the z integration arises from very small values of z, the z integration may be cut off at an upper limit of unity and the expression obtained from the μ integration may be expanded in powers of z, with only the lowest order terms retained. To terms of order less than z^3, the result of the integration over μ is given by

$$\int_{-1}^{1}d\mu\frac{P\mu-z}{(P\mu-z)^2+v^2}=-\frac{2z}{P^2+v^2}. \tag{55}$$

Before performing the integration over z, the expression for the dielectric constant can be simplified by expanding in powers of z and keeping only the lowest order terms. In terms of the parameters z and v, this approximate expression for the dielectric constant may be written

$$\epsilon=1+\frac{3}{16\gamma^2z^2}(1-v\tan^{-1}v^{-1}). \tag{56}$$

Substituting Eq. (55) and Eq. (56) into Eq. (54), integrating over z from zero to unity, and keeping only the leading term gives

$$E_{\mathrm{corr}}{}^{\mathrm{line}}(\mathbf{p})=\mathrm{ry}\frac{4\ln r_s}{\pi^3}\int_0^{\infty}\frac{1-v\tan^{-1}v^{-1}}{P^2+v^2}dv. \tag{57}$$

Differentiating Eq. (57) with respect to P and evaluating at the Fermi surface gives

$$\begin{aligned}\left.\frac{dE_{\mathrm{corr}}{}^{\mathrm{line}}}{dP}\right|_{P=1}&=-\mathrm{ry}\frac{8\ln r_s}{\pi^3}\int_0^{\infty}\frac{1-v\tan^{-1}v^{-1}}{(1+v^2)^2}dv,\\ &=-\mathrm{ry}\pi^{-2}\ln r_s,\end{aligned} \tag{58}$$

which (as stated in Sec. V), is of higher order in r_s than the terms included in the specific heat.

Evaluating the right-hand member of Eq. (57) at $p=p_0$ or $P=1$ gives $E_{\mathrm{corr}}{}^{\mathrm{line}}(p_0)=2\pi^{-2}\,\mathrm{ry}(1-\ln 2)\ln r_s$. This is only the leading term in r_s of an expression for $E_S-E_0-E_{\mathrm{ex}}(p_0)$ of the form $(A_0+A_1r_s+\cdots)\ln r_s+(B_0+B_1r_s+\cdots)$. Thus

$$A_0=2\pi^{-2}(1-\ln 2)\ \mathrm{ry}. \tag{59}$$

To determine the higher order terms the following corrections to $E_{\mathrm{corr}}{}^{\mathrm{line}}(p_0)$ must be taken into account: (1) the z integration in $E_{\mathrm{corr}}{}^{\mathrm{line}}(p_0)$ must be performed more accurately, (2) contributions of higher order self-energy graphs must be included, and (3) the dielectric constant must be evaluated to higher order in the expansion parameter e^2. Although the first item involves no essential difficulty the second and third steps introduce greater complications, depending on the accuracy desired, and have not yet been carried out. Consequently we do not attempt to go beyond the lowest order term here, although, as explained below, computation of the constant B_0 term would be equivalent to the calculation of Gell-Mann and Brueckner and should not be too difficult. By carrying it along we will be able to determine it from comparison with their results. Collecting terms, our expression for the negative of the separation energy is

$$E_S(r_s)\approx \mathrm{ry}\left[\frac{1}{\alpha^2 r_s^2}-\frac{2}{\pi\alpha r_s}+\frac{2}{\pi^2}(1-\ln 2)\ \ln r_s+B_0\right], \quad (60)$$

where, following the above authors, we denote the number $(4/9\pi)^{\frac{1}{3}}$ by α.

A theorem of Seitz[28] relates the correlation energy of a particle at the top of the Fermi sea to the average correlation energy per particle. We shall determine the total ground state energy per particle in the same way, that is by relating it to the total energy of a particle at the Fermi surface, or the negative of the separation energy. The average energy per particle of a degenerate Fermi gas consisting of N particles enclosed in a volume V, depends only on the density and can be expressed as some function $g(r_s)$. The total energy of the system is therefore $Ng(r_s)$. Both N and V are essentially infinite. Introducing one additional particle of momentum p_0 into the system while holding the volume constant changes the density and consequently the average energy per particle by an infinitesimal amount. Let r_s' describe the density of the system of $N+1$ particles in the volume V; then r_s' is related to r_s by the equation

$$r_s'=r_s-r_s/3N. \quad (61)$$

The average energy per particle for the system of $N+1$ particles is $g(r_s')$ and is related to $g(r_s)$ by the equation

$$g(r_s')=g(r_s)-\frac{r_s}{3N}g'(r_s), \quad (62)$$

where the prime on $g(r_s)$ signifies differentiation with respect to r_s. The total energy of the system of $N+1$ particles is $(N+1)[g(r_s)-\frac{1}{3}N^{-1}r_s g'(r_s)]$. The change in the energy of the system caused by introducing the additional particle is the difference between this quantity and $Ng(r_s)$, and must equal E_S:

$$g(r_s)-\tfrac{1}{3}r_s g'(r_s)=E_S(r_s). \quad (63)$$

By means of the integrating factor r_s^{-3} the left-hand member of Eq. (63) can be written as $-\frac{1}{3}r_s^4 d[r_s^{-3}g(r_s)]/dr_s$. Solving for the ground state energy per particle gives in general

$$g(r_s)=-3r_s^3\int^{r_s}E_S(r_s')r_s'^{-4}dr_s'. \quad (64)$$

A shortcoming of the self-energy method appears at this point: because of the constant of integration in Eq. (64) the coefficient of r_s^3 in the expansion for $g(r_s)$ remains undetermined. This will not, however, constitute any practical disadvantage until all the lower order terms have been evaluated. In principle the constant of integration is fixed by replacing the indefinite integral $-\int^{r_s}$ by the definite integral $\int_{r_s}^{\infty}$, since $g(r_s)$ must vanish in the limit $r_s\rightarrow\infty$. It does not appear feasible to use this prescription, however, because of the difficulty of computing the separation energy in the low density (strong interaction) range. In connection with the integration over the series expansion of the separation energy, a peculiar feature of Eq. (64) may be noted. When one carries out the integration, the term $A_3 r_s'^3 \ln r_s'$ in $E_S(r_s')$ leads to a term $-(3/2)A_3 r_s^3 \times(\ln r_s)^2$ in $g(r_s)$. The appearance of such a term in the expansion for the ground-state energy does not seem to have been allowed for by Gell-Mann and Brueckner.[4] Its absence would require $A_3=0$. For the special case of the electron gas, substituting from Eq. (60), integrating, and neglecting higher terms, we obtain

$$g(r_s)\approx \mathrm{ry}\left[\frac{3}{5}\frac{1}{\alpha^2 r_s^2}-\frac{3}{2\pi\alpha r_s}\right.$$
$$\left.+\frac{2}{\pi^2}(1-\ln 2)(\ln r_s+\tfrac{1}{3})+B_0\right]. \quad (65)$$

The r_s^{-2} and r_s^{-1} terms represent the average kinetic and exchange energies, respectively, while the $\ln r_s$ term in the correlation energy is in agreement with well-known results of other investigators. By identifying the constant term with that computed by Gell-Mann and Brueckner, we obtain $B_0=-0.117$. Alternatively, it is clear that a computation of B_0 by means of the self-energy method presented here would be equivalent to the work of these authors.

The significance of the second term in the left-hand member of Eq. (63) is easily seen by considering the pressure, which is the negative of the derivative of the total energy with respect to the volume:

$$P(r_s)=-\frac{d[Ng(r_s)]}{dV}=-n\frac{r_s}{3}g'(r_s), \quad (66)$$

where $n=N/V$ is the average particle density. Thus Eq. (63) can be written in the form

$$E_S(r_s)=g(r_s)+n^{-1}P(r_s). \quad (67)$$

An alternative derivation of Eq. (67) is provided by keeping the density instead of the volume constant. Then the additional term represents just the work required to expand the system against the external pressure in order to make room for the added particle. If, instead of the electron gas with a uniform background of positive charge, we consider actual metals at equilibrium, then the pressure is zero and Eq. (67) reduces to

$$E_S(r_s)=g(r_s), \quad (68)$$

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[28] See reference 6, p. 343.

where r_s corresponds to the equilibrium density. This equality between the negative separation energy and the energy per particle is simply Bethe's theorem.[14] By means of it the separation energy gives the ground state energy directly, without an integration over density. Of course, for real metals the effect of the lattice on E_S must be included, perhaps for monovalent metals by the Wigner-Seitz cellular method.[29] The integration over density is required only for the determination of the equilibrium value of r_s. Once this is fixed, either by minimizing the function $g(r_s)$ defined by Eq. (64) or by referring to the empirical density, the correlation energy is given by Eq. (68). Furthermore, differentiating Eq. (63) shows that at equilibrium all higher derivatives of $g(r_s)$ can be expressed in terms of derivatives of the separation energy. For example, the compressibility depends upon the first derivative:

$$g''(r_s) = -\frac{3}{r_s} E_S'(r_s). \tag{69}$$

VIII. SUMMARY

The correlation energy of a degenerate electron gas at high density has been evaluated by a new simplified method in which attention is focused on a single electron at the surface of the Fermi sea. The self-energy of such an electron was evaluated by making use of Lindhard's frequency and wave-number dependent dielectric constant. In the less ideal case of a real metal it might be possible to substitute an empirically determined function, or perhaps a simplified approximate one which would make the integrations easier. The self-energy led to the high-density expression for the correlation energy, while the derivative of the self-energy with respect to momentum yielded the correlation correction to the specific heat. Although these results were already known, an interesting by-product of the calculation was the optical potential, in analogy to the similar momentum dependent quantity in nuclear physics. The imaginary part of the optical potential gave a mean free path inversely proportional to the square of the distance (in momentum space) of the electron or hole from the Fermi surface. Thus the concept of a single-particle excitation of the electron gas is valid for small values of the excitation energy but breaks down for larger values. Although the calculations have been carried through only for lowest order in the density parameter r_s, the relatively simple and intuitive nature of the self-energy approach should make it feasible to carry the calculations to higher order, corresponding to lower density.

In conclusion, one of us (RAF) wishes to acknowledge several valuable and stimulating discussions with D. F. Dubois during initial states of this investigation.

APPENDIX I. MOMENTUM-EXCITON AS PROPAGATOR OF THE POLARIZATION

The external charge distribution $\rho(\mathbf{x}_1,t_1)$ discussed in Sec. III sets up an electrostatic potential

$$\textstyle\sum_{\mathbf{k}} \Phi(\mathbf{k},t_1) e^{i\mathbf{k}\cdot\mathbf{x}_1}.$$

The Fourier transform $\Phi(\mathbf{k},t_1)$ is related to the charge distribution by the equation

$$\Phi(\mathbf{k},t_1) = \frac{4\pi}{k^2 V} \int e^{-i\mathbf{k}\cdot\mathbf{x}_1} \rho(\mathbf{x}_1,t_1) d^3x_1. \tag{70}$$

The interaction Hamiltonian representing the perturbation on the electron gas caused by this electrostatic potential is

$$H' = -e \textstyle\sum_{\mathbf{k}} \Phi(\mathbf{k},t_1) \sum_i e^{i\mathbf{k}\cdot\mathbf{x}_i}, \tag{71}$$

where the sum over the index i is over all electrons in the gas. The action of H' on the electron gas can cause a transition from the ground state Ψ_0 to an excited state Ψ_n. According to the momentum-exciton model[8] the excited state wave function Ψ_n is a linear superposition of one-electron excited states Φ_i:

$$\Psi_n = \textstyle\sum_i A_{in}(\mathbf{k}) \Phi_i. \tag{72}$$

The matrix element H_{n0}' is given by

$$H_{n0}' = -e\Phi(\mathbf{k},t_1) C_n^*(\mathbf{k}), \tag{73}$$

where

$$C_n(\mathbf{k}) = \textstyle\sum_i A_{in}(\mathbf{k}). \tag{74}$$

Although Eq. (72) holds only when the backward-going graphs are neglected, Eqs. (73) and (74) are independent of this approximation. $A_{in}(\mathbf{k})$ of Eq. (74) is the sum of the forward and backward propagating amplitudes $A_{in}^{\pm}(\mathbf{k})$. The amplitude of the intermediate state n at time t_2 is given by

$$\mathcal{A}_n(\mathbf{k},t_2) = -i \int_{-\infty}^{t_2} H_{n0}' e^{-iE_n t_{21}} dt_1. \tag{75}$$

Substitution from Eqs. (73) and (70) gives

$$\mathcal{A}_n(\mathbf{k},t_2) = i\frac{4\pi e}{V} \int d^3x_1 \int_{-\infty}^{t_2} dt_1 \frac{1}{k^2} e^{-i\mathbf{k}\cdot\mathbf{x}_1 - iE_n t_{21}} \times C_n^*(\mathbf{k}) \rho(\mathbf{x}_1,t_1). \tag{76}$$

The effective electrostatic potential at point 2 in space-time can be determined from the scattering of a test charge. The gas is in some excited state Ψ_n with amplitude $\mathcal{A}_n(\mathbf{k},t_2)$. The test charge can cause a transition back to the ground state Ψ_0 in the scattering process. The effective electrostatic potential acting on the test charge is given by

$$\varphi(\mathbf{x}_2,t_2) = \sum_{n,\mathbf{k}} \left(\Psi_0, \sum_i \frac{-e}{|\mathbf{x}_2 - \mathbf{x}_j|} \mathcal{A}_n(\mathbf{k},t_2) \Psi_n \right). \tag{77}$$

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[29] E. P. Wigner and F. Seitz, Phys. Rev. 43, 804 (1933); 46, 509 (1934).

By means of Eqs. (72) and (76), $\varphi(\mathbf{x}_2,t_2)$ can be written

$$\varphi(\mathbf{x}_2,t_2)=\int d^3x_1\int_{-\infty}^{t_2}dt_1\left[-i\left(\frac{4\pi}{V}\right)^2e^2\times\sum_{\mathbf{k},n}\frac{1}{k^4}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-iE_nt_{21}}|C_n(\mathbf{k})|^2\right]\rho(\mathbf{x}_1,t_1).\quad(78)$$

The quantity in the brackets appearing in the integrand of Eq. (78) is by definition the propagator $G_+(2,1)$:

$$G_+(2,1)=-i\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2}M(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}_{21}}\times\sum_n|C_n(\mathbf{k})|^2e^{-iE_nt_{21}},\quad(79)$$

where the sum over $\mathbf{k}$ has been replaced by an integral and $M(\mathbf{k})=4\pi e^2/k^2V$ [Eq. (4) of reference 8].

Let us now concentrate on the continuum contribution, set $E_n\equiv\omega$, and replace the sum over n by an integral. This gives

$$G_+^{\text{cont}}(2,1)=-i\int\frac{4\pi}{k^2}|C_n(\mathbf{k})|^2\times2\pi\rho(\omega)M(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}\frac{d^3kd\omega}{(2\pi)^4},\quad(80)$$

where $\rho(\omega)$ is the density of states. The quantity $|C_n(\mathbf{k})|^2$ can be determined from the momentum-exciton model. The coefficients $A_{in}(\mathbf{k})$ are defined by the equations $A_{in}=A_{in}^++A_{in}^-$ and

$$A_{in}^{\pm}(\mathbf{k})=\frac{M(\mathbf{k})C_n(\mathbf{k})}{\pm E_n-T_i^{\pm}},\quad(81\text{a})$$

where $T_i^{\pm}=E(\mathbf{k}_i\pm\mathbf{k})-E(\mathbf{k}_i)$ [Eqs. (17) and (18) of reference 8]. Although the wave function Ψ_n is normalized according to

$$\sum_i[|A_{in}^+(\mathbf{k})|^2-|A_{in}^-(\mathbf{k})|^2]=1\quad(81\text{b})$$

[Eq. (20) of reference 8], it is easily seen that the backward propagating amplitudes do not contribute to the normalization for the continuum states. Consequently $|C_n(\mathbf{k})|^2$ is given by

$$|C_n(\mathbf{k})|^2=M(\mathbf{k})^{-2}\left[\sum_i\frac{1}{(E_n-T_i^+)^2}\right]^{-1}.\quad(82)$$

By performing a summation over i on both sides of Eq. (81a), one arrives at the relation

$$\sum_i\left[\frac{1}{E_n-T_i^+}-\frac{1}{E_n+T_i^-}\right]=M(\mathbf{k})^{-1}.\quad(83)$$

Since the propagator can be expected to be independent of the detailed spectrum of the $T_i^{\pm}$, we may assume for convenience that they are uniformly spaced[30] in the vicinity of E_n corresponding to the density of states $\rho(E_n)$. The deviation of the spectrum from this constant state density is expressed by a Cauchy principle value integral:

$$\sum_i\left[\frac{1}{E_n-T_i^+}-\frac{1}{E_n+T_i^-}\right]=-\rho(E_n)\sum_{m=-\infty}^{\infty}\frac{1}{m-\lambda}+\frac{V}{(2\pi)^3}P\int d^3k_i\left[\frac{1}{E_n-T_i^+}-\frac{1}{E_n+T_i^-}\right],\quad(84)$$

where $\lambda=(E_n-E')/\Delta E$, $T_i=E'+m\Delta E$, m is an integer, $\Delta E=\rho(E_n)^{-1}$ is the spacing between consecutive energy levels, and E' is one of the T_i in the vicinity of E_n. Solving Eqs. (84) and (83) for the sum over m gives

$$\sum_{m=-\infty}^{\infty}\frac{1}{m-\lambda}=-\frac{1}{M\rho}\left\{1-\frac{MV}{(2\pi)^3}P\int d^3k_i\times\left[\frac{1}{E_n-T_i^+}-\frac{1}{E_n+T_i^-}\right]\right\}=-\frac{\epsilon_1(\mathbf{k},\omega)}{M(\mathbf{k})\rho(\omega)},\quad(85)$$

where the quantity in curly braces has been identified as Lindhard's expression for the real part of the dielectric constant. Now a well-known expression for the above sum as a function of the variable λ is

$$\sum_{m=-\infty}^{\infty}\frac{1}{m-\lambda}=-\pi\cot\pi\lambda,\quad(86)$$

which by substitution into Eq. (85) fixes the eigenvalues λ precisely. Differentiation of Eq. (86) followed by substitution from Eq. (85) leads to the useful result

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m-\lambda)^2}=\pi^2\csc^2\pi\lambda=\pi^2+\frac{\epsilon_1^2}{M^2\rho^2}.\quad(87)$$

The sum in the right-hand member of Eq. (82) can consequently be expressed as

$$\sum_i(E_n-T_i^+)^{-2}=\rho^2\sum_{m=-\infty}^{\infty}\frac{1}{(m-\lambda)^2}=M^{-2}[(\pi M\rho)^2+\epsilon_1^2]=M^{-2}|\epsilon|^2,\quad(88)$$

where we have substituted from Eq. (87) and have used the equality

$$\epsilon_2=\pi M\rho.\quad(89)$$

ϵ_2 is the imaginary part of ϵ, the complex dielectric constant. Inserting this result into Eq. (82) gives

$$|C_n(\mathbf{k})|^2=|\epsilon(\mathbf{k},\omega)|^{-2}.\quad(90)$$

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[30] This treatment is suggested by a simplified model of radiation damping which is worked out in the paper of V. Weisskopf and E. P. Wigner [Z. Physik 63, 54 (1930)].

Returning now to the Green's function, we substitute from Eqs. (89) and (90) to obtain

$$G_+{}^{\mathrm{cont}}(2,1)=\int_L \frac{d^3k d\omega}{(2\pi)^4}\frac{4\pi}{k^2\epsilon(\mathbf{k},\omega)}e^{i\mathbf{k}\cdot\mathbf{x}_{21}-i\omega t_{21}}. \quad (91)$$

The subscript L on the integral sign signifies that the ω integration is to be carried out over the part of the loop shown in Fig. 3(c) which encircles the cut. In arriving at Eq. (91), we have used the fact that ϵ_2 changes sign across the cut while ϵ_1 remains unchanged. For those values of $\mathbf{k}$ for which plasma oscillations exist there is an additional discrete contribution to $G_+(2,1)$ corresponding to plasmon excitation, whose energy we denote by E_1. According to Eqs. (79), (81a), and (81b), the magnitude of this contribution to the propagator is determined by

$$|C_1(\mathbf{k})|^{-2}=M(\mathbf{k})^2\textstyle\sum_i[(E_1-T_i^+)^{-2}-(E_1+T_i^-)^{-2}], \quad (92)$$

which differs from Eq. (82) in that the effect on the normalization of the backward propagating amplitudes in the excited state of the gas can no longer be neglected. The sum in Eq. (92) must be evaluated somewhat differently from that in Eq. (82). Again we consider the real part of Lindhard's expression for the dielectric constant:

$$\epsilon_1(\mathbf{k},\omega)=1-M(\mathbf{k})P\sum_i\left(\frac{1}{\omega-T_i^+}-\frac{1}{\omega+T_i^-}\right), \quad (93)$$

but now for ω in the vicinity of E_1 we are clear of the continuum and can drop the P which indicates the principal value sum. Designating the partial derivative with respect to ω by a prime, we then have

$$\epsilon_1'(\mathbf{k},\omega)=M(\mathbf{k})\textstyle\sum_i[(\omega-T_i^+)^{-2}-(\omega+T_i^-)^{-2}], \quad (94)$$

so

$$|C_1(\mathbf{k})|^2=\frac{1}{M\epsilon_1'(\mathbf{k},E_1)}. \quad (95)$$

Substitution into Eq. (79) gives for the plasmon contribution to the Green's function

$$\begin{aligned}G_+{}^{\mathrm{plasm}}(2,1)&=-i\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^2\epsilon_1'(\mathbf{k},E_1)}e^{i(\mathbf{k}\cdot\mathbf{x}_{21}-E_1t_{21})}\\&=\oint_{\mathrm{cw}}\frac{d^3kd\omega}{(2\pi)^4}\frac{4\pi}{k^2\epsilon(\mathbf{k},\omega)}e^{i(\mathbf{k}\cdot\mathbf{x}_{21}-\omega t_{21})}, \quad (96)\end{aligned}$$

where the integration is clockwise (cw) on a small circle about the root of the dielectric constant and is illustrated by the right-hand portion of Fig. 3(b). Addition of the continuum and plasmon contributions yields a contour of integration equivalent to that shown in Fig. 3(c) and consequently leads to Eq. (21) for the polarization propagator.

APPENDIX II. PROPERTIES OF THE OPTICAL POTENTIAL

In Sec. VII the optical potential, which we write as $\mathrm{RE}(p-p_0)+i\,\mathrm{IM}(p-p_0)$ was defined for $p<p_0$ as the negative of the self-energy of a hole. The latter quantity is represented by the Feynman graphs which result from reversal of the directions of the arrows in Fig. 1. Consider specifically the process which is represented by reversing the direction of the arrows in Fig. 1(a). Let the initial and final states consist of a hole of momentum $-\mathbf{p}$ and the intermediate state of a hole of momentum $-\mathbf{p}'$ and in addition the disturbance in the electron gas of momentum $\mathbf{p}'-\mathbf{p}$. Also interchange the labels "1" and "2" on the space-time points of interaction. What actually happens at time t_2 is that an electron of momentum $+\mathbf{p}'$ makes a jump into the vacancy in momentum state $+\mathbf{p}$ while at the same time giving rise to a disturbance in the electron gas. At time t_1 the electron jumps back into the state of momentum $+\mathbf{p}'$ and absorbs the disturbance. The factor in the matrix element for the entire process due to these jumps is

$$\begin{aligned}&\sum_{p'<p_0}[e^{i\mathbf{p}\cdot\mathbf{x}_2-iE(\mathbf{p})t_2}]^*e^{i\mathbf{p}'\cdot\mathbf{x}_2-iE(\mathbf{p}')t_2}\\&\qquad\times[e^{i\mathbf{p}'\cdot\mathbf{x}_1-iE(\mathbf{p}')t_1}]^*e^{i\mathbf{p}\cdot\mathbf{x}_1-iE(\mathbf{p})t_1}, \quad (97)\end{aligned}$$

where we have summed over the intermediate states. Consequently the self-energy of a hole of momentum $-\mathbf{p}$, distinguished from the corresponding quantity for an electron by a prime, is given by

$$\begin{aligned}\mathrm{S.E.}'(-\mathbf{p})=e^2\int d^3x_1dt_1\,&e^{-i\mathbf{p}\cdot\mathbf{x}_2+iE(\mathbf{p})t_2}\\&\times[\sum_{p'<p_0}e^{i\mathbf{p}'\cdot\mathbf{x}_{21}-iE(\mathbf{p}')t_{21}}]G_+(2,1)\\&\times e^{i\mathbf{p}\cdot\mathbf{x}_1-iE(\mathbf{p})t_1}+\cdots, \quad (98)\end{aligned}$$

where the dots indicate a similar integral in which the sum is over positive energy intermediate states. The quantity in brackets in Eq. (98) can be identified with the negative-energy portion of $-K_+(2,1)$, the negative of the electron propagator. With a similar identification of the positive-energy terms, Eq. (98) becomes

$$\begin{aligned}\mathrm{S.E.}'(-\mathbf{p})=-e^2\int d^3x_1dt_1\,&e^{-i\mathbf{p}\cdot\mathbf{x}_2+iE(\mathbf{p})t_2}\\&\times K_+(2,1)G_+(2,1)e^{i\mathbf{p}\cdot\mathbf{x}_1-iE(\mathbf{p})t_1}. \quad (99)\end{aligned}$$

The right-hand side of Eq. (99) is simply the negative of the function S.E.(p), which has been found [Eq. (23)] to represent the self-energy of an electron of momentum $+\mathbf{p}$ for the case $p>p_0$. This function can be extended into the interval $p<p_0$, and Eq. (99) can be written

$$\mathrm{S.E.}'(-\mathbf{p})=-\mathrm{S.E.}(\mathbf{p}). \quad (100)$$

This must be compared with the negative of the vacuum self-energy of an electron of momentum $\mathbf{p}$, which we can term purely for convenience the "vacuum self-energy of a hole," and which we define formally by the equation

$$[\text{S.E.}'(-\mathbf{p})]_0 = -\text{S.E.}(\mathbf{p})_0. \qquad (101)$$

The difference represents the observable self-energy of a hole of momentum $-\mathbf{p}$:

$$\text{S.E.}'(-\mathbf{p}) - [\text{S.E.}'(-\mathbf{p})]_0 = -\{\text{S.E.}(\mathbf{p}) - [\text{S.E.}(\mathbf{p})]_0\} = -\Delta E(\mathbf{p}), \qquad (102)$$

where the renormalized self-energy function $\Delta E(\mathbf{p})$ has already been studied for $p > p_0$ in Secs. IV–VII. Because of the minus sign in the definition of the optical potential for $p < p_0$, we have the equality

$$\text{RE}(p - p_0) + i\,\text{IM}(p - p_0) = \Delta E(\mathbf{p}) \qquad (103)$$

for all values of p. The real and negative imaginary parts of Eq. (103) are the quantities plotted as functions of p in Fig. 6.

The function $\text{RE}(p-p_0)$ is, according to Eqs. (103) and (36), given by

$$\text{RE}(p - p_0) = E_{\text{corr}}{}^{\text{line}}(\mathbf{p}) + E_{\text{ex}}{}'(\mathbf{p}) + E_{\text{corr}}{}^{\text{res}}(\mathbf{p}). \qquad (104)$$

As shown in Sec. VII the slope of $E_{\text{corr}}{}^{\text{line}}(\mathbf{p})$ at the Fermi surface is negligible in the high density limit. Therefore, for momenta $\mathbf{p}$ for which $p - p_0$ is small, $E_{\text{corr}}{}^{\text{line}}(\mathbf{p})$ can be set equal to its value at the Fermi surface. The second term is given by Eq. (32) and clearly exhibits the continuity at the Fermi surface required of RE and its derivative. For $p > p_0$, $E_{\text{corr}}{}^{\text{res}}(\mathbf{p})$ is defined by Eq. (34) but for $p < p_0$, the pole which gives rise to $E_{\text{corr}}{}^{\text{res}}(\mathbf{p})$ is in the third quadrant and deforming the contour encircles this pole in a clockwise direction. This contributes an additional minus sign to the integral which is otherwise a function only of the absolute value $|E| = v_0|p - p_0|$. Consequently Eq. (39) holds in general for all momentum vectors near the Fermi surface, inside as well as outside, and all the terms in $\text{RE}(p-p_0)$ have the required continuity of value and slope expressed by Eq. (52b).

The imaginary part of the optical potential for $p < p_0$ is given by

$$\text{IM}(p - p_0) = E_{\text{corr}}{}^{\text{im}}(\mathbf{p}) = -e^2 \int_{E \leq E' \leq 0} \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} \,\text{Im} \frac{1}{\epsilon(\mathbf{k}, E - E' - i\delta)}, \qquad (105)$$

where as explained in the preceding paragraph, the additional minus sign relative to Eq. (34) results from encircling the pole in the opposite direction. By means of Eq. (22) it is readily seen that the integrals in Eqs. (105) and (34) are the same function of the absolute value $|E| = v_0|p - p_0|$ and consequently that near the Fermi surface IM is an odd function of $p - p_0$, as expressed by Eq. (52a).