Klein-Gordon and Dirac Equations in General Relativity

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(Received June 2, 1958)

It is shown, that when account is taken of the gravitational field of a point charge, the Klein-Gordon and Dirac equations for the motion of a charged particle in a Coulomb field do not possess solutions which can be expressed as series of terms proportional to positive integral powers of the gravitational constant.

THE purpose of this note is to point out that, when account is taken of the gravitational field of a point charge, the Klein-Gordon and Dirac equations for the motion of a charged particle in a Coulomb field do not possess solutions which can be expressed as series of terms proportional to positive integral powers of G, the gravitational constant. If the gravitational field of a point charge is expanded in positive integral powers of G, to first order the Klein-Gordon equation contains a term proportional to r^{-4} and the Dirac equation contains an r^{-3} term. Terms of higher order are more singular. The analysis is based on the rigorous solution of the combined system of Einstein's and Maxwell's equations for the electrostatic and gravitational fields of a point charge.[1] In the usual spherical polar coordinates, we have for the fields of a particle of charge e and mass M,

$$g_{00}=e^{\nu},\quad g_{11}=-e^{-\nu},\quad g_{22}=-r^2,\quad g_{33}=-r^2\sin^2\theta, \tag{1}$$

where

$$e^{\nu}=1-\frac{2GM}{c^2r}+\frac{Ge^2}{c^4r^2}, \tag{2}$$

and, for the scalar potential, $\phi=-\phi_4=e/r$.

Consider first the Klein-Gordon equation for a charge q and mass m in the preceding fields. It can be written covariantly as

$$\frac{-\hbar^2}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^k}\left((-g)^{\frac{1}{2}}g^{kl}\frac{\partial\psi}{\partial x^l}\right)+\frac{2i\hbar q}{c}\phi^k\frac{\partial\psi}{\partial x^k}+\frac{q^2}{c^2}\phi_k\phi^k\psi=m^2c^2\psi, \tag{3}$$

in which ϕ_k is the covariant potential 4-vector. In this case, we find for a state of angular momentum l the following radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right)+\frac{d\nu}{dr}\frac{dR_l}{dr}-\frac{e^{-\nu}l(l+1)}{r^2}R_l=\frac{e^{-\nu}}{\hbar^2}\left[m^2c^2-\frac{e^{-\nu}}{c^2}(E-q\phi)^2\right]R_l. \tag{4}$$

If the gravitational field can be regarded as small, we can expand $e^{-\nu}$,

$$e^{-\nu}=1+\frac{2GM}{c^2r}-\frac{Ge^2}{c^4r^2}. \tag{5}$$

Upon substituting this expression, r^{-4} singularities are found in the terms involving $e^{-\nu}/r^2$ and $e^{-\nu}\phi^2$. Terms containing r^{-3} are also present. It may be concluded from Fuchs' theorem that the equation does not possess a power series expansion about $r=0$.

A similar situation prevails in the Dirac equation. The Dirac equation can be written as

$$\gamma^{\alpha}\nabla_{\alpha}\psi+\mu\psi=0\quad(\mu=mc/\hbar), \tag{6}$$

where ∇_{α} produces the covariant derivative of the spinor ψ. From the work of Brill and Wheeler,[2] the radial equations for the usual F and G functions can be written

$$\begin{aligned}&[e^{-\nu}(h-V)+\mu e^{-\nu/2}]F+\frac{dG}{dr}-\frac{k}{r}Ge^{-\nu/2}=0,\\&[e^{-\nu}(h-V)-\mu e^{-\nu/2}]G+\frac{dF}{dr}-\frac{k}{r}Fe^{-\nu/2}=0,\end{aligned} \tag{7}$$

in which $h=E/\hbar c$, $V=-qe/\hbar cr$, and k is the eigenvalue of the angular K operator. From (5) we see that to first order in the gravitational constant there is an r^{-3} term in the effective potential due to the presence of the r^{-2} term in $e^{-\nu}$.

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[1] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1942), p. 204. Attention is called to an error in Bergmann's derivation of the electrostatic potential which actually is e/r, instead of the form given. The Swarzschild singularity does not exist in this solution if $(e/m)>G^{\frac{1}{2}}$.

[2] D. Brill and J. A. Wheeler, Revs. Modern Phys. 29, 465 (1957).