field, and the problem of elementary particle physics is shifted from the question of the number of elementary particles to the number of elementary fields. It would appear that the future task of fundamental theory would be to look for criteria which specify elementary fields.

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Conservation Laws in General Relativity as the Generators of Coordinate Transformations

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The components of the so-called canonical energy-stress pseudotensor in general relativity may be thought of as the generators of infinitesimal coordinate transformations corresponding to a rigid parallel displacement of the coordinate origin, just as in Lorentz-covariant theories. In this paper it is shown that the canonical expressions, as well as the expressions proposed by Landau and Lifshitz and the expressions for the angular momentum density, are all special cases of an infinity of conservation laws whose pseudovectors generate arbitrary curvilinear coordinate transformations. This approach enables us to construct the transform of every one of these conservation laws under an arbitrary (finite) coordinate transformation. Finally it is shown that every one of these conservation laws may be used to obtain a surface integral relationship that describes the motion of singularities in a general-relativistic theory. It is concluded that there is an infinite number of parameters that describes a singularity of the field, a fact that had previously been in doubt.

1. INTRODUCTION

THROUGHOUT mechanics and field theories, it is well known that the fundamental conservation laws are related to the universal invariance properties of physical laws, e.g., the conservation of linear momentum to the invariance with respect to displacement of the coordinate origin; the conservation of energy depends likewise on invariance with respect to the choice of the origin of the time scale (the instant $t=0$), and the conservation of angular momentum on the invariance with respect to orthogonal transformations in three-space. The structure of conservation laws in general relativity and in general-relativistic theories differs from that in nonrelativistic and in Lorentz-covariant theories because of the much wider scope of coordinate transformations in general relativity. It was discovered a long time ago that the so-called conservation laws of energy and linear momentum in general relativity,

$$\mathfrak{t}^{\rho}{}_{\mu,\rho}=0, \qquad \mathfrak{t}^{\rho}{}_{\mu}\equiv g_{\alpha\beta,\mu}\frac{\partial\mathfrak{L}}{\partial g_{\alpha\beta,\rho}}-\delta^{\rho}{}_{\mu}\mathfrak{L}, \tag{1.1}$$

which hold only insofar as the field equations of the theory are satisfied, are related to a set of identities, the so-called "strong" conservation laws,[1,2]

$$\mathfrak{T}^{\rho}{}_{\mu,\rho}\equiv 0, \qquad \mathfrak{T}^{\rho}{}_{\mu}=\mathfrak{U}^{[\rho\sigma]}{}_{\mu,\sigma}. \tag{1.2}$$

The quantities $\mathfrak{T}^{\rho}{}_{\mu}$ equal $\mathfrak{t}^{\rho}{}_{\mu}$ when the field equations are satisfied. The "superpotentials" $\mathfrak{U}^{[\rho\sigma]}{}_{\mu}$, which were first discovered by von Freud,[3] can also be constructed in general-relativistic theories that differ in detail from Einstein's 1916 theory.[4,5] The existence of the strong laws leads to the (partial) determination of the equations of motions of singularities by the surrounding field.

The canonical energy-stress components do not form a tensor density, nor even a geometric object. Being formed of the components of the metric tensor and its first derivatives, all components can be made to vanish simultaneously at any one world point, though not in a whole region. Moreover, the integrals over the energy-stress expressions that in Lorentz-covariant theories would be interpreted as the whole energy and as the whole linear momentum, respectively, transform as the components of a four-vector only with respect to a very restricted group of coordinate transformations. This elusive character of the energy-stress tensor has rendered the physical interpretation of the corresponding constants of the motion dubious.

This somewhat unsatisfactory situation has been complicated further by the discovery of another expression in general relativity which also obeys a set of equations of continuity, by Landau and Lifshitz.[6] Gold-

[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).
[2] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).
[3] P. von Freud, Ann. Math. **40**, 417 (1939).
[4] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
[5] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).
[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Publishing Company, Inc., Reading, 1951), p. 316 of the English translation.

berg has investigated the relationship between the canonical and the Landau-Lifshitz expressions[5]; in the course of this investigation he has discovered a possible expression for the angular momentum density of the gravitational field, which is different from one earlier suggested by Bergmann and Thomson.[7] He has also constructed whole classes of additional conservation laws, which are generalizations of the canonical and of the Landau-Lifshitz expressions.

In this paper we shall construct a whole gamut of conservation laws as the generators of all curvilinear infinitesimal coordinate transformations. All the conservation laws enumerated above are included in this general class. Each of these conservation laws can be exploited to yield a condition on the motion of singularities. Hence, a singularity cannot be fully characterized by a finite number of parameters. It is also possible to construct a transformation theory for the properties of singular regions.

2. COORDINATE TRANSFORMATIONS AND THEIR GENERATORS

In this section we shall construct the generators of infinitesimal coordinate transformations in the general theory of relativity, but the method used is usable for any theories whose field equations can be derived from a least-action principle.

Infinitesimal invariant transformations in a field theory and their generators are connected by the relationship[4,8]

$$L^A\delta y_A+C^\rho{}_{,\rho}\equiv 0. \tag{2.1}$$

L^A symbolizes the field equations, and δy_A represents the infinitesimal invariant transformation of the field variables y_A (i.e., a transformation of variables that leaves the functional dependence of the Lagrangian density L on its arguments $y_A, y_{A,\rho}$ unchanged). C^ρ might be called the generating density, and the volume integral $\int C^0 d^3x$ is the generating functional. Given an infinitesimal transformation δy_A, the generating density C^ρ is determined only up to a completely arbitrary curl field, so that with C^ρ the field $\bar{C}^\rho$,

$$\bar{C}^\rho=C^\rho+V^{[\rho\sigma]}{}_{,\sigma}, \tag{2.2}$$

with an arbitrary field $V^{[\rho\sigma]}$, is also an admissible generating density.

We shall now turn to Einstein's (1916) theory of gravitation. An infinitesimal transformation $\delta g_{\mu\nu}$ will represent an invariant transformation if there exists a field C^ρ such that

$$\mathfrak{G}^{\mu\nu}\delta g_{\mu\nu}+C^\rho{}_{,\rho}\equiv 0. \tag{2.3}$$

We shall now consider the transformations of the metric field induced by infinitesimal coordinate transformations, transformations that we know are invariant. The change in the metric field (considered as a set of functions of the coordinates) is

$$\bar{\delta}g_{\mu\nu}=-(\xi_{\mu;\nu}+\xi_{\nu;\mu}),\qquad \xi_\mu=g_{\mu\rho}\delta x^\rho. \tag{2.4}$$

It follows that the generator in Eq. (2.3) is determined by the requirement

$$C^\rho{}_{,\rho}\equiv 2\mathfrak{G}^{\mu\nu}\xi_{\mu;\nu}\equiv 2(\mathfrak{G}^{\mu\rho}\xi_\mu)_{;\rho}\equiv 2(\mathfrak{G}^{\mu\rho}\xi_\mu)_{,\rho}. \tag{2.5}$$

This conversion into a divergence is possible because of the (contracted) Bianchi identities. Hence one possible choice of the generating density is

$$C^\rho=2\mathfrak{G}^{\rho\sigma}\xi_\sigma=2\mathfrak{G}^\rho{}_\sigma\delta x^\sigma. \tag{2.6}$$

In this sense infinitesimal coordinate transformations are generated by an expression that vanishes.[8]

However, in accordance with Eq. (2.2) the choice of the generator (2.6) is not unique; we can add a curl. In availing ourselves of this freedom of choice we can correct one shortcoming of the expression (2.6). The right-hand side, though zero where the field equations are satisfied, is of the second differential order in the field variables. We shall add a curl chosen so that the resulting expression contains no higher than first derivatives of the metric tensor. In general relativity the "superpotentials" appearing in Eq. (1.2) satisfy[9]:

$$2\mathfrak{G}^\rho{}_\sigma\equiv \mathfrak{t}^\rho{}_\sigma-\mathfrak{U}^{[\rho\tau]}{}_{\sigma,\tau}. \tag{2.7}$$

Hence, if we add to the right-hand side of Eq. (2.6) the curl

$$\bar{C}^\rho=C^\rho+(\mathfrak{U}^{[\rho\tau]}{}_\sigma\delta x^\sigma)_{,\tau}, \tag{2.8}$$

we obtain the alternative generating density

$$\bar{C}^\rho=\mathfrak{t}^\rho{}_\sigma\delta x^\sigma+\mathfrak{U}^{[\rho\tau]}{}_\sigma\delta x^\sigma{}_{,\tau}. \tag{2.9}$$

This new expression is manifestly free of second-order derivatives of the metric tensor. Regardless of the choice of the δx^σ, the divergence $\bar{C}^\rho{}_{,\rho}$ vanishes if the field equations are satisfied. The variety of conservation laws obtained in this manner is as great as the variety of conceivable vector fields $\delta x^\sigma(x^\alpha)$ in a four-dimensional continuum.

3. STRONG CONSERVATION LAWS AND THEORY OF MOTION

The quantities (2.9) obey a weak conservation law. The corresponding strong law is

$$D^\rho{}_{,\rho}\equiv 0,\quad D^\rho\equiv(\mathfrak{U}^{[\rho\tau]}{}_\sigma\delta x^\sigma)_{,\tau}\equiv\bar{C}^\rho-2\mathfrak{G}^\rho{}_\sigma\delta x^\sigma. \tag{3.1}$$

That is to say, curl D^ρ equals $\bar{C}^\rho$ wherever the field equations are satisfied. The combination of weak and strong conservation laws permits us to formulate conditions that are satisfied by spatially isolated singular domains. The following derivation is formed exactly after earlier derivations of more specialized laws.[10]

[7] P. G. Bergmann and R. Thomson, Phys. Rev. 89, 400 (1953).
[8] Bergmann, Goldberg, Janis, and Newman, Phys. Rev. 103, 807 (1956).
[9] Reference 5, Eq. (3.7).
[10] J. N. Goldberg, Phys. Rev. 89, 263 (1953).

We integrate the divergence (3.1) over a three-dimensional domain $x^0=t=$ constant, separating the time from the spatial derivatives,

$$\int(\dot{D}^0+D^s{}_{,s})\equiv 0. \tag{3.2}$$

Because

$$D^0\equiv(\mathfrak{U}^{[0n]}{}_\sigma\delta x^\sigma)_{,n},\; n=1,2,3, \tag{3.3}$$

we can convert each of the two terms in the three-dimensional volume integral (3.2) into a two-dimensional surface integral,

$$\frac{d}{dt}\oint\mathfrak{U}^{[0n]}{}_\sigma\delta x^\sigma dS_n+\oint D^n dS_n\equiv 0. \tag{3.4}$$

Equation (3.4) is an identity. We can convert it into a dynamical law if we specify that the surface of integration, though enclosing a singular domain, lies wholly in a region in which the field equations of the vacuum are satisfied. Then, in the second term, we may replace D^n by $\bar{C}^n$, obtaining thus a weak law, that is one that exploits the validity of the field equations on the closed surface of integration,

$$\frac{d}{dt}\oint\mathfrak{U}^{[0n]}{}_\sigma\delta x^\sigma dS_n+\oint(\mathfrak{t}^n{}_\sigma\delta x^\sigma+\mathfrak{U}^{[n\tau]}{}_\sigma\delta x^\sigma{}_{,\tau})=0. \tag{3.5}$$

The first integrand is linear in the first derivatives of the metric, the second integrand quadratic. The explicit expressions for the $\mathfrak{U}^{[\mu\nu]}{}_\sigma$ are[5]

$$\mathfrak{U}^{[\mu\nu]}{}_\sigma=\mathfrak{g}_{\sigma\tau}(\mathfrak{g}^{\tau\mu}\mathfrak{g}^{\rho\nu}-\mathfrak{g}^{\tau\nu}\mathfrak{g}^{\rho\mu})_{,\rho}. \tag{3.6}$$

The first integral represents some conserved property of the singular region, whose nature depends on the choice of the four weighting functions δx^σ. The second integral represents a corresponding flux across the bounding surface. In general the value of the first integral, and its derivative with respect to time, depends on the choice of bounding surface. For a given metric one can also construct δx^σ such that the value of the integral is independent of the surface.

4. TRANSFORMATION PROPERTIES OF THE CONSERVED QUANTITIES. SPECIAL CASES

In the preceding two sections we have related generating densities whose divergences vanish weakly to infinitesimal coordinate transformations. Whereas the transformation laws of the quantities $\bar{C}^\rho$ are relatively involved, those of the vector field δx^σ are very simple. Accordingly we may state that under a (finite) coordinate transformation that converts the vector field δx^σ into the new field $\delta x^{\sigma\prime}$, the conservation law $\bar{C}^\rho{}_{,\rho}=0$ goes over into the new law $\bar{C}^{\rho\prime}{}_{,\rho}=0$, both $\bar{C}^\rho$ and $\bar{C}^{\rho\prime}$ being given by Eq. (2.9). In view of the fact that through an appropriate choice of coordinate system every vector field can be given the same standard form, e.g., (1,0,0,0), the totality of all conservation laws $\bar{C}^\rho{}_{,\rho}=0$ in one coordinate system is equivalent to one of them, stated in terms of all conceivable coordinate systems.

The expressions (2.9) assume exactly the canonical form if we set

$$\delta x^\sigma=k^\sigma, \tag{4.1}$$

a set of four constants. We obtain the Landau-Lifshitz expression if we set, instead,

$$\delta x^\sigma=\mathfrak{g}^{\sigma\alpha}k_\alpha, \tag{4.2}$$

where the k_α are again constants. The so-called angular momentum expressions are obtained if we choose

$$\delta x_\sigma=(\mathfrak{g}^{\sigma\alpha}x^\beta-\mathfrak{g}^{\sigma\beta}x^\alpha)J_{\alpha\beta} \tag{4.3}$$

or some similar expression, where the $J_{\alpha\beta}$ are again constants.

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