Energies of Ground and Excited Nuclear Configurations in the First $p_{\frac{1}{2}}$ Region*

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Shell-model analysis is applied to configurations containing both $p_{\frac{3}{2}}$ and $s_{\frac{1}{2}}$ nucleons. Energy levels of excited configurations (often recognized by their parity being opposite to that of the ground state) as well as binding energies are considered. Good agreement is obtained which gives support to the jj-coupling scheme. The results obtained are used for calculating other levels. In particular, the modes of excitation of the first excited 0+ level in O^{16} and the $\frac{1}{2}-$ level in F^{19} are determined.

I. INTRODUCTION

RECENTLY a new approach to shell-model calculations has been used.[1–4] This approach, is based upon the jj-coupling shell model and involves no specific assumption on the nuclear forces. We assume only that (a) the potential energy of the nucleus is due to *two-body forces* between nucleons. Similarly, we do not take the radial parts of the single-nucleon wave functions to be of any specific form. We assume only that (b) these radial functions are the same in a definite subshell independent of the number of nucleons (at least for all nuclei in which the same shells are being filled). Whenever we consider protons and neutrons together we take the nuclear forces to be charge-independent. We also assume, in these cases, the same radial functions for neutrons and protons. We thus (c) take the total isotopic spin T to be a good quantum number.

We treat in every case a group of nuclei, each containing extra nucleons outside the same closed shells. In these nuclei we consider the energy levels obtained by the different modes of coupling of the extra nucleons. Under our assumptions all these energies, both of ground states (binding energies) and of excited levels, can be expressed as combinations of a smaller number of parameters. These parameters are expectation values of the nuclear interaction in two-body configurations (or linear combinations thereof) and single-nucleon energies. The latter we interpret as the sum of the kinetic energy of a single nucleon and its interaction with the closed shells. We check the validity of our assumptions (and the configuration assignment of the levels) by looking for values of the parameters which will reproduce the experimental energies. If we succeed in finding such a set of values from which the energy levels can be calculated accurately enough, our assumptions seem to be justified. We can further use the parameters thus obtained for calculating positions of other levels which have not been found or have not been identified.

In the papers mentioned above,[1–4] energies of both ground states and excited levels have been considered. However, in all those cases, all levels were members of ground configurations, i.e., configurations to which the ground states belong. By configuration we mean, as usual, the numbers of nucleons in the various j subshells (those in closed shells are usually omitted). The excited states observed in the spectrum of any nucleus belong both to the ground and to excited configurations. The main difficulty in handling excited states is that we do not know to which configurations they belong. We know that configurations overlap, their levels appearing alternately (this does not necessarily mean that they get mixed). Therefore, even when spins and parities are known we cannot be sure of the configuration assignment. However, there are cases in which it is rather easy to assign a configuration to excited levels. If a low-lying level has a parity opposite to that of the ground state we know that its configuration was obtained from the ground configuration by raising an odd number of nucleons to higher orbits of an opposite parity. In many cases it is just one nucleon which is excited into a higher orbit. It is easy to guess the correct configuration since the occupied orbits in the ground state as well as the neighboring higher orbits are known from shell-model interpretations of experimental data. Such low-lying levels with opposite parity are found in nuclei where an oscillator shell is being completed. In this case all the available neighboring orbits have a parity opposite to that of the last orbit being filled.

In the present work we apply this analysis to energy levels of nuclear configurations which contain both $0p_{\frac{3}{2}}$ and $1s_{\frac{1}{2}}$ nucleons. Some of the configurations we treat are ground configurations while in others there are nucleons excited from the $p_{\frac{3}{2}}$ to the $s_{\frac{1}{2}}$ orbit. Of the three subshells of the oscillator third shell, the $d_{\frac{3}{2}}$ subshell is considerably higher than the $s_{\frac{1}{2}}$ and $d_{\frac{5}{2}}$ subshells and therefore excitations into this subshell probably involve higher energy. In some simple cases we can distinguish between excitations into the $d_{\frac{5}{2}}$ and $s_{\frac{1}{2}}$ subshells by the value of the resulting spin. We can thus handle separately the configurations with $s_{\frac{1}{2}}$ nucleons and those with $d_{\frac{5}{2}}$ nucleons. In this paper we treat only the former

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[1] S. Goldstein and I. Talmi, Phys. Rev. **102**, 589 (1956); S. P. Pandya, Phys. Rev. **103**, 956 (1956).
[2] I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956).
[3] S. Goldstein and I. Talmi, Phys. Rev. **105**, 995 (1957).
[4] R. D. Lawson and J. L. Uretsky, Phys. Rev. **106**, 1369 (1957). I. Talmi, Phys. Rev. **107**, 326 (1957); **107**, 1601 (1957); *Proceedings of the Rehovoth Conference on Nuclear Structure* (North-Holland Publishing Company, Amsterdam, 1958), p. 31.

configurations. Excitations into $d_{\frac{5}{2}}$ orbits will be considered in a subsequent paper. Only in a few cases will this restriction prevent us from making definite statements.

The problem of the center-of-mass motion in the shell model appears in the present case more drastically than in the cases of ground configurations. The difficulty encountered is that of spurious states[5] which differ from states of ground configurations only by a different center-of-mass motion. For central fields of arbitrary shape the center-of-mass motion cannot be separated in shell-model wave functions. Only if harmonic-oscillator wave functions are used can the ground-state wave function be written as a product of a function which describes the center-of-mass motion in the $0s$ orbit of an oscillator potential and an internal wave function. It could be argued that this internal wave function describes also the ground state of a real nucleus, in which the center of mass moves in a uniform motion. This separation holds for all levels of the ground configuration and those configurations in which extra nucleons are excited into orbits in the same oscillator shell. States of other configurations might well contain contributions from spurious states in which the internal wave function is that of states which belong to the configurations mentioned above while the center-of mass is raised into a higher orbit of the oscillator potential. In particular all states of configurations in which a p nucleon is raised into the next s or d orbit might contain contributions from spurious states in which the center of mass moves in a $0p$ orbit of the oscillator potential.

In the jj-coupling scheme this difficulty does not appear if a $p_{\frac{1}{2}}$ nucleon is raised into $d_{\frac{5}{2}}$ orbit. All such spurious states can be obtained by operating with $\mathbf{R}=(1/N)\sum\mathbf{r}_i$ on all wave functions of configurations with the closed $0s$ shell and nucleons in the $0p$ shell. In the configurations we consider all the $p_{\frac{3}{2}}$ and $0s_{\frac{1}{2}}$ nucleons do not change orbits whereas a $p_{\frac{1}{2}}$ nucleon is raised into the $d_{\frac{5}{2}}$ orbit. Obviously such configurations cannot be obtained by operating with $\mathbf{R}$. Every $\mathbf{r}_i$ is a vector and therefore can change the total j of a nucleon by one unit only. A $p_{\frac{1}{2}}$ nucleon can be moved by $\mathbf{R}$ only into $d_{\frac{3}{2}}$ or $s_{\frac{1}{2}}$ orbits.

States of excited configurations with $s_{\frac{1}{2}}$ nucleons considered here can well contain spurious states. In this case there are also simple selection rules on the operation of $\mathbf{R}$ which in some cases prevent the appearance of spurious states. In all other cases we calculated the weight of the spurious states in the shell-model wave functions on the assumption of harmonic-oscillator wave functions. The method of calculation is presented in the appendix. The weight found was less than 10% in all cases. However, the agreement obtained in the present paper between theory and experiment is very good. We do not have any explanation for this fact. It might indicate that a few percent of spurious states do not do much harm to the calculation. In any case, the difficulty of spurious center-of-mass motion is present when using any central field other than that of the harmonic oscillator even in ground configurations. Perhaps understanding this difficulty in our case will have to wait for the understanding of those simpler cases.

Our analysis show very good agreement between the simple jj-coupling shell model and the experimental data. As a result we obtain expectation values of the effective two-body interaction and single-nucleon energies. These parameters can now be used to treat others, less certain, cases. Of special interest are the $\frac{1}{2}$ level of F^{19} which is only 0.11 Mev above the ground state and the first excited 0+ level of O^{16}. We find the $\frac{1}{2}-$ level of F^{19} in very good agreement with the calculated level obtained by exciting one $p_{\frac{1}{2}}$ nucleon into the $s_{\frac{1}{2}}$ orbit. The case of O^{16} is more complicated. Since the 6.06 level has even parity it can be obtained only by raising an even number of $p_{\frac{1}{2}}$ nucleons into $s_{\frac{1}{2}}$ or $d_{\frac{5}{2}}$ orbits. We find that all levels due to excitation of *two* $p_{\frac{1}{2}}$ nucleons lie a few Mev higher than the observed level. This is a result of the large attraction of two protons and two neutrons in the same orbit ($p_{\frac{1}{2}}$, $s_{\frac{1}{2}}$, or $d_{\frac{5}{2}}$) as compared to twice the attraction of two such nucleons. If two nucleons out of four are raised from one orbit ($p_{\frac{1}{2}}$ say) into another one ($s_{\frac{1}{2}}$ or $d_{\frac{5}{2}}$), the attraction between the two pairs is rather small and does not compensate the attraction lost by the separation.

We therefore calculated the position of an 0+ level obtained by exciting all four $p_{\frac{1}{2}}$ nucleons. If these nucleons are excited into the $d_{\frac{5}{2}}$ orbit the level seems still to lie too high. But if the excitation is to the $s_{\frac{1}{2}}$ orbit good agreement is obtained. On the basis of our model we assign the first excited 0+ level of O^{16} to a configuration in which there are four $1s_{\frac{1}{2}}$ nucleons and no $0p_{\frac{1}{2}}$ present. This mode of excitation somewhat recalls the α-particle model. It is interesting to find out whether experiments will verify or disprove this assignment.

After completing this work we were pleased to find that the configuration assignment we made in these cases is precisely the one suggested a few years ago by Christy and Fowler.[6] Also the $\frac{1}{2}-$ level of O^{17} and F^{17} to be treated in Sec. IV was assigned the same configuration by us as by these authors who made a rough estimate of the energies neglecting the $p_{\frac{1}{2}}-s_{\frac{1}{2}}$ interaction.

In this paper, as well as in previous work,[1–4] only the energies of states are considered. Since the energy has a stationary value at the correct wave function, even our approximate wave functions can give energies in agreement with experiments. Matrix elements of other operators, such as magnetic moments and transition probabilities, do not have the stationary property and

[5] J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A232**, 561 (1955).

[6] R. F. Christy and W. A. Fowler, Phys. Rev. **96**, 851(A) (1954).

are rather sensitive to small admixtures in the wave functions. It is the authors opinion that any theory must at least predict correct energies. If and only if this requirement is satisfied we can hope to go to higher approximations which will enable us to calculate more sensitive quantities.

II. METHOD OF CALCULATION

As already mentioned, we do not calculate the energy levels with a specific interaction or radial functions. We have to calculate the expressions of the energy levels in the many-nucleon configurations in terms of expectation value in two-nucleon configurations. Let us now discuss the total energy of the levels.

We consider nuclei in which the $0s_{\frac{1}{2}}$ and $0p_{\frac{3}{2}}$ subshells are closed and extra nucleons are in the $0p_{\frac{1}{2}}$ and $1s_{\frac{1}{2}}$ orbits. The sum of the kinetic energy and mutual interaction of the nucleons in the closed shells (the core) is constant, according to our assumptions, and independent of the number of extra nucleons. This core energy is equal to the binding energy of the nucleus in which there are no extra nucleons, namely C^{12}. In order to analyze the contribution of the extra nucleons, we therefore first subtract the binding energy of C^{12} from the experimental total energies of all the levels considered.

The result of this subtraction is the sum of the kinetic energies of the extra nucleons, their interaction with nucleons in closed shells and their mutual interaction. Because of the spherical symmetry of the closed shells the interaction of a single nucleon with the core is independent of the orientation of its spin j in space. Obviously, this is true also for its kinetic energy. Thus, the sum of the kinetic energy of the extra nucleons and their interaction with the closed shells can be expressed as a sum of single-nucleon energies.

The mutual interaction can be expressed in terms of the expectation values of the effective interaction in the two-nucleon configurations $p_{\frac{1}{2}}^2$, $p_{\frac{1}{2}}s_{\frac{1}{2}}$, and $s_{\frac{1}{2}}^2$. If there is only one $s_{\frac{1}{2}}$ nucleon present, there is no need of the expectation values in the $s_{\frac{1}{2}}^2$ configuration. All the states of the $p_{\frac{1}{2}}^m s_{\frac{1}{2}}^n$ configurations can be written as linear combinations of wave functions in which the "stripped configuration" $p_{\frac{1}{2}}^m$ is in a definite state characterized by T and J. The kinetic energy of the m $p_{\frac{1}{2}}$ nucleons, their interaction with the core and their mutual interaction are independent of the presence of the $s_{\frac{1}{2}}$ nucleons. The sum of these energies can, therefore, be taken from nuclei with no $s_{\frac{1}{2}}$ nucleons. Energies of $p_{\frac{1}{2}}^m$ nuclear configurations were analyzed in reference 2 and very good agreement was obtained. We take these energies, due to the $p_{\frac{1}{2}}^m$ configurations, from reference 2 and subtract them too from the total energies. If there are several $s_{\frac{1}{2}}$ nucleons, their mutual interaction can be taken care of in the same manner. What finally remains is the sum of single $s_{\frac{1}{2}}$ nucleon energies and their interaction with the m $p_{\frac{1}{2}}$ nucleons.

We now define the single-nucleon energies. Let A_s denote the sum of the kinetic energy and interaction with the closed shells of a single $s_{\frac{1}{2}}$ *neutron*. This will be taken as a free parameter to be determined by the experimental data. The corresponding single *proton* energy differs from A_s by the electrostatic repulsion between the extra proton and the core (the difference in radial functions of neutrons and protons also contributes). In previous papers the Coulomb energy was expressed in terms of a single parameter $e^2(\nu/\pi)^{\frac{1}{2}}$ by using harmonic-oscillator wave functions. We use here a slightly more convenient procedure. The single $s_{\frac{1}{2}}$ proton energy A_s' will be taken as an independent parameter. The rest of the Coulomb energy is due to the mutual electrostatic interaction of the extra protons. This we calculate from the oscillator model, taking the value of $e^2(\nu/\pi)^{\frac{1}{2}}$ from reference 2 to be 0.349 Mev. We subtract also this part of the Coulomb energy from the total energies of the levels. This way we are left with energies equal to a combination of the free parameters A_s, A_s', and those of the $s_{\frac{1}{2}}-p_{\frac{1}{2}}$ interaction.

Some care must be given to the Coulomb energies. Since all the states have a definite isotopic spin we sometimes cannot say whether nucleons excited into $s_{\frac{1}{2}}$ orbits are protons or neutrons. In order to evaluate the Coulomb energy and take the correct linear combinations of A_s and A_s' we have to consider the way the states are built. We work in a scheme that the m $p_{\frac{1}{2}}$-nucleons are in a state of definite T_1 and J_1 and the n $s_{\frac{1}{2}}$ nucleons are in a state with definite T_2 and J_2. Every state with a definite T and J built from such states is a linear combination with Clebsch-Gordan coefficients of products of states with T_1, M_{T_1} and T_2, M_{T_2}. The values of M_{T_1} and M_{T_2} along with m and n give us the numbers of $p_{\frac{1}{2}}$ protons and $s_{\frac{1}{2}}$ protons and so the Coulomb energy can be computed for every term in this linear combination. For example in the case of a single $s_{\frac{1}{2}}$ nucleon we can write the wave function of a state with a definite $M_T=(Z-N)/2$ as

$$\psi(p_{\frac{1}{2}}^m(T_1J_1)s_{\frac{1}{2}};\, TM_TJ) = \sum_{M_{T_1}m_t} \psi(p_{\frac{1}{2}}^m(T_1M_{T_1}J)s_{\frac{1}{2}}m_t;\, TM_TJ) \times (T_1M_{T_1}\tfrac{1}{2}m_t|\,T\tfrac{1}{2}TM_T). \quad (1)$$

Thus we see that the $s_{\frac{1}{2}}$ nucleon has a probability $(T_1M_T-\frac{1}{2}\frac{1}{2}\frac{1}{2}|\,T\frac{1}{2}TM_T)^2$ to be a proton and $(T_1M_T+\frac{1}{2}\frac{1}{2}-\frac{1}{2}|\,T\frac{1}{2}TM_T)^2$ to be a neutron. In the same way we have to take the Coulomb energy of the $p_{\frac{1}{2}}^m$ configuration in the states with the various M_T values with weights given by the squares of these vector addition coefficients.

The last term to be calculated is the $s_{\frac{1}{2}}-p_{\frac{1}{2}}$ interaction. The calculation is almost identical to that of reference 3. The only difference is that here states are characterized also by T in addition to J. This means that unlike the case in reference 3, we have to anti-

symmetrize the wave function in the proton and neutron coordinates, including the isospin coordinates. However, the change due to antisymmetrization is only the introduction of appropriate exchange terms in the expectation values of the $p_{\frac{1}{2}}s_{\frac{1}{2}}$ *two-nucleon configurations*. Since we do not calculate explicitly these expectation values we have only to characterize them with the correct value of T.[7]

The formulas we get are easily obtained from those of Sec. II in reference 3. The expressions there, where the various J values appear, should be multiplied by the same expressions with the appropriate values of the isospin T. For completeness we give here the derivation of the results in the case of a single $s_{\frac{1}{2}}$-nucleon. The configuration is $p_{\frac{1}{2}}{}^m s_{\frac{1}{2}}$ and the interaction considered is $\sum_{k=2}{}^{m+1} V_{1k}$ where nucleon number 1 is the $s_{\frac{1}{2}}$ nucleon. We write j for $p_{\frac{1}{2}}$ and j' for $s_{\frac{1}{2}}$ and calculate the expectation value of the interaction in the $j^m j'$ configuration. We have to express it in terms of the expectation values in the states with T and J of the jj' configuration.

We calculate diagonal as well as nondiagonal matrix elements of the interaction in a scheme where the m j nucleons are coupled to form a state with definite T_1 and J_1. In this scheme we obtain

$$\langle j^m(T_1J_1)j'TJ|\sum_{k=2}^{m+1} V_{1k}|j^m(T_1'J_1')j'TJ\rangle = m\langle j^m(T_1J_1)j'TJ|V_{12}|j^m(T_1'J_1')j'TJ\rangle. \quad (2)$$

Because of the total antisymmetry of the wave function in the coordinates of the m j nucleons, it is possible to calculate the contribution of one term of the interaction and multiply by m. Since we chose the term V_{12} it is desirable to rewrite the j^m wave function in order to have nucleon number 2 in a more convenient position. This is achieved by writing the totally antisymmetric j^m wave function in terms of fractional parentage coefficients:

$$\psi(j^mT_1J_1)=\sum_{T_2J_2}\psi(j^{m-1}(T_2J_2)j_2T_1J_1)\times(j^{m-1}(T_2J_2)jT_1J_1\|j^mT_1J_1). \quad (3)$$

Introducing these expressions into both sides of the matrix element in (2), we obtain the following form

$$m\sum_{T_2J_2,T_2'J_2'}(j^{m-1}(T_2J_2)jT_1J_1\|j^mT_1J_1)\times(j^mT_1'J_1'\|j^{m-1}(T_2'J_2')jT_1'J_1')\times\langle j^{m-1}(T_2J_2)j_2(T_1J_1)j_1'TJ|V_{12}|j^{m-1}\times(T_2'J_2')j_2(T_1'J_1')j_1'TJ\rangle. \quad (4)$$

In order to obtain the expectation value of V_{12} in the jj'-configuration in states with definite T and J, we have to change the coupling scheme. On the left-hand side of the matrix element in (4), T_2 and J_2 are coupled first to $t_2=\frac{1}{2},j_2=j$ to give T_1J_1 which then are coupled to $t_1=\frac{1}{2},j_1'=j'$ to give the total TJ. We go over into the scheme where $t_2=\frac{1}{2},j_2=j$ are first coupled to $t_1=\frac{1}{2},j_1'=j'$ to give some $T'J'$ which are coupled to T_2J_2 to give the total TJ. In this scheme the matrix elements of V_{12} can be written down immediately. The transformation we have to make is a product of two simple transformations, one for the J's and one for the T's. Every coefficient of the simple transformation contains a Racah coefficient.[8] On carrying out these transformations on both sides of the matrix element in (4) we can integrate over all nucleon coordinates other than 1 and 2, since only these appear in V_{12}. As a result, in order to get a nonvanishing contribution, we must have $T_2'=T_2J_2'=J_2$. Because of the invariance of V_{12} under rotations in ordinary and spin spaces and in the isospin space, T' and J' must have the same values as both sides of the V_{12} matrix element. We thus obtain

$$m\sum_{T'J'}\langle jj'T'J'|V_{12}|jj'T'J'\rangle\times\sum_{T_2J_2}(j^{m-1}(T_2J_2)jT_1J_1\|j^mT_1J_1)\times(j^mT_1'J_1'\|j^{m-1}(T_2J_2)jT_1'J_1')\times[(2T_1+1)(2T_1'+1)(2J_1+1)(2J_1'+1)]^{\frac{1}{2}}(2T'+1)\times(2J'+1)W(T_2\tfrac{1}{2}T\tfrac{1}{2};T_1T')W(T_2\tfrac{1}{2}T\tfrac{1}{2};T_1'T')\times W(J_2jJj';J_1J')W(J_2jJj';J_1'J'). \quad (5)$$

This is the desired expression of expectation values and nondiagonal matrix elements of the interaction in the j^mj' configuration in terms of the expectation values in the two nucleon jj' configuration. Upon using this result with $j=\frac{1}{2}$ and $j'=\frac{1}{2}$, all the matrix elements required for this work were calculated.

III. CONFIGURATIONS WITH ONE $s_{\frac{1}{2}}$ NUCLEON

Nuclear configurations $p_{\frac{1}{2}}{}^m s_{\frac{1}{2}}$, which contain m $p_{\frac{1}{2}}$-nucleons and one $s_{\frac{1}{2}}$-nucleon outside the C^{12} core, occur in nuclei from C^{13} up to F^{17}. In many cases the levels of these configurations can be easily identified. First by the parity of the states which is even (or odd) according to m being even (or odd). Then the spin of the state often determines whether the even parity nucleons is in a $s_{\frac{1}{2}}$ or $d_{\frac{5}{2}}$ orbit. We first treat all the cases in which there is no ambiguity in the configuration assignment. The experimental data were taken from the Ajzenberg and Lauritsen review article,[9] from Nuclear Data Cards,[10] and from Mattauch's review article.[11]

[7] In the appendix there is a simple example of a specific calculation and there we take care of the exchange terms explicitly.

[8] G. Racah, Phys. Rev. **63**, 367 (1943), Eq. (4).

[9] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

[10] Nuclear Data Cards (National Research Council, Washington, D. C.).

[11] Mattauch, Waldmann, Bieri, and Everling, Z. Naturforsch. **11a**, 525 (1956).

TABLE I. Experimental and calculated energies of $p_{\frac{1}{2}}{}^m s_{\frac{1}{2}}$ configurations in Mev.

Nucleus	Position of level above g.s.	J, parity, T	Configuration	Total energy minus B.E.(C^{12})	Percent weight of spurious states	Energy in terms of the parameters	Exp. energy treated	Calculated energy
$_6C_7^{13}$	3.09	$\frac{1}{2}+\frac{1}{2}$	$s_{\frac{1}{2}}$	1.86	5.1	A_s	1.86	1.64
$_7N_6^{13}$	2.37	$\frac{1}{2}+\frac{1}{2}$	$s_{\frac{1}{2}}$	−0.42	5.1	A_s'	−0.42	−0.52
$_6C_8^{14}$	6.89	0−1	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	6.23	···	$A_s + V_0'$	0.75	0.87
$_6C_8^{14}$	6.09	1−1	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	7.03	3.2	$A_s + V_1'$	1.55	1.65
$_7N_7^{14}$	8.70	0−1	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	3.79	···	$\frac{1}{2}A_s+\frac{1}{2}A_s' + V_0'$	−0.17	−0.21
$_7N_7^{14}$	8.06	1−1	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	4.43	3.2	$\frac{1}{2}A_s+\frac{1}{2}A_s' + V_1'$	0.47	0.57
$_7N_7^{14}$	4.91	0−0	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	7.58	9.5	$\frac{1}{2}A_s+\frac{1}{2}A_s'+2V_0- V_0'$	3.61	3.65
$_7N_7^{14}$	6.23	1−0	$p_{\frac{1}{2}}s_{\frac{1}{2}}$	6.26	6.3	$\frac{1}{2}A_s+\frac{1}{2}A_s' +2V_1 - V_1'$	2.29	2.37
$_6C_9^{15}$	g.s.	$\frac{1}{2}+\frac{3}{2}$	$p_{\frac{1}{2}}{}^2s_{\frac{1}{2}}$	14.33	···	$A_s +\frac{1}{2}V_0'+\frac{3}{2}V_1'$	1.22	1.27
$_7N_9^{16}$	0.12	0−1	$p_{\frac{1}{2}}{}^3s_{\frac{1}{2}}$	25.87	···	$A_s +V_0 +\frac{1}{2}V_0'+\frac{3}{2}V_1'$	2.45	2.43
$_7N_9^{16}$	0.39	1−1	$p_{\frac{1}{2}}{}^3s_{\frac{1}{2}}$	25.60	···	$A_s +V_1+\frac{1}{2}V_0'+\frac{3}{2}V_1'$	2.18	2.18
$_8O_9^{17}$	0.87	$\frac{1}{2}+\frac{1}{2}$	$p_{\frac{1}{2}}{}^4s_{\frac{1}{2}}$	38.72	···	$A_s +\frac{1}{2}V_0+\frac{3}{2}V_1+\frac{1}{2}V_0'+\frac{3}{2}V_1'$	3.33	3.22
$_9F_8^{17}$	0.51	$\frac{1}{2}+\frac{1}{2}$	$p_{\frac{1}{2}}{}^4s_{\frac{1}{2}}$	35.53	···	$A_s'+\frac{1}{2}V_0+\frac{3}{2}V_1+\frac{1}{2}V_0'+\frac{3}{2}V_1'$	1.06	1.06

In C^{13} and N^{13}, with one nucleon outside the C^{12} core, we interpret the lowest $\frac{1}{2}+$ levels as due to excitation of the single extra nucleon from the $p_{\frac{1}{2}}$ to the $s_{\frac{1}{2}}$ orbit. In C^{14} and N^{14}, with two extra nucleons, odd parity low-lying levels belong to the $p_{\frac{1}{2}}s_{\frac{1}{2}}$ or $p_{\frac{1}{2}}d_{\frac{5}{2}}$ configurations. Of these, states with spins $J=0$ and $J=1$ belong to the former and states with spins $J=2$ and $J=3$ belong to the latter configuration. Experimentally, there are known low-lying 0− and 1− levels in C^{14} (and $T=1$ such levels in N^{14}). In N^{14} there are in addition $T=0$, 0−, and 1− observed levels. The experimental situation is less certain in the case of C^{15}. We expect the ninth neutron to be in a $d_{\frac{5}{2}}$ or $s_{\frac{1}{2}}$ orbit. Before the analysis is completed, it is impossible to conclude from C^{13} (and N^{13}) that the $s_{\frac{1}{2}}$ orbit is lower than the $d_{\frac{5}{2}}$ orbit in C^{15}. Similarly it would be wrong to arrive at the opposite conclusion from O^{17} (and F^{17}). The ground state of C^{15} β decays into the $\frac{1}{2}-$ ground state of N^{15} with a logft value of ~ 6. This can be at most an ordinary first forbidden transition with $\Delta J=0$, 1 but not $\Delta J=2$. This is an indication that the ground-state spin of C^{15} is $\frac{1}{2}+$.[12] The energy of this state agrees very well with our calculated value which lends further support to this assignment. In N^{16} the ninth neutron is again either in a $d_{\frac{5}{2}}$ or a $s_{\frac{1}{2}}$ orbit. When it is coupled to the $p_{\frac{1}{2}}$ proton the resulting spin can be 2− or 3− in the first case and 0− or 1− in the second. In fact, the lowest four states of N^{16} are known experimentally to have these spin values. In O^{17} and F^{17} there is one nucleon outside the closed shells of O^{16}. It can therefore be in a $d_{\frac{5}{2}}$ orbit (experimentally the ground state) or in the $s_{\frac{1}{2}}$ orbit (experimentally observed as the first excited state). In all these cases there is no ambiguity in the configuration assignments. In some of the nuclei mentioned protons and neutrons are in different shells and the analysis of reference 3 could have been made. Instead, we prefer to consider all configurations together by treating protons and neutrons alike with the isotopic spin formalism.

[12] This assignment is made for other reasons by G. A. Bartholomew *et al.*, Can. J. Phys. **34**, 147 (1956).

The experimental situation is summarized in Table I. The levels are listed according to the nucleus in which they occur (column 1), their position above the ground state of that nucleus (column 2) and their spin, parity, and isotopic spin (column 3). The configuration assignments for the nucleons outside the closed shells of C^{12} are given in column 4. The total energies of the levels (being the binding energies of the nucleus minus the corresponding values of column 2) from which the binding energy of C^{12} was subtracted are listed in column 5. The percentage weight of spurious states in the wave function of each level, calculated with harmonic-oscillator wave functions, is given in column 6.

As already explained, we are interested in the single-nucleon energy in the $s_{\frac{1}{2}}$ orbit (A_s or A_s') and in the interaction of this single nucleon with the $p_{\frac{1}{2}}$ nucleons. The actual parameters we use for this interaction are defined in terms of the expectation values in the $p_{\frac{1}{2}}s_{\frac{1}{2}}$ configuration as follows:

$$V_J=\tfrac{1}{2}\{\langle p_{\frac{1}{2}}s_{\frac{1}{2}}T=0\,J|V_{12}|p_{\frac{1}{2}}s_{\frac{1}{2}}T=0\,J\rangle + \langle p_{\frac{1}{2}}s_{\frac{1}{2}}T=1\,J|V_{12}|p_{\frac{1}{2}}s_{\frac{1}{2}}T=1\,J\rangle\},$$

$$V_J'=\langle p_{\frac{1}{2}}s_{\frac{1}{2}}T=1\,J|V_{12}|p_{\frac{1}{2}}s_{\frac{1}{2}}T=1\,J\rangle.$$

The physical meaning of these parameters is very simple. V_J is the interation in a state with spin J of a $p_{\frac{1}{2}}$ proton and an $s_{\frac{1}{2}}$ neutron (or a $p_{\frac{1}{2}}$ neutron and an $s_{\frac{1}{2}}$ proton). V_J' is simply the interaction in a state with spin J of two protons or two neutrons (or a proton and a neutron with $T=1$), one in a $p_{\frac{1}{2}}$ and the other in a $s_{\frac{1}{2}}$ orbit. The expression of the part of the energy we treat in terms of A_s, A_s', V_J, and V_J' is given in column 7. The corresponding experimental energies are listed in column 8. These values were obtained from the experimental energies of column 4 by subtracting $E(p_{\frac{1}{2}}{}^m)$, the energy due to the $p_{\frac{1}{2}}{}^m$ configuration, and the electrostatic interaction between the $p_{\frac{1}{2}}$ and $s_{\frac{1}{2}}$ protons. Energies of $p_{\frac{1}{2}}{}^m$ configurations were taken from an analysis of the experimental binding energies in the $p_{\frac{1}{2}}$ shell according to reference 2. The Coulomb energy we calculated with harmonic-oscillator wave functions taking the value of $e^2(\nu/\pi)^{\frac{1}{2}}$ from reference 2.

We now equate the expressions in column 7 to the corresponding experimental energies in column 8. In this way 13 equations with 6 unknown parameters are obtained. We look for values of the parameters which will fulfil best these equations by a least square fit. The best values (in Mev) obtained for the parameters are

$$A_s=1.636, \quad A_s'=-0.522,$$
$$V_0=1.162, \quad V_0'=-0.770,$$
$$V_1=0.914, \quad V_1'=0.011.$$

We check the agreement by plugging these values in the expressions of column 7. This way we obtain the calculated energies (column 9) which should be compared with the experimental energies in column 8. As easily realized, the agreement is very good. The accuracy obtained can be measured by the root-mean-square (rms) deviation. This is defined as $[\sum_{i=1}^{N}\Delta_i^2/(N-k)]^{\frac{1}{2}}$, where Δ_i denote the deviations of the calculated energies from the experimental, N is the number of equations, and k is the number of free parameters. The rms deviation computed from Table I is only 0.13 Mev.

The difference of the values obtained for A_s and A_s' is well accounted for by the electrostatic repulsion.

The parameters thus obtained can be used to determine positions of levels in other, less certain, cases. Our results can be immediately applied to the O^{16} nucleus with a $p_{\frac{1}{2}}^4$ ground configuration. There should be four odd-parity levels which belong to the excited $p_{\frac{1}{2}}^3 s_{\frac{1}{2}}$ configuration. That part of the energy which depends on the parameters for these levels is given below:

$$T=0,\ J=0: \quad \tfrac{1}{2}A_s+\tfrac{1}{2}A_s' \qquad +3V_1+\tfrac{3}{2}V_0'-\tfrac{3}{2}V_1'.$$
$$T=0,\ J=1: \quad \tfrac{1}{2}A_s+\tfrac{1}{2}A_s'+V_0+2V_1-\tfrac{1}{2}V_0'+\tfrac{1}{2}V_1'.$$
$$T=1,\ J=0: \quad \tfrac{1}{2}A_s+\tfrac{1}{2}A_s'+V_0 \qquad +\tfrac{1}{2}V_0'+\tfrac{3}{2}V_1'.$$
$$T=1,\ J=1: \quad \tfrac{1}{2}A_s+\tfrac{1}{2}A_s' \qquad + V_1+\tfrac{1}{2}V_0'+\tfrac{3}{2}V_1'.$$

If we insert the values of our parameters and add the rest of the energy, we obtain the total energies. The values of these turn out to be, in the same order, 23.55, 25.36, 22.77, and 22.53 Mev. These correspond to 0− levels with $T=0$ at 11.89 and with $T=1$ at 12.67 Mev above the ground state. Similarly the 1− levels are calculated to lie 10.08 ($T=0$) and 12.91 Mev ($T=1$) above the ground state of O^{16}.

One 0− level has been identified in O^{16}, 12.78 Mev above the ground state. This seems to agree well with the predicted 12.67 Mev. Still no isospin assignment has been made experimentally. No other 0− level has yet been found at ~12 Mev but there are already many levels in this region and a new one might well be soon found.

On the other hand, 1− levels are known at 7.12, 9.58, and 13.09 Mev above the ground state. The level at 9.58 appears as a resonance in $C^{12}+\alpha$ scattering which probably indicates its having $T=0$. This would agree fairly well with the calculated position of the $T=0$ level at 10.08 Mev. The agreement between the experimental value of 12.91 and the calculated 13.09 Mev is rather good. However, the experimental situation should be better known before definite conclusions are drawn. The main difficulty in O^{16} is the occurrence of a low-lying 1− level at 7.12 Mev. This is lower by 3 Mev than the lowest predicted value. It is very difficult to reconcile such a big shift, if due to configuration interaction, with our model. This would involve extremely large nondiagonal matrix elements. The weight of spurious states in these 1− states of $p_{\frac{3}{2}}^3 s_{\frac{1}{2}}$ is not bigger than in the previous cases. We therefore conclude that this low 1− state of O^{16} is due to a different mode of excitation.

A slightly more complicated case is that of the $\frac{1}{2}+$ levels in N^{15} and O^{15}. All the states considered until now were built from a single state of the $p_{\frac{1}{2}}^m$ configuration (simply because $m=1$ and $m=3$ correspond to a $p_{\frac{1}{2}}$ nucleon and a $p_{\frac{1}{2}}$ hole, respectively). Some of the low-lying $\frac{1}{2}+$ states in N^{15} and O^{15} presumably belong to the $p_{\frac{1}{2}}^2 s_{\frac{1}{2}}$ configuration. Two states with $T=\frac{1}{2}$ and $J=\frac{1}{2}$ can be obtained by adding the $s_{\frac{1}{2}}$ nucleon to the two $p_{\frac{1}{2}}^2$ states. These are the $T=0$ $J=1$ and $T=1$ $J=0$ states of the $p_{\frac{1}{2}}^2$ configuration. If there were no $p_{\frac{1}{2}}-s_{\frac{1}{2}}$ forces, the two states thus obtained would diagonalize the interaction matrix. Since these forces exist (and are quantitatively determined by our analysis), we have to calculate the nondiagonal element and diagonalize the two by two matrix. The energy matrix E, for O^{15}, is given below in terms of the parameters. The corresponding matrix for N^{15} is slightly different, since $E(p_{\frac{1}{2}}^2)$ includes the Coulomb energy. Also A_s and A_s' should be interchanged.

$$\langle p_{\frac{1}{2}}^2(T_1J_1)s_{\frac{1}{2}}T=\tfrac{1}{2}\ J=\tfrac{1}{2}\,|E|\,p_{\frac{1}{2}}^2(T_1'J_1')s_{\frac{1}{2}}T=\tfrac{1}{2}\ J=\tfrac{1}{2}\rangle.$$

T_1J_1 \ $T_1'J_1'$	10	01
10	$7.59+\frac{2}{3}A_s+\frac{1}{3}A_s'+\frac{3}{4}V_0+(9/4)V_1-\frac{1}{4}V_0'-\frac{3}{4}V_1'$	$\frac{3}{4}(V_0-V_1-V_0'+V_1')$
01	$\frac{3}{4}(V_0-V_1-V_0'+V_1')$	$11.98+A_s'+\frac{3}{4}V_0+\frac{1}{4}V_1+\frac{3}{4}V_0'+\frac{1}{4}V_1'$

We substitute the values we obtained for the parameters and diagonalize the matrix. The eigenvalues which are the total energies (relative to C^{12}) of the $\frac{1}{2}+$ states turn out to be 11.01 and 12.59 Mev. These correspond to levels 8.83 and 7.25 Mev above the ground states of O^{15}. The levels in N^{15} are calculated to be at 7.55 and 9.11 Mev.

To these predictions should be added that of the

$T=\frac{1}{2}$ $J=\frac{3}{2}$ level of the same $p_{\frac{1}{2}}^2 s_{\frac{1}{2}}$ configuration. The total energy of this level (relative to C^{12}) is

$$11.98+A_s'+V_1+V_1',$$

in the case of O^{15} and

$$12.44+A_s+V_1+V_1',$$

in N^{15}. Thus these levels should lie 7.46 and 8.33 Mev above the respective ground states.

Levels with spins $\frac{1}{2}+$ and $\frac{3}{2}+$ are observed in O^{15} and N^{15}. However, the experimental situation is far from being clear cut so that we shall not attempt to compare calculated with experimental levels. Our model predicts also another $\frac{1}{2}+$ state in O^{15} and N^{15} in the same neighborhood. This state which belongs to the $s_{\frac{1}{2}}^3$ configuration will be treated later.

These were all the cases in which only one nucleon occupies the $s_{\frac{1}{2}}$ orbit. The treatment of more $s_{\frac{1}{2}}$ nucleons will be carried in the next section. This will include the cases of the 0+ level in O^{16} and the $\frac{1}{2}-$ level in F^{19}.

IV. CONFIGURATIONS CONTAINING SEVERAL $s_{\frac{1}{2}}$ NUCLEONS

In the last section we treated $p_{\frac{1}{2}}^m s_{\frac{1}{2}}$ configurations. We could get information on the $p_{\frac{1}{2}}-s_{\frac{1}{2}}$ interaction by subtracting from the total energies the energies due to the $p_{\frac{1}{2}}^m$ configurations. We would now like to consider configurations of the type $p_{\frac{1}{2}}^m s_{\frac{1}{2}}^n$ with $n>1$. To do this we have to know the interaction energy in the $s_{\frac{1}{2}}^n$ configurations. The values of these energies, however, cannot be taken from reference 2. This is because the $s_{\frac{1}{2}}^n$ configurations treated there occur as ground configurations only beyond Si^{28} where the $d_{\frac{5}{2}}$ shell is closed. There is no good reason to believe that the *effective forces* between $s_{\frac{1}{2}}$ nucleons will be the same in the absence of the closed $d_{\frac{5}{2}}$ shell. In fact, if we take the interaction parameters from reference 2, the $s_{\frac{1}{2}}^3$ configuration which is probably the ground configuration[13] of F^{19}, is calculated to lie 7 Mev higher than observed.

We can determine the effective interactions in $s_{\frac{1}{2}}^n$ configurations by analyzing nuclei with these configurations outside the closed shells of O^{16}. This procedure will give us the necessary interaction parameters. The difficulty is to find out where the configuration is $s_{\frac{1}{2}}^n$ and not $d_{\frac{5}{2}}^n$, say. In O^{17} and F^{17} the single nucleon $\frac{1}{2}+$ states are well known. We compare the level schemes of F^{18} and F^{19} (or Ne^{19}) to those of Al^{26} and Al^{25} (or Mg^{25}) with corresponding numbers of holes in the $d_{\frac{5}{2}}$ shells. From the fact that there is no similarity we conclude that the $J=1$ ground state of F^{18} belongs to the $s_{\frac{1}{2}}^2$ configuration. Similarly, we take the ground state of F^{19} (and Ne^{19}) with $J=\frac{1}{2}$ to belong to the $s_{\frac{1}{2}}^3$ configuration. These cases are sufficient to determine the interaction parameters. However, apart from the Coulomb energy there are as many equations as unknowns. The $s_{\frac{1}{2}}^n$ interaction parameters thus obtained are, in the notation of reference 2, in Mev,

$$a=3.57 \quad \text{and} \quad b=-1.91.$$

These values are considerably bigger than the values 1.98 and -0.32 obtained in reference 2 from nuclei beyond Si^{28}.

With these parameters the $T=1$ $J=0$ state of the $s_{\frac{1}{2}}^2$ configuration is calculated to lie 3 Mev above the experimental ground state of O^{18}. This result is, however, satisfactory. There are indications (e.g., position of the 2+ excited state) that the O^{18} ground state as well as the corresponding $T=1$ state at 0.95 Mev in F^{18} belong predominantly to the $d_{\frac{5}{2}}^2$ configuration.[14,15] The $s_{\frac{1}{2}}^2$ $T=1$ $J=0$ level in F^{18} is calculated to lie 3.8 Mev above the ground state.

We can also calculate with these parameters the energy of the $J=0$ state of the $s_{\frac{1}{2}}^4$ configuration. The result, 33.00 Mev, is in excellent agreement with the experimental binding energy of Ne^{20} which is 33.05 Mev (these energies are relative to O^{16}). This might be a strong argument in favor of our procedure here. However, we do not know the correct configuration assignment for the ground state of Ne^{20}. We hope to return soon to this discussion in more detail. Nevertheless we shall tentatively use the quoted values of the interaction parameters. As we shall presently see, the rather strong interaction between the $s_{\frac{1}{2}}$ nucleons gives rise to interesting modes of excitation.

The procedure adopted can be simply described as using the experimental binding energies of F^{18}, F^{19} (and Ne^{19}) and Ne^{20}. In dealing with these energies it does not matter what the exact configurations are. However, in addition to these binding energies we must use the parameters of the interaction between the $p_{\frac{1}{2}}$ nucleons and nucleons in the next higher shell. The values of these parameters for $s_{\frac{1}{2}}$ nucleons are different from those for $d_{\frac{5}{2}}$ nucleons. Thus the results depend on the predominant configurations assigned to the states. This is the reason for the preceding discussion. In the cases treated below, fair agreement is obtained only with the previously determined $p_{\frac{1}{2}}-s_{\frac{1}{2}}$ interaction parameters. We have enough information on the $p_{\frac{1}{2}}-d_{\frac{5}{2}}$ parameters to exclude the possibility of getting similar agreement by using them.

The first nucleus in which two $s_{\frac{1}{2}}$ nucleons can occur is N^{14} (or C^{14} with levels corresponding to the $T=1$

[13] In our model configuration interaction is assumed to be negligible and thus $s_{\frac{1}{2}}^3$ must be the ground configuration since it is energetically preferred to $d_{\frac{5}{2}}^2 s_{\frac{1}{2}}$. Also in other treatments, where rather strong configuration interaction is assumed, the $s_{\frac{1}{2}}^3$ configuration is postulated to lie rather low and it has a large amplitude in the ground-state wave function of F^{19}.

[14] O. M. Bilaniuk and P. V. C. Hough, Phys. Rev. **108**, 305 (1957).

[15] The experimental values for configuration mixing given in reference 14 are obtained by assuming that 81% of the $J=4$ state in O^{18} belongs to $d_{\frac{5}{2}}^2$, whereas the rest belongs to the $d_{\frac{5}{2}}d_{\frac{3}{2}}$ configuration. Having in mind the large $d_{\frac{5}{2}}-d_{\frac{3}{2}}$ separation and the pairing effect, we can assume instead an almost pure $d_{\frac{5}{2}}^2$ configuration for the $J=4$ state. Then the experimental weight of the $d_{\frac{5}{2}}^2$ configuration in the $J=0$ ground state of O^{18} turns out to be about 95%.

levels of N^{14}). If the two $p_{\frac{1}{2}}$ nucleons are raised into the $s_{\frac{1}{2}}$ orbit, the configuration is simply $s_{\frac{1}{2}}^2$ and there are two levels with $T=0\ J=1$ and $T=1\ J=0$. The total energies of these are simply

$$T=0,\ J=1: A_s+A_s'+a-\tfrac{3}{2}b=7.54 \text{ Mev},$$
$$T=1,\ J=0: \quad A_s+A_s'+a+\tfrac{1}{2}b=3.73 \text{ Mev}.$$

The numerical values were obtained by inserting the values of the parameters. The corresponding levels are at 4.95 ($T=0\ J=1$) and 8.76 Mev ($T=1\ J=0$) above the ground state of N^{14}. Experimentally there are levels with $J=1+$ at 5.10 and with $T=1\ J=0+$ at 8.62 Mev. It is tempting to identify these levels with the predicted ones. The agreement obtained is, thus, very good and encouraging to treat in this manner further cases.

The next pair of nuclei to be considered is the O^{15}-N^{15} pair. The ground configuration of these is $p_{\frac{1}{2}}^3$. The $p_{\frac{1}{2}}^2 s_{\frac{1}{2}}$ configuration was treated in the previous section. If one more $p_{\frac{1}{2}}$ nucleon is excited the configuration becomes $p_{\frac{1}{2}} s_{\frac{1}{2}}^2$. The states of this configuration lie rather high where there is not enough experimental information. If the last $p_{\frac{1}{2}}$ nucleon is also excited the $s_{\frac{1}{2}}^3$ configuration is obtained. The energy of the only state $T=\frac{1}{2}\ J=\frac{1}{2}+$ can be easily calculated with the values of A_s, A_s', and the $s_{\frac{1}{2}}^n$ interaction parameters. This $\frac{1}{2}+$ level in O^{15} is calculated to lie 6.52 Mev above the ground state. This would agree fairly well with the experimental level at 6.82 Mev with positive parity and possible spins $\frac{1}{2}$ and $\frac{3}{2}$. On the other hand, in N^{15} the corresponding $\frac{1}{2}+$ level turns out to lie 7.00 Mev above the ground state. Again there are experimentally even-parity levels with possible spin $\frac{1}{2}$ at 7.16 and 7.31 Mev. However, there is in N^{15} an additional low-lying level, 5.31 Mev above the ground state, with a possible spin $\frac{1}{2}+$. Therefore, it is not yet possible to conclude whether there is agreement in this case. The experimental situation should be cleared up first and particularly the correspondence between the mirror nuclei O^{15} and N^{15}.

A more interesting case is O^{16} in which the first excited state, at 6.06 Mev, is a 0+ state. If two $p_{\frac{1}{2}}$ nucleons are raised into $s_{\frac{1}{2}}$ orbits, two states with $T=0\ J=0$ can be formed. In the scheme we use, these two states are $p_{\frac{1}{2}}^2(T_1=1\ J_1=0)s_{\frac{1}{2}}^2(T_2=1\ J_2=0)T=0\ J=0$ and $p_{\frac{1}{2}}^2(T_1=0\ J_1=1)s_{\frac{1}{2}}^2(T_2=0\ J_2=1)T=0\ J=0$. The energy matrix defined by these states has a nondiagonal element due to the $p_{\frac{1}{2}}-s_{\frac{1}{2}}$ interaction. The eigenvalues of this matrix give two 0+ states, 13.81 and 16.91 Mev above the ground state of O^{16}. Both these values are much higher than the experimental 6.06 Mev. Preliminary calculations show that also the excitation of two $p_{\frac{1}{2}}$ nucleons into $d_{\frac{5}{2}}$ orbits gives levels in about the same range.

These energies turn out to be so high because of the following reason. The attraction between a pair of nucleons in the $p_{\frac{1}{2}}$ orbit and a pair in the $s_{\frac{1}{2}}$ (or $d_{\frac{5}{2}}$) orbit, as determined by our parameters, is much smaller than the attraction of two pairs in the same orbit. Thus we were led to try the excitation of all four $p_{\frac{1}{2}}$ nucleons into the $s_{\frac{1}{2}}$ orbit. The total energy (relative to C^{12}) of the resulting $T=0\ J=0$ state turns out to be 28.54 Mev. This corresponds to a 0^+ level at 6.90 Mev above the ground state of O^{16}. This is reasonably close to the experimental value. If the four $p_{\frac{1}{2}}$ nucleons are raised into the $d_{\frac{5}{2}}$ orbit the resulting level is much higher. We thus adopt the assignment of the first excited 0^+ state in O^{16} to the $s_{\frac{1}{2}}^4$ configuration with no $p_{\frac{1}{2}}$ nucleons present. It would be very interesting to verify or disprove experimentally this mode of excitation. The occurrence of the electric monopole transition from the excited 0+ level to the ground state indicates admixtures to our wave function. However, it is rather difficult to determine quantitatively such admixtures.

A related mode of excitation gives a low-lying state also in O^{17} and F^{17}. In these nuclei the $\frac{1}{2}+$ state, of the $p_{\frac{1}{2}}^4 s_{\frac{1}{2}}$ configuration, is the first excited state. The lowest state ($J=\frac{1}{2}-$) of the $p_{\frac{1}{2}}^3 s_{\frac{1}{2}}^2$ configuration is calculated to lie 6.69 Mev above the ground state of O^{17} (and 6.32 Mev above the ground state of F^{17}). However, a still lower $\frac{1}{2}-$ level belongs to the $p_{\frac{1}{2}} s_{\frac{1}{2}}^4$ configuration. The total energy of this level is easily calculated from the interaction parameters and turns out to be 35.58 Mev in O^{17} and 31.67 Mev in F^{17} (relative to C^{12}). The corresponding $\frac{1}{2}-$ levels should therefore be at 4.01 Mev in O^{17} and 4.37 Mev in F^{17}. Experimentally, there are $\frac{1}{2}-$ levels in O^{17} and F^{17} at 3.06 and 3.10 Mev above the respective ground states. The agreement, although significant, is not very good, the deviation being about 1 Mev (which should be compared to the calculated value of 36 (or 32) Mev). If we would take for this level the $p_{\frac{1}{2}} d_{\frac{5}{2}}^4$ configuration, the agreement would turn much worse. Still this result might be considered satisfactory since this mode of excitation causes a great change in the occupation of orbits in the nucleus. These considerations should be applied also in the similar case of O^{16}.

In view of the previous cases, the occurrence of the very low-lying $\frac{1}{2}-$ level in F^{19} could be well understood. The ground configuration of F^{19} is $p_{\frac{1}{2}}^4 s_{\frac{1}{2}}^3$ and its total energy (of its only level $\frac{1}{2}+$) is 55.61 Mev (relative to C^{12}). The simplest excited configuration is $p_{\frac{1}{2}}^3 s_{\frac{1}{2}}^4$ and its energy can be simply calculated in the same manner as before. This energy turns out to be 55.79 Mev which is even slightly bigger than the ground-state energy. The $\frac{1}{2}-$ state would thus be predicted to be $\sim$0.2 Mev below the $\frac{1}{2}+$ ground state. Actually it lies $\sim$0.1 Mev above it. In the mirror nucleus Ne^{19} the $\frac{1}{2}-$ state is calculated to lie 0.23 Mev above the $\frac{1}{2}+$ ground state. Experimentally it lies 0.26 or 0.29 Mev above it. These $\frac{1}{2}-$ states were usually interpreted as due to a $p_{\frac{1}{2}}$ hole.[6] Now this interpretation has been given a quantitative basis.

APPENDIX

There is a simple way to calculate the weight of spurious states in the wave functions used in this work. In the ground state the center of mass is moving in a $0s$ oscillator orbit and its total energy is $\frac{3}{2}\hbar\omega$. If one nucleon is excited into a higher oscillator shell there might be some motion of the center of mass in the $0p$ orbit. If x^2 is the weight of the spurious states it is also the amount of the $0p$ motion. Thus, the energy of the center-of-mass motion will be $(1-x^2)(\frac{3}{2}\hbar\omega)+x^2(\frac{5}{2}\hbar\omega)$. If we calculate the energy of the center-of-mass motion, we can, in these simple cases, obtain right away the weight of the spurious states.

Instead of calculating the total energy we calculate the expectation value of the potential energy. This is well known to be equal to half the total energy in the harmonic-oscillator motion. This potential energy V has no cross terms between states with $0s$ motion and $0p$ motion and therefore it can be used in the cases we treat here. Let N denote the number of nucleons and m the mass of each. We then have

$$V=\tfrac{1}{2}(Nm)\omega(\sum\mathbf{r}_i/N)^2=(1/2N)m\omega(\sum\mathbf{r}_i)^2. \quad \text{(A1)}$$

This is equal to a sum of potential energies of single nucleons and simple two-body terms:

$$V=\frac{1}{N}\left[\sum\frac{m\omega}{2}r_i^2+m\omega\sum_{i<j}(\mathbf{r}_i\cdot\mathbf{r}_j)\right]. \quad \text{(A2)}$$

The first term within the brackets is one half the sum of the energies of the single nucleons in the various orbits of the oscillator potential. The second term can also be easily evaluated.

In practice, things are even simpler when we take into account the fact that in ground states $\langle V\rangle=\frac{3}{4}\hbar\omega$. If there are N_0 nucleons in closed shells and lowest unfilled shells, we can rewrite V as

$$V=(N_0/N)(1/2N_0)m\omega(\sum_{i=1}^{N_0}\mathbf{r}_i)^2 +\frac{m\omega}{N}(\sum_{i=1}^{N_0}\mathbf{r}_i)\cdot(\sum_{j=N_0+1}^{N}\mathbf{r}_j)+\frac{m\omega}{2N}(\sum_{i=N_0+1}^{N}\mathbf{r}_i)^2. \quad \text{(A3)}$$

The first term is simply $(N_0/N)(\frac{3}{4}\hbar\omega)$. The third term represents single-nucleon terms and two-nucleon terms of the nucleons in excited orbits, while the second term represents their "interaction" with the N_0 nucleons in the lowest orbits.

The "interaction" $\mathbf{r}_1\cdot\mathbf{r}_2$ has very simple selection rules. Taken between nucleon states l_1j_1, l_2j_2 and $l_1'j_1'$, $l_2'j_2'$ it vanishes unless the triangular conditions are satisfied by $(j_1j_1'1)$, $(j_2j_2'1)$, $(l_1l_1'1)$ and $(l_2l_2'1)$. Furthermore the parities of l_1 and l_1' must be opposite and the same must hold for l_2 and l_2'. This means that only exchange terms contribute to $\langle V\rangle$. Our results in Sec. II give any two-body interaction in terms of the expectation values in the two-nucleon configuration. Thus, we have only to calculate these for the special type of interaction we deal with here.

Only V_J' contains exchange terms and therefore we have to consider its value. Writing explicitly the nucleon *coordinates* 1 and 2, we have

$$V_J'=\langle jj'T=1\,J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,jj'T=1\,J\rangle =\langle j_1j_2'J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,j_1j_2'J\rangle-\langle j_1j_2'J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,j_2j_1'J\rangle. \quad \text{(A4)}$$

We change the order of j and j' in the right-hand side of the exchange term in order to get an ordinary matrix element. We obtain from the symmetry properties of the Clebsch-Gordan coefficients the expression

$$V_J'=\langle j_1j_2'J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,j_1j_2'J\rangle -(-1)^{j+j'-J}\langle j_1j_2'J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,j_1'j_2J\rangle. \quad \text{(A5)}$$

Here nucleons 1 and 2 appear in the same order throughout and we can drop these indices. We get from the second term, only which contributes, by use of well-known techniques:[16]

$$V_J'=(-1)^{j+j'-J+1}\langle jj'J\,|\,\mathbf{r}_1\cdot\mathbf{r}_2\,|\,j'jJ\rangle =(-1)^{j+j'-J+1}(-1)^{j+j-J}(j\|r\|j')(j'\|r\|j) \times W(jj'j'j;J1)=-(j\|r\|j')^2W(jj'j'j;J1). \quad \text{(A6)}$$

The value of the reduced matrix element is readily given by[16]

$$(nl\tfrac{1}{2}j\|r\|n'l'\tfrac{1}{2}j')=(-1)^{\frac{1}{2}+1-l'-j}(nl\|r\|n'l') \times[(2j+1)(2j'+1)]^{\frac{1}{2}}W(ljl'j';\tfrac{1}{2}1). \quad \text{(A7)}$$

Obviously $2n+l$ must differ from $2n'+l'$ by one only for the matrix element not to vanish. Because of the parity, l and l' must differ by one and n and n' can differ at most by one. For such values of the single-nucleon quantum numbers we have

$$(nl\|r\|n'l')=(l\|C^{(1)}\|l')\int_0^\infty R_{nl}(r)R_{n'l'}(r)rdr, \quad \text{(A8)}$$

where R_{nl} are the harmonic-oscillator wave functions[17,18] and the angular part is[16]

$$(l\|C^{(1)}\|l')=[(2l+1)(2l'+1)\tfrac{1}{2}C_{l0'1}]^{\frac{1}{2}}. \quad \text{(A9)}$$

We write down the final expression

$$V_J'=-(2j+1)(2j'+1)(2l+1)(2l'+1)\tfrac{1}{2}C_{l0'1} \times W(jj'j'j;J1)W^2(ljl'j';\tfrac{1}{2}1)\left[\int R_{nl}R_{n'l'}rdr\right]^2. \quad \text{(A10)}$$

[16] G. Racah, Phys. Rev. **62**, 438 (1942).
[17] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[18] R. Thieberger, Nuclear Phys. **2**, 533 (1956/57). In this paper there is a slight error on page 534: V_{1l} should be equal to $1\mid(4\nu/2l+3)r^2$.

The term which has the form of the interaction of an excited j' nucleon with the closed shells can be written down in a close form. We have to add contributions from the appropriate shells and only from nucleons identical to the excited one. We get for a closed shell with $2j+1$ identical nucleons, by using the technique of Sec. II,

$$\langle j^{2j+1}(0)j' \mid (\sum_{i=1}^{2j+1} \mathbf{r}_i)\cdot\mathbf{r}' \mid j^{2j+1}(0)j'\rangle$$
$$=(2j+1)\langle j^{2j+1}(0)j' \mid \mathbf{r}_1\cdot\mathbf{r}' \mid j^{2j+1}(0)j'\rangle$$
$$=(2j+1)\,\langle j^{2j}(j)j_1(0)j' \mid \mathbf{r}_1\cdot\mathbf{r}' \mid j^{2j}(j)j_1(0)j'\rangle$$
$$=(2j+1)\sum_J (2J+1)W^2(jjj'j';0J)V_J'$$
$$=\frac{1}{2j'+1}\sum_J (2J+1)V_J'. \quad \text{(A11)}$$

Inserting the values of V_J' from (A6) and (A7), we obtain

$$-\frac{1}{2j'+1}\sum_J (2J+1)(nl\|r\|n'l')^2(2j+1)(2j'+1)$$
$$\times W^2(ljl'j';\tfrac{1}{2}1)W(jj'j'j;J1)$$
$$=-(2j+1)(nl\|r\|n'l')^2W^2(ljl'j';\tfrac{1}{2}1)$$
$$\times\sum_J (2J+1)W(jj'j'j;J1). \quad \text{(A12)}$$

Using Eq. (43) reference 16, we can carry out the summation over J:

$$\sum_J (2J+1)W(jj'j'j;J1)=-[(2j+1)(2j'+1)]^{\frac{1}{2}}$$
$$\times(-1)^{j+j'}\sum_J(-1)^{J+1}W(jjj'j';0J)$$
$$\times W(jj'j'j;J1)=-(-1)^{j+j'}[(2j+1)(2j'+1)]^{\frac{1}{2}}$$
$$\times W(jjj'j';01)=1. \quad \text{(A13)}$$

Thus the interaction of a $n'l'j'$ nucleon with the closed nlj shell is

$$-(2j+1)(nl\|r\|n'l')^2W^2(ljl'j';\tfrac{1}{2}1). \quad \text{(A14)}$$

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Gamma-Ray Threshold Method and the $O^{18}(d,n\gamma)F^{19}$ Reaction

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A neutron threshold technique involving observations of the gamma rays from the de-excitation of the residual excited states is described. A study of the $O^{18}(d,n\gamma)F^{19}$ reaction with this technique has resulted in the observation of neutron thresholds at bombarding energies of 346±8, 525±8, and 584±10 kev. These thresholds correspond to states at 6.048±0.014, 6.210±0.014, and 6.262±0.015 Mev in the F^{19} nucleus, respectively. A detailed gamma-ray spectrum was obtained at a bombarding energy of 1.00 Mev using a single-crystal spectrometer, and another detailed spectrum was obtained at 1.40 Mev using a three-crystal pair spectrometer. Coincidence measurements were made for several of the cascade gamma rays.

I. INTRODUCTION

NEUTRON threshold techniques have been used to observe a number of nuclear reactions[1] in order to determine the energies of excited states in the residual nuclides. The essence of the neutron threshold method is the use of a detector which discriminates against the "fast" neutrons in favor of the "slow" neutrons (a few kev of energy) which are emitted just above their threshold. Since these techniques generally have utilized enriched BF_3 or boron-lined proportional counters surrounded by a small amount of paraffin moderator, this discrimination against the faster neutrons is purely a statistical matter involving the B^{10} neutron-capture cross section as a function of neutron energy and the scattering cross section of hydrogen, also as a function of neutron energy. In addition, the fractional energy loss of the neutron per collision in the paraffin is also a statistical function, but is independent of neutron energy. Thus the ratio of "slow" to "fast" efficiencies is never large, as contrasted with electronic pulse-height discrimination employed in modern scintillation detectors. A further complication in the above method of threshold detection is that a neutron which leaves the target as a fast neutron might enter the detector as a slow neutron because of degradation of its energy by the floor and walls of the laboratory, or by the target assembly and the detector equipment.

A technique which involves the observation of the resulting gamma rays instead of the neutrons obviates many of the inherent disadvantages of neutron detec-

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