Structure of Particles in Linearized Gravitational Theory*

R. SACHS† AND P. G. BERGMANN
Department of Physics, Syracuse University, Syracuse, New York
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We have examined the invariant character of restrictions imposed on singularities or other sources of the gravitational fields in the Einstein-Infeld-Hoffmann theory of motion in general relativity. We have succeeded in providing a complete classification of sources that can occur in the linearized theory only, in terms of properties that are invariant under Lorentz and "gauge" transformations (the latter designation refers to linearized curvilinear coordinate transformations). Except for several explicitly known solutions, all solutions can be derived from a "supermetric" corresponding to the Hertz potential of electrodynamics. One method of classification is in terms of gauge-invariant integrals over spatial closed surfaces completely surrounding the particles. The motion of each source is determined by its own intrinsic angular momentum and dipole moment. The results do not depend on any particular assumed form of the stress-energy tensor of the sources.

1. INTRODUCTION

SINCE 1938, a great deal of attention has been given to the problem of deriving the equations of motion for gravitating particles from Einstein's field equations of general relativity.[1–5] The fact that in general relativity many different metric fields represent the same physical situation leads to peculiar difficulties of interpretation which so far have been resolved only in part. A similar many-to-one correspondence between the mathematics and the physics of a given situation in Lorentz-invariant theories causes no serious difficulties because the multiplicity of mathematical representations is much smaller; in general relativity, on the other hand, we do not have available a complete set of "true observables,"[6] whose numerical values would determine[2] the physical situation completely and would be, in turn, completely determined by it. To this extent the physical interpretation of a given metric depends on *ad hoc* procedures; likewise, the construction of a metric to represent a given physical situation is often rendered more difficult by the fact that we do not know how to cast our physical notions into the relevant mathematical form.

We may possibly avoid these difficulties by restricting all our considerations to a single coordinate system, or to a set of coordinate systems whose multiplicity is not essentially greater than the number of different Lorentz systems.[7] The corresponding procedure in electrodynamics has been very successful in dealing with the problems raised by gauge invariance; however, in gravitational theory its usefulness is still doubtful. We shall adopt an alternative approach, which consists of attempting to formulate all physically significant relationships in a coordinate-invariant fashion. Ideally, in this approach, we should be able to classify invariantly the various possible structures a single particle might possess, and to describe invariantly how a set of such particles moves in their joint gravitational field.

As a contribution to such a program we have obtained an invariant description of the structures and motions of particles in the linear approximation of general relativity. The linear approximation is the first step in a systematic approximation procedure, which consists of the expansion of the metric in powers of its deviation from the flat Lorentz metric.[8,9] We shall show that in the linearized theory all those solutions of the field equations that correspond to particles can be generated in a simple fashion; their structures and motions can be classified in a way that is Lorentz covariant and invariant under "gauge" transformations, the latter being those curvilinear coordinate transformations whose products with Lorentz transformations yield the full group of coordinate transformations appropriate to the linearized theory.

In addition to emphasizing invariance properties, we have formulated our results in a way that does not depend on an arbitrarily assumed stress-energy tensor for the particles. To this end we introduce in space-time a number of "world tubes"—four-dimensional regions of finite spatial cross section bounded by timelike lines. Within the world tubes there are sources of the field: either the metric becomes singular or the field equations acquire nonvanishing right-hand sides. Outside the tubes the metric obeys the vacuum field equations. Such a treatment reflects our actual ignorance of the precise structure of the matter tensor. Incidentally, our results remain valid if within the world tubes the gravitational field is so large that the linear approximation breaks down there but remains valid outside. In such a treat-

* Supported in part by the National Science Foundation and the Air Force Office of Scientific Research.
† Present address: Institute for Theoretical Physics (Professor P. Jordan), University of Hamburg, Hamburg, Germany.
1 Einstein, Infeld, and Hoffmann, Ann. Math. **39**, 66 (1938).
2 A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).
3 L. Infeld and A. Schild, Revs. Modern Phys. **21**, 408 (1949).
4 B. Bertotti, Nuovo cimento **12**, 226 (1954).
5 L. Infeld and J. Plebianski, Bull. intern. acad. polon. sci., Class III, 4, 757 (1956).
6 A. Komar has now reported such a set; Phys. Rev. **111**, 1182 (1958).
7 V. Fock, Revs. Modern Phys. **29**, 325 (1957).
8 S. N. Gupta, Phys. Rev. **96**, 1683 (1954).
9 E. Newman and P. G. Bergmann, Revs. Modern Phys. **29**, 443 (1957).

ment the appropriate quantities for classifying the sources are the fields outside the world tubes; we develop suitable techniques for the classification in the next two sections.

By linearizing the field equations we destroy some of their important qualitative features. In particular, two different particles do not interact in this approximation; the motion of any one particle depends only on its own structure, and any particle which does not grow a dipole moment is constrained to move with constant velocity. Likewise, if we define the energy density of the field either by means of the canonical stress-energy tensor or as in the next section, we find that the first-order total energy density contains contributions only from the sources, not from the field itself. It is also quite possible that some of the solutions of the linearized field equations are not approximations to any exact solution. On the other hand, some important qualitative features of the full theory, such as the Bianchi identities, do have their counterparts in the linear approximation. The linear theory is thus useful in allowing us to see precisely to what extent the Bianchi identities between the field equations limit the nature and motion of the field sources. Furthermore, in searching for invariant properties in the exact theory, or in higher approximations, we can limit ourselves to those quantities whose linear approximations are gauge-invariant (or zero).

2. PRELIMINARY RESULTS IN THE FULL THEORY

In this section we shall introduce some definitions and relations which are valid in general relativity independently of any approximation scheme; later sections will deal with the corresponding linear approximations. We use units such that the velocity of light is unity; the symbol k designates the gravitational constant.

The tensor density of weight two,

$$H^{[\mu\alpha][\nu\beta]}=\frac{1}{16\pi k}(\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta}-\mathfrak{g}^{\mu\beta}\mathfrak{g}^{\alpha\nu}), \tag{2.1}$$

has the symmetry properties

$$\begin{aligned} &H^{[\mu\alpha][\nu\beta]}=H^{[\nu\beta][\mu\alpha]}=-H^{[\beta\nu][\mu\alpha]},\\ &H^{[\mu\alpha][\nu\beta]}+H^{[\mu\beta][\alpha\nu]}+H^{[\mu\nu][\beta\alpha]}=0. \end{aligned} \tag{2.2}$$

From it we can form two identically conserved "pseudotensors" (e.g., quantities which transform as tensors under linear coordinate transformations)[10]:

$$\mathcal{T}^{\mu\nu}=H^{[\alpha\mu][\beta\nu]}{}_{,\alpha\beta},\quad \mathcal{T}^{\mu\nu}{}_{,\nu}\equiv 0,\quad \mathcal{T}^{\mu\nu}=\mathcal{T}^{\nu\mu}, \tag{2.3}$$

$$L^{\mu[\nu\rho]}=\mathcal{T}^{\mu\nu}x^{\rho}-\mathcal{T}^{\mu\rho}x^{\nu},\quad L^{\mu[\nu\rho]}{}_{,\mu}\equiv 0. \tag{2.4}$$

The identity sign will frequently be used, as in the above equations, to emphasize that the equality holds whether or not the field equations have vanishing right-hand sides. We shall call $\mathcal{T}^{\mu\nu}$ and $L^{\mu[\nu\rho]}$ the total energy density and total angular momentum density, respectively,[11] without intending to imply that their physical interpretation in the exact theory has as yet been fully settled. Actually there are many different identically conserved quantities any one of which might conceivably be considered the stress-energy pseudotensor[12] and many pseudotensors which might conceivably be identified with the angular momentum density. Regardless of how we interpret $\mathcal{T}^{\mu\nu}$ and $L^{\mu[\nu\rho]}$, the linear approximations to the surface integrals (2.5) and (2.6) below are very useful for classifying solutions of the linearized field equations.

The total energy or angular momentum contained in a finite portion of a three-dimensional space-like hypersurface S can be measured by means of integrals over a closed two-dimensional surface:

$$\begin{aligned} &P^{\mu}=\int_S dS_{\rho}\,\mathcal{T}^{\mu\rho}\equiv\oint_f df^{*}{}_{[\rho\sigma]}\,H^{[\rho\sigma][\mu\alpha]}{}_{,\alpha},\\ &L^{[\mu\nu]}=-\int_S dS_{\rho}\,L^{\rho[\mu\nu]}\equiv\oint_f df^{*}{}_{[\rho\sigma]}\\ &\qquad\qquad\times\{u^{[\rho\sigma]\nu}x^{\mu}-u^{[\rho\sigma]\mu}x^{\nu}+H^{[\rho\sigma][\mu\nu]}\}, \end{aligned} \tag{2.5}$$

$$u^{[\rho\sigma]\mu}=H^{[\rho\sigma][\mu\alpha]}{}_{,\alpha}. \tag{2.6}$$

where f is the surface bounding S. Here we have used the notation of reference 10. If S is a region of three space at constant time, Eqs. (2.5) and (2.6) are simply statements of Gauss' theorem: for this case $\int_S dS_{\mu}\,\phi^{\mu}\to\int_V \phi^4 dV$, where dV and V are the three-dimensional volume element and volume, respectively; and $\oint_f df^{*}{}_{[\rho\sigma]}\,\phi^{[\rho\sigma]}\to\int_A \phi^{[4s]}n_s dA$, where n_s is the unit normal in three space, dA is the element of surface area, and A the total surface area.

We next introduce a quantity which we shall call the supermetric, in analogy to the Hertz superpotentials of electrodynamics. Given a metric that satisfies the field equations and is nonsingular everywhere outside certain world tubes (which surround the trajectories of particles), we can construct within the world tubes a metric which is also nonsingular, but does not satisfy the vacuum field equations, by extending the given outside metric with sufficiently differentiable and otherwise arbitrary functions throughout the interior of the world tubes. If we introduce a coordinate system in which the extended metric obeys the De-Donder coordinate conditions,

$$\mathfrak{g}^{\mu\nu}{}_{,\nu}\equiv 0, \tag{2.7}$$

we can find in this coordinate system a $V^{\mu[\nu\alpha]}$ such that

$$\mathfrak{g}^{\mu\nu}=V^{\mu[\nu\alpha]}{}_{,\alpha}. \tag{2.8}$$

Furthermore, because of the symmetry of the metric

[10] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Inc., Cambridge, 1951).

[11] The angular momentum density includes the dipole moment density.

[12] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).

$\mathfrak{g}^{\mu\nu}$ there exists a supermetric $M^{[\mu\alpha][\nu\beta]}$ such that

$$\mathfrak{g}^{\mu\nu}=M^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}, \tag{2.9}$$

with

$$M^{[\mu\alpha][\nu\beta]}=M^{[\nu\beta][\mu\alpha]}.$$

In the linearized theory (2.9) can be used to generate solutions of the field equations by a suitable choice of $M^{[\mu\alpha][\nu\beta]}$. We can add to $M^{[\mu\alpha][\nu\beta]}$ any term of the form $F^{[\mu\alpha\rho][\nu\beta\sigma]}{}_{,\rho\sigma}$, with $F^{[\mu\alpha\rho][\nu\beta\sigma]}=F^{[\nu\beta\sigma][\mu\alpha\rho]}$, without changing the value of the metric, $\mathfrak{g}^{\mu\nu}$. Also, if

$$M'^{[\mu\alpha][\nu\beta]}=M^{[\mu\alpha][\nu\beta]}-(1/24)\epsilon^{\mu\alpha\nu\beta}\epsilon_{\rho\sigma\gamma\tau}M^{[\rho\sigma][\gamma\tau]}, \tag{2.10}$$

then $\mathfrak{g}'^{\mu\nu}=\mathfrak{g}^{\mu\nu}$, and

$$M'^{[\mu\alpha][\nu\beta]}+M'^{[\mu\beta][\alpha\nu]}+M'^{[\mu\nu][\beta\alpha]}\equiv 0. \tag{2.11}$$

In general $M^{[\mu\alpha][\nu\beta]}$ has 21 independent components whereas $M'^{[\mu\alpha][\nu\beta]}$ has only 20.

3. THE LINEARIZED THEORY

To formulate the linearized theory we suppose that the metric tensor is expanded in the powers of some parameter λ,

$$\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}+\lambda\gamma^{\mu\nu}+\cdots. \tag{3.1}$$

$\eta^{\mu\nu}$ is the Lorentz metric, with diagonal components -1, -1, -1, 1; indices raised or lowered with the Lorentz metric will be underlined and remain in their original positions. The usual physical interpretation of λ is that $\lambda=k\bar{m}$ where $\bar{m}$ is a suitable sum or average of the masses appearing in a given problem; λ then has the dimensions of a length. The expansion (3.1) induces in all other quantities of interest, such as the curvature tensor or the integrals discussed in Sec. II, a corresponding expansion,

$$Q={}_0Q+\lambda\,{}_1Q+\cdots, \tag{3.2}$$

where Q is any quantity with any number of superscripts or subscripts. All further calculations in this paper will be accurate only to the first order in λ.

Coordinate transformations which lead from a metric of the form (3.1) to another whose zeroth approximation is the Lorentz metric have the form:

$$x'^{\rho}=\omega_{\nu}{}^{\rho}x^{\nu}+\lambda\xi^{\rho}(x). \tag{3.3}$$

$\omega_{\nu}{}^{\rho}$ represents a Lorentz transformation and ξ^{ρ} is arbitrary. Within the world tubes where the metric is singular or unspecified we allow singular or unspecified ξ^{ρ}. The transformation group (3.3) contains a normal subgroup of "gauge"[13] transformations for which $\omega_{\nu}{}^{\rho}=\delta_{\nu}{}^{\rho}$. In the linearized theory only quantities which are gauge invariant have direct physical significance. Under a gauge transformation, the $\gamma^{\mu\nu}$ transform as follows:

$$\gamma'^{\mu\nu}(x)=\gamma^{\mu\nu}(x)+\eta^{\nu\sigma}\xi^{\mu}{}_{,\sigma}+\eta^{\mu\sigma}\xi^{\nu}{}_{,\sigma}-\eta^{\mu\nu}\xi^{\rho}{}_{,\rho}. \tag{3.4}$$

[13] There will be no confusion with the true gauge transformations, as we shall be concerned with purely gravitational fields in this paper.

The first-order field equations

$${}_1G^{\mu\nu}=\tfrac{1}{2}[\eta^{\rho\sigma}\gamma^{\mu\nu}{}_{,\sigma\rho}-\eta^{\rho\nu}\gamma^{\mu\alpha}{}_{,\rho\alpha}-\eta^{\mu\rho}\gamma^{\mu\alpha}{}_{,\rho\alpha}+\eta^{\mu\nu}\gamma^{\alpha\beta}{}_{,\alpha\beta}], \tag{3.5}$$

and the linearized curvature tensor

$$\begin{aligned}{}_1R^{\mu\nu\alpha\beta}=\tfrac{1}{2}[&\eta^{\nu\rho}\eta^{\sigma\beta}\gamma^{\mu\alpha}{}_{,\rho\sigma}+\eta^{\rho\mu}\eta^{\sigma\alpha}\gamma^{\mu\beta}{}_{,\rho\sigma}\\&-\eta^{\mu\rho}\eta^{\beta\sigma}\gamma^{\nu\rho}{}_{,\rho\sigma}-\eta^{\mu\rho}\eta^{\beta\sigma}\gamma^{\nu\alpha}{}_{,\rho\sigma}\\&-\tfrac{1}{4}[\eta^{\mu\alpha}\eta^{\nu\rho}\eta^{\beta\lambda}\gamma^{\sigma\sigma}{}_{,\rho\lambda}+\eta^{\nu\beta}\eta^{\rho\mu}\eta^{\alpha\lambda}\gamma^{\sigma\sigma}{}_{,\rho\lambda}]\\&+\tfrac{1}{4}[\eta^{\mu\beta}\eta^{\rho\nu}\eta^{\lambda\alpha}\gamma^{\sigma\sigma}{}_{,\rho\lambda}+\eta^{\nu\alpha}\eta^{\rho\mu}\eta^{\lambda\beta}\gamma^{\sigma\sigma}{}_{,\rho\lambda},\end{aligned} \tag{3.6}$$

are thus both gauge-invariant.

The linear approximations to the integrals (2.5) and (2.6),

$${}_1P^{\mu}=\int_S dS_{\rho}\,{}_1\mathcal{T}^{\mu\rho}\equiv\oint_f df^*{}_{[\rho\sigma]}\,{}_1\mathfrak{u}^{[\rho\sigma]\mu}, \tag{3.7}$$

and

$$\begin{aligned}{}_1L^{[\mu\nu]}&=-\int_S dS_{\rho}\,{}_1L^{\rho[\mu\nu]}\\&\equiv\oint_f df^*{}_{[\rho\sigma]}\{{}_1\mathfrak{u}^{[\rho\sigma]\nu}x^{\mu}-{}_1\mathfrak{u}^{[\rho\sigma]\mu}x^{\nu}+{}_1H^{[\rho\sigma][\mu\nu]}\},\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}&{}_1\mathfrak{u}^{[\rho\sigma]\mu}={}_1H^{[\rho\sigma][\mu\alpha]}{}_{,\alpha},\\&{}_1H^{[\rho\sigma][\mu\alpha]}=\frac{1}{16\pi k}(\eta^{\rho\mu}\gamma^{\sigma\alpha}+\eta^{\sigma\alpha}\gamma^{\rho\mu}-\eta^{\rho\alpha}\gamma^{\sigma\mu}-\eta^{\sigma\mu}\gamma^{\rho\alpha}),\\&{}_1\mathcal{T}^{\mu\nu}={}_1\mathfrak{u}^{[\mu\alpha]\nu}{}_{,\alpha}=\frac{1}{8\pi k}\,{}_1G^{\mu\nu},\\&{}_1L^{\mu[\nu\rho]}={}_1\mathcal{T}^{\mu\nu}x^{\rho}-{}_1\mathcal{T}^{\mu\rho}x^{\nu},\end{aligned} \tag{3.9}$$

are also gauge-invariant. The only change in the volume integrals comes from changes in the shape or location of the volume of integration and changes in x^{ρ}; these contributions are quadratic in λ. Using Stokes' theorem we can deduce the gauge invariance of the surface integrals directly; hence their invariance does not depend on the transformation properties of the field within the volume of integration. If $\phi^{[\rho\sigma\alpha]}$ is any quantity antisymmetric in all three indices and ds a parameter along the line l bounding the finite two-dimensional surface f, then Stokes' theorem states:

$$\int_l df^*{}_{[\rho\sigma]}\phi^{[\rho\sigma\alpha]}{}_{,\alpha}\equiv\oint_f dl^*{}_{[\rho\sigma\alpha]}\phi^{[\rho\sigma\alpha]}, \tag{3.10}$$

with

$$dl^*{}_{[\rho\sigma\alpha]}=\tfrac{1}{3}\epsilon_{\rho\sigma\alpha\beta}[dx^{\beta}(s)/ds]ds.$$

If, in particular, f is closed, then the integrals vanish. On subjecting the integrands of the surface integrals (3.7) and (3.8) to a gauge transformation, we find that the added terms have just the form $\phi^{[\rho\sigma\alpha]}{}_{,\alpha}$ and thus integrate to zero. In the exact theory, or in higher approximations, the simple transformation properties described in the two preceding paragraphs are not valid.

From Eqs. (3.7), (3.8), and (3.9) we see that the surface integrals have another special property in the linear approximation: the integrals over two different surfaces which enclose the same sources have the same values. In other words, all the energy-momentum and all the angular momentum-dipole moment is carried by the sources. It is therefore meaningful to define the intrinsic angular momentum and dipole moment of a source. We choose within one world tube a timelike line $z^\alpha(\tau)$. Then if f surrounds just one source, the intrinsic angular momentum and dipole moment of the source are

$$N^{[\mu\nu]}(\tau)=\oint_f df^*{}_{[\rho\sigma]}\{{}_1u^{[\rho\sigma]\nu}\Delta x^\mu(\tau) - {}_1u^{[\rho\sigma]\mu}\Delta x^\nu(\tau)+{}_1H^{[\rho\sigma][\mu\nu]}\}, \quad (3.11)$$

with

$$\Delta x^\nu(\tau)=x^\nu-z^\nu(\tau).$$

Because of their invariance properties the surface integrals (3.7), (3.8), and (3.10) provide a preliminary classification—according to their energy and angular momentum—of the solutions of the linearized field equations (3.5). To distinguish between inequivalent solutions with the same energy and angular momentum we can use the curvature tensor (3.6). Since the curvature tensor is gauge invariant, two metrics which differ only by a gauge transformation have the same curvature tensor. Moreover, in the linearized theory the equality of two curvature tensors obtained from two different metrics is just the integrability condition for the existence of a gauge transformation that carries one metric into the other. Thus there is a one-to-one correspondence (*modulo* Lorentz transformations) between the different curvature tensors and inequivalent metrics.

To bring the curvature tensor into a form which can be handled easily, we rewrite it as a two-index quantity by the following identification of pairs of antisymmetric indices with a single index[14]:

$$[12]\leftrightarrow 3,\quad [23]\leftrightarrow 1,\quad [31]\leftrightarrow 2,$$
$$[14]\leftrightarrow 4,\quad [24]\leftrightarrow 5,\quad [34]\leftrightarrow 6. \quad (3.12)$$

When the field equations are satisfied, the result is

$${}_1R^{[\mu\alpha][\nu\beta]}\leftrightarrow R_{CD},$$
$$R_{CD}=\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & -\mathbf{A}\end{pmatrix}, \quad (3.13)$$

where $\mathbf{A}$ and $\mathbf{B}$ are symmetric three-by-three matrices with vanishing trace. Under spatial rotations $\mathbf{A}$ and $\mathbf{B}$ transform independently of each other as tensors of rank two.

[14] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).

4. SOLUTIONS OF THE LINEARIZED EQUATIONS

Though the approximate field equations, Eq. (3.5), are linear, the construction of their general solutions is complicated by the fact that more than one unknown function appears in each equation. If we adopt the linearized De-Donder conditions the field equations are thereby separated into ten independent D'Alembert's equations, but from among the solutions of the D'Alembert's equations we must yet select those solutions that obey the coordinate conditions. Therefore we shall proceed indirectly, by first exhibiting three particular types of solutions to the field equations and subsequently showing that the most general particle-like solution can be obtained from these three prototypes by linear superposition.

The A-type solution is one generated by a mass point which accelerates itself along a curved world line $z^\alpha(\tau)$ by growing a time-dependent intrinsic angular momentum and dipole moment. Let $b^\alpha=dz^\alpha/d\tau]_{\tau=\tau_r}$; τ be normalized to be the proper time, $b^\alpha b_\alpha=1$; $l^\alpha(\tau_r)=x^\alpha-z^\alpha(\tau_r)$; $\tau_r(x)$ be defined by $l^\alpha(\tau_r)l_\alpha(\tau_r)=0$ and $l^4(\tau_r)\geq 0$; $n=l^\alpha b_\alpha$; $\Delta z^\alpha=z^\alpha(\tau_r)-z^\alpha(\tau_0)$; and C^μ be four constants. Then the A-type solution is[15]

$$\gamma^{\mu\nu}=A^{\mu\nu}[x^\alpha;z^\alpha(\tau);z^\alpha(\tau_0);C^\alpha] =\frac{b^\mu C^\nu+b^\nu C^\mu}{n} +\left[\frac{C^\rho(b^\mu\Delta z^\nu+b^\nu\Delta z^\mu)-\Delta z^\rho(b^\mu C^\nu+b^\nu C^\mu)}{n}\right]_{,\rho}. \quad (4.1)$$

To see that this solution represents a self-accelerated monopole we evaluate its energy and angular momentum with the appropriate surface integrals (3.7), (3.8), and (3.11),

$$\begin{aligned}{}_1P^\mu&=C^\mu/2k,\\ {}_1L^{[\mu\nu]}&=\text{constant},\\ N^{[\mu\nu]}&=\frac{1}{2k}[\Delta z^\mu(\tau)C^\nu-\Delta z^\nu(\tau)C^\mu].\end{aligned} \quad (4.2)$$

Even if the mass point does not accelerate, its dipole moment in the rest system, $N^{[4s]}$, grows secularly unless $b^\nu\sim C^\nu$; if $b^\nu\sim C^\nu$ then Eq. (4.1) gives the usual solution for a stationary mass point.

The B-type solution is also singular and represents a disembodied angular momentum[16]:

$$\gamma^{\mu\nu}=B^{\mu\nu}[x^\alpha;\Lambda^{[\alpha\rho]};z^\alpha(\tau)] =\left(\frac{\Lambda^{[\mu\rho]}b^\nu+\Lambda^{[\nu\rho]}b^\mu}{n}\right)_{,\rho}, \quad (4.3)$$
$${}_1P^\mu=0,$$
$${}_1L^{[\mu\nu]}=N^{[\mu\nu]}(\tau)=\Lambda^{[\mu\nu]}/2k=\text{constant}.$$

[15] This solution has been considered by Dr. P. Havas (private communication).

[16] J. Lubanski, Acta Phys. Polon. **6.4**, 356 (1937).

The third type of solution is one that can be derived from a supermetric that obeys D'Alembert's equation:

$$\gamma^{\mu\nu}={}_1M^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}, \tag{4.4}$$

with

$$\eta^{\rho\sigma}\,{}_1M^{[\mu\alpha][\nu\beta]}{}_{,\sigma\rho}=0. \tag{4.5}$$

The right-hand sides of Eq. (4.5) may either vanish everywhere or else vanish only outside several world tubes. If f is any closed surface lying wholly within the region where the right-hand sides of (4.5) vanish, then the net energy in the three-dimensional hypersurface bounded by f also vanishes:

$$\begin{aligned}{}_1P^{\mu}=&\oint_f df^*{}_{[\rho\sigma]}\{\eta^{\lambda\sigma}\,{}_1M^{[\mu\beta][\rho\alpha]}{}_{,\beta\lambda}\\&-\eta^{\lambda\rho}\,{}_1M^{[\mu\beta][\sigma\alpha]}{}_{,\beta\lambda}+\eta^{\lambda\alpha}\,{}_1M^{[\mu\beta][\sigma\alpha]}{}_{,\alpha\lambda}\\=&\oint_f df^*{}_{[\rho\sigma]}\{\eta^{\lambda\sigma}\,{}_1M^{[\mu\beta][\rho\alpha]}{}_{,\beta\lambda}\\&-\eta^{\lambda\rho}\,{}_1M^{[\mu\beta][\sigma\alpha]}{}_{,\beta\lambda}+\eta^{\lambda\alpha}\,{}_1M^{[\mu\beta][\sigma\rho]}{}_{,\beta\lambda}\}_{,\alpha}\\=&0.\end{aligned} \tag{4.6}$$

The last equality comes from Stokes' theorem. A similar calculation shows that the net angular momentum, around any point, of the sources of the field (4.4) also vanishes. Thus, the metric (4.4) represents either a source-free radiation field or else the field due to quadrupole and higher multipole sources.

A general metric which corresponds to a collection of particles is one that is defined and obeys the linearized field equations everywhere within a fundamental region R that consists of all space-time except for N world tubes, and is asymptotically flat at infinity. For simplicity of presentation we shall assume that the world tubes do not collide or split in the course of time; the generalization of our discussion to cases in which tubes do collide or split involves no new principle. In each world tube a timelike world line, ${}^iz^{\alpha}({}^i\tau)$ with ${}^i\tau$ the proper time, can be chosen. Then to each tube we can assign a constant energy ${}^i{}_1P^{\mu}$ and an intrinsic angular momentum ${}^iN^{[\mu\nu]}({}^i\tau)$ as measured by surface integrals which surround only the ith world tube. We shall show that such a metric can be written in the form

$$\begin{aligned}\gamma^{\mu\nu}=\sum_{i=1}^{N}\{&{}^iA^{\mu\nu}[x^{\alpha};\,{}^iC^{\alpha};\,{}^iz^{\alpha}({}^i\tau);\,{}^iz^{\alpha}({}^i\tau_0)]\\&+{}^iB^{\mu\nu}[x^{\alpha};\,{}^i\Lambda^{[\alpha\beta]};\,{}^iz^{\alpha}]\}+M^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}\\&+\eta^{\rho\nu}\xi^{\mu}{}_{,\rho}+\eta^{\mu\rho}\xi^{\nu}{}_{,\rho}-\eta^{\mu\nu}\xi^{\rho}{}_{,\rho},\end{aligned} \tag{4.7}$$

where $M^{[\mu\alpha][\nu\beta]}$ satisfies D'Alembert's equation throughout R,

$$\eta^{\sigma\rho}M^{[\mu\alpha][\nu\beta]}{}_{,\sigma\rho}=0, \tag{4.8}$$

${}^iA^{\mu\nu}$ and ${}^iB^{\mu\nu}$ are the A-type and B-type solutions, and ξ^{μ} is arbitrary in R.

Let us choose the constants in the A-type and B-type solutions by requiring the sources of these two types to carry all the energy and angular momentum:

$$\begin{aligned}{}^i{}_1P^{\mu}&={}^iC^{\mu}/2k,\\{}^iN^{[\mu\nu]}({}^i\tau_0)&={}^i\Lambda^{[\mu\nu]}/2k.\end{aligned} \tag{4.9}$$

Define

$$\Delta^{\mu\nu}=\gamma^{\mu\nu}-\sum_{i=1}^{N}({}^iA^{\mu\nu}+{}^iB^{\mu\nu}), \tag{4.10}$$

and

$$H_{\Delta}{}^{[\mu\alpha][\nu\beta]}=\eta^{\mu\nu}\Delta^{\alpha\beta}+\eta^{\alpha\beta}\Delta^{\mu\nu}-\eta^{\mu\beta}\Delta^{\alpha\nu}-\eta^{\alpha\nu}\Delta^{\mu\beta}. \tag{4.11}$$

Because of Eq. (4.9) the sources of $\Delta^{\mu\nu}$ have no net energy or angular momentum; thus if f lies in R,

$$\oint_f df^*{}_{[\rho\sigma]}H_{\Delta}{}^{[\rho\sigma][\mu\alpha]}{}_{,\alpha}=0, \tag{4.12}$$

and

$$\begin{aligned}\oint_f df^*{}_{[\rho\sigma]}\{&H_{\Delta}{}^{[\rho\sigma][\mu\alpha]}{}_{,\alpha}x^{\nu}\\&-H_{\Delta}{}^{[\rho\sigma][\nu\alpha]}{}_{,\alpha}x^{\mu}-H^{[\rho\sigma][\mu\nu]}\}=0.\end{aligned} \tag{4.13}$$

It does not matter around which point we evaluate the angular momentum.

On comparing Eq. (4.12) with Eq. (4.6) we might conjecture that the metric can be derived from a supermetric which obeys D'Alembert's equation in R. This conjecture is actually correct. In fact, from Eq. (4.12) we can infer the existence of a $C^{[\rho\sigma\alpha]\mu}$ which obeys the equation[17]

$$H_{\Delta}{}^{[\rho\sigma][\mu\alpha]}{}_{,\alpha}=C^{[\rho\sigma\alpha]\mu}{}_{,\alpha}. \tag{4.14}$$

The deduction of Eq. (4.14) from Eq. (4.12) is an application, in a higher number of dimensions, of the same mathematical theorem that allows us to infer in magnetostatics the equation $\mathbf{B}=\nabla\times\mathbf{A}$ from the equation, valid for any closed surface in three space, $\oint\mathbf{B}\cdot d\mathbf{A}=0$. Using Eq. (4.14) in Eq. (4.13) and eliminating two terms by means of Stokes' theorem, we obtain

$$\oint_f df^*{}_{[\rho\sigma]}\{C^{[\rho\sigma\nu]\mu}-C^{[\rho\sigma\mu]\nu}+H_{\Delta}{}^{[\rho\sigma][\mu\nu]}\}=0. \tag{4.15}$$

From Eq. (4.15), valid for any closed surface in R, we can in turn conclude the existence of a $D^{[\mu\nu][\rho\sigma\alpha]}$ such that[17]

$$C^{[\rho\sigma\nu]\mu}-C^{[\rho\sigma\mu]\nu}+H_{\Delta}{}^{[\rho\sigma][\mu\nu]}=D^{[\mu\nu][\rho\sigma\alpha]}{}_{,\alpha}. \tag{4.16}$$

Using the cyclic identity (2.2) on $H_{\Delta}{}^{[\rho\sigma][\mu\nu]}$ we can solve Eq. (4.16) for $C^{[\rho\sigma\nu]\mu}$ in terms of $D^{[\mu\nu][\rho\sigma\alpha]}$. When the

[17] G. de Rham and Kunihiko Kodaira, *Harmonic Integrals*, lecture notes, Princeton Institute for Advanced Study, revised (1953) edition, p. 41.

expression for $C^{[\rho\sigma\nu]\mu}$ is resubstituted into Eq. (4.16), we get

$$H_\Delta{}^{[\mu\alpha][\nu\beta]}=\tfrac{1}{6}\{2D^{[\alpha\mu][\beta\nu\tau]}+2D^{[\beta\nu][\alpha\mu\tau]}+D^{[\alpha\beta][\mu\nu\tau]}+D^{[\nu\alpha][\mu\beta\tau]}+D^{[\mu\nu][\alpha\beta\tau]}+D^{[\mu\beta][\nu\alpha\tau]}\}_{,\tau}, \quad (4.17)$$

valid in R.

Equations (4.12) through (4.17) are valid in any gauge frame. Now extend $\Delta^{\mu\nu}$, which is defined by Eq. (4.10) only in R, through the N world tubes in a sufficiently differentiable but otherwise arbitrary fashion. Retain the definition (4.11) for $H_\Delta{}^{[\mu\alpha][\nu\beta]}$ even within the world tubes. Independently extend $D^{[\mu\nu][\rho\sigma\alpha]}$ through the world tubes; in general the relation between $D^{[\mu\nu][\rho\sigma\alpha]}$ and $H_\Delta{}^{[\mu\alpha][\nu\beta]}$, Eq. (4.17), will be violated outside R. We can go to a coordinate frame in which $\Delta^{\mu\nu}$ obeys the linearized De-Donder conditions,

$$\Delta^{\mu\nu}{}_{,\nu}\equiv 0. \quad (4.18)$$

Finally, we define $M^{*[\mu\alpha][\nu\beta]}$ and $E^{[\mu\nu][\rho\sigma\alpha]}$ by the equations

$$\eta^{\epsilon\lambda}M^{*[\mu\alpha][\nu\beta]}{}_{,\epsilon\lambda}=H_\Delta{}^{[\mu\alpha][\nu\beta]}$$

and

$$\eta^{\epsilon\lambda}E^{[\mu\nu][\rho\sigma\alpha]}{}_{,\epsilon\lambda}=D^{[\mu\nu][\rho\sigma\alpha]} \quad (4.19)$$

everywhere. Then a possible supermetric is

$$M^{[\mu\alpha][\nu\beta]}=M^{*[\mu\alpha][\nu\beta]}-\tfrac{1}{6}\{2E^{[\mu\alpha][\nu\beta\tau]}+2E^{[\nu\beta][\mu\alpha\tau]}+E^{[\alpha\beta][\mu\nu\tau]}+E^{[\nu\alpha][\mu\beta\tau]}+E^{[\mu\nu][\alpha\beta\tau]}+E^{[\mu\beta][\nu\alpha\tau]}\}_{,\tau}. \quad (4.20)$$

For this supermetric we find

$$\Delta^{\mu\nu}=M^{*[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}=M^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}, \quad (4.21)$$

$$\eta^{\rho\epsilon}M^{[\mu\alpha][\nu\beta]}{}_{,\epsilon\rho}=0 \quad \text{in} \quad R. \quad (4.22)$$

From Eqs. (4.10), (4.21), and (4.22) we obtain the desired Eqs. (4.7) and (4.8), after transforming back to an arbitrary coordinate frame.

A useful property of those supermetrics that obey Eq. (4.22) is that they can be chosen in such a way that they have all the algebraic properties of the linearized curvature tensor in empty space. In fact, suppose we have a supermetric which obeys Eq. (4.22) and is already chosen to obey the cyclic identity (2.11); we can then define a "conformal" supermetric:

$$M''^{[\mu\alpha][\nu\beta]}=M'^{[\mu\alpha][\nu\beta]}-\tfrac{1}{2}(\eta^{\mu\nu}M'^{\alpha\beta}+\eta^{\alpha\beta}M'^{\mu\nu}-\eta^{\mu\beta}M'^{\alpha\nu}-\eta^{\alpha\nu}M'^{\mu\beta})+\tfrac{1}{6}(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\beta}\eta^{\alpha\nu})M', \quad (4.23)$$

where $M'^{\alpha\beta}=M'^{[\mu\alpha][\rho\beta]}\eta_{\mu\rho}$, $M'=M'^{\alpha\rho}\eta_{\alpha\rho}$. This new supermetric has vanishing trace

$$\eta_{\mu\rho}M''^{[\mu\alpha][\rho\beta]}\equiv 0, \quad (4.24)$$

and the metric generated by it differs from the metric generated by the original supermetric by at most a gauge transformation.

The "equations of motion" in the linear approximation are rather trivial; from the defining equations (3.7) and (3.11) we have

$$d\,{}^iN^{[\mu\nu]}/d\,{}^i\tau=-{}^ib^\mu\,{}^i{}_1P^\nu+{}^ib^\nu\,{}^i{}_1P^\mu. \quad (4.25)$$

If we regard the intrinsic angular momentum and dipole moment as being determined by the internal dynamics of the sources, Eq. (4.25) shows how a variable dipole moment is related to the self-acceleration. For the singular solution (4.1) it is possible to choose the world tube surrounding the singularity as having an arbitrarily small diameter; thus the world lines ${}^iz^\mu$ cannot in general be chosen in such a way that the left-hand side of Eq. (4.25) vanishes. If we arbitrarily require the left-hand side of Eq. (4.25) to vanish, or make the restrictions[18]

$${}^i{}_1P_\mu\,{}^iN^{[\mu\nu]}=0 \quad \text{or} \quad {}^ib_\mu\,d\,{}^iN^{[\mu\nu]}/d\,{}^i\tau=0, \quad (4.26)$$

then the sources are constrained to move with constant velocity unless ${}^i{}_1P^\mu=0$. If ${}^i{}_1P^\mu=0$, the motion is arbitrary.

5. STATIC LINEARIZED METRICS AND THEIR CLASSIFICATION

In this section we illustrate the classification of metrics by means of the curvature tensor for the particular case of static linearized metrics which have point sources and are asymptotically flat at infinity. Greek indices will run from 1 to 4, lower case Latin indices from 1 to 3, and capital Latin indices from 1 to 6.

In De-Donder coordinates the linearized field equations for static fields are

$$\eta^{st}\gamma^{\mu\nu}{}_{,st}=0, \qquad \gamma^{\mu s}{}_{,s}=0. \quad (5.1)$$

Let m, p^s, and $m^{[rs]}=-m^{[sr]}$ be constants; let $\Psi^{rs}=\Psi^{sr}$, $\phi^{r[st]}=-\phi^{r[ts]}$ and ${}_1M^{[rs][pq]}=-{}_1M^{[sr][pq]}={}_1M^{[pq][rs]}$ be sets of harmonic functions zero at spatial infinity and singular at the origin; and represent the three-dimensional radius vector by r. Then the basic solutions of Eq. (5.1), from which all others can be obtained by superposition, are summarized in the Table I. The designations "electric" and "magnetic" are natural if we draw an analogy between γ^{44} and the scalar potential of electrostatics or between γ^{4s} and the vector potential of magnetostatics; we shall see that the solutions (III c) are equivalent to the solutions (I c) under gauge transformations. That there are, for example, no magnetic-type monopole solutions follows either from the general considerations of the previous section or from applying De Rham's theorem[17] to the equation, in-

[18] See F. A. E. Pirani, Bull. intern. acad. polon. sci., Class III, 5, 143 (1957).

TABLE I. Basic solutions of Eq. (5.1).

	(a) Monopole	(b) Dipole	(c) Higher poles
I. "Electric" type $\gamma^{4s}=\gamma^{rs}=0$	$\gamma^{44}=m/r$	$\gamma^{44}=(p^s/r)_{,s}$	$\gamma^{44}=\Psi^{rs}{}_{,rs}$
II. "Magnetic" type $\gamma^{44}=\gamma^{rs}=0$		$\gamma^{4s}=(m^{[sr]}/r)_{,s}$	$\gamma^{4s}=\phi^{r[st]}{}_{,rt}$
III. "Electric" type $\gamma^{44}=\gamma^{4s}=0$			$\gamma^{rs}={}_1M^{[rt][sp]}{}_{,tp}$

variant under time-independent gauge transformations,

$$\oint_l d^*l_{[st]}(\eta^{rt}\gamma^{4s}{}_{,r}-\eta^{rs}\gamma^{4t}{}_{,r})=0. \tag{5.2}$$

Here l is any closed one-dimensional loop, in three-dimensional space, on which the metric is nonsingular.

How many of these solutions are inequivalent? By calculating the energy and angular momentum of the monopole and dipole solutions we find that they are static versions of the A-type and B-type solutions discussed in Sec. IV. The two types of dipoles are distinguished by the fact that the constants in Eq. (4.3) obey the inequalities $\Lambda^{[\mu\rho]}\Lambda_{[\mu\rho]}<0$ and $\Lambda^{[\mu\rho]}\Lambda_{[\mu\rho]}>0$ for electric and magnetic dipoles, respectively. Thus, these three solutions are all distinct and none of them are equivalent to the solutions in the third column of the table, which have vanishing energy and vanishing angular momentum.

The classification of the higher poles follows from examining their curvature. We first note that by writing ${}_1M^{[4s][4r]}\equiv\Psi^{sr}$ and ${}_1M^{[4r][st]}\equiv\phi^{r[st]}$ we have the solutions in the third column expressed in terms of supermetrics. We next add to the supermetrics suitable terms so that the new supermetrics obey the algebraic identities (2.11) and (4.24); this procedure is equivalent to a gauge transformation. We can then write the supermetrics in the six-dimensional notation discussed in Sec. III and verify the following simple relation between the curvature tensor (also expressed in six-dimensional notation) and the supermetric.

$$M^{[\mu\alpha][\nu\beta]}\leftrightarrow M_{AB}=\begin{pmatrix}\mathbf{E} & \mathbf{F}\\ \mathbf{F} & -\mathbf{E}\end{pmatrix},$$
$$R^{[\mu\alpha][\nu\beta]}\leftrightarrow R_{AB}=\begin{pmatrix}\mathbf{C} & \mathbf{D}\\ \mathbf{D} & -\mathbf{C}\end{pmatrix}, \tag{5.3}$$

where the matrices $\mathbf{E}$ and $\mathbf{F}$ are harmonic functions—with vanishing trace according to the prescription of Sec. II—and

$$C^{rs}\sim E^{pq}{}_{,pqt l}\eta^{rt}\eta^{sl},\quad D^{rs}\sim F^{pq}{}_{,pqtl}\eta^{rt}\eta^{sl}. \tag{5.4}$$

In the last equation we have taken advantage of the fact that $\mathbf{E}$ and $\mathbf{F}$, or $\mathbf{C}$ and $\mathbf{D}$, transform independently of each other as tensors of rank two under purely spatial rotations to let the indices on $\mathbf{F}$ and on $\mathbf{D}$ run from 1 to 3 according to the prescription $4\leftrightarrow1$, $5\leftrightarrow2$, and $6\leftrightarrow3$. Equation (5.4) is valid only when the metric is time-independent, the supermetric obeys Laplace's equation, and the supermetric has been standardized to obey Eqs. (2.11) and (4.24). Applying Eqs. (5.3) and (5.4) to the higher multipole solutions, we find that the two apparently distinct electric-type multipoles I and III have not only the same curvature but even the same standardized supermetric, with $\mathbf{F}=0$ and $\mathbf{E}$ arbitrary; the magnetic multipoles, on the other hand, have a supermetric with $\mathbf{E}=0$ and $\mathbf{F}$ arbitrary so that, from Eq. (5.4), their curvature is different from the curvature of the electric multipoles. To classify the time-dependent solutions we could proceed exactly as we did in the time-independent case, but the results are not very transparent in view of the lack of a simple relation between the supermetric and the curvature.

6. DISCUSSION

We have shown that any particle-type solution of the linear field equations has a well-defined, gauge-invariant, total energy-momentum and intrinsic angular momentum-dipole moment associated with it, both when its sources are singularities and when the sources consist of an extended matter tensor. The energy is constant and the intrinsic angular momentum is related to it by an "equation of motion." The actual motion of a source is unrestricted unless we impose conditions, in no way implied by the theory itself, on the intrinsic angular momentum. In the linear theory, these additional restrictions are gauge-invariant. Beside its energy and angular momentum, a source has also a number of higher moments, specified by arbitrary functions of the time which are totally unrelated to the motion of the particle. The totality of all solutions can be obtained by the superposition of two basic solutions and of solutions obtained by solving homogenous, separated D'Alembert's equations; there are gauge-invariant methods for classifying inequivalent solutions.

A question of fundamental importance remains whether a similar invariant treatment can be given for the higher orders of the approximation scheme or the full theory. In any order beyond the first, the Riemann tensor is no longer invariant under curvilinear coordinate transformations, and an invariant formulation becomes far more difficult.

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