PHYSICAL REVIEW VOLUME 112, NUMBER 1 OCTOBER 1, 1958

Relativistic Field Theory of Unstable Particles

P. T. MATTHEWS AND ABDUS SALAM
Imperial College, London, England
(Received June 9, 1958)

An attempt is made to clarify the fundamental physical basis of the intrinsic characteristics of unstable elementary particles and to replace the conventional, purely phenomenological, description in terms of decaying states and complex energy levels, by definitions which are consistent with general requirements of relativistic quantum mechanics like Hermiticity, unitarity, and causality.

1. INTRODUCTION

Of the sixteen known "elementary" particles, all but four are unstable. In spite of this, no theory of unstable particles exists which is consistent with the general properties of relativistic quantum mechanics. This is particularly unfortunate since most of the recent significant progress in elementary particle theory has been made through the application of these general principles.

The conventional treatment of decays is a kind of "adiabatic" approximation. The particle is first tacitly assumed to have a definite mass, M. The rate of decay per unit time, λ, is calculated like any transition between stable particles. A beam of such particles will then be attenuated according to the relation

$$dn(\tau)/d\tau = -\lambda(M)n(\tau), \tag{1.1}$$

where $n(\tau)$ is the number of particles in the beam at proper time τ. This is interpreted as the individual particles of the beam having a wave function in their own rest system,

$$\psi(\tau) = \exp[+iM\tau - \tfrac{1}{2}\lambda|\tau|]. \tag{1.2}$$

Although this scheme is generally accepted as giving the "right answer" if the interaction producing the decay is sufficiently weak, there exists no exact theory to which this is supposed to be an approximation. Furthermore, none of the assumptions made are consistent with a complete theory of elementary particles. It is intuitively clear that an unstable particle does not have a definite mass, since it does not live long enough for an exact measurement to be made. The wave function (1.2) is certainly a useful approximation, but the complex mass and consequent lack of conservation of probability of the wave function violate general features of any complete theory, and are typical of an approximate one-particle treatment of what is essentially a many-particle problem.

Our object is to propose a theory of unstable particles, which is consistent with general requirements of a relativistic quantum field theory. The aspect of an unstable particle on which we shall concentrate is the uncertainty in the mass. In a rough way, this is already inherent in the conventional treatment. The wave function (1.2) can be written in terms of its Fourier transform

$$\psi(\tau) = \frac{\lambda}{2\pi}\int \frac{\exp[-im\tau]}{(m-M)^2+\frac{1}{4}\lambda^2}dm, \tag{1.3}$$

which may be interpreted as describing a distribution of mass values, with a spread, Δm, related to the mean life $\Delta\tau(=1/\lambda)$, by an uncertainty relation

$$\Delta m \Delta\tau \sim 1, \tag{1.4}$$

or

$$\Delta Q \Delta\tau \sim 1,$$

where ΔQ is the uncertainty in the Q value of the decay. With this as starting point, we build in subsequent sections a local field theory of unstable particles which can adequately describe their "static" characteristics, like mean mass and mean life. In the course of this development we shall give a very direct physical interpretation to the spectral function introduced by Källén[1] and Lehmann.[2,3]

Nishijima and Zimmermann[4] have recently shown that a local field theory can be set up for a stable bound state which is similar to that for stable "elementary" particles so far as the axiomatic formulation is concerned. By the same token the theory of unstable particles we propose here incorporates the theory of unstable bound states. These appear as resonances in scattering processes.

---

[1] G. Källén, Helv. Phys. Acta **25**, 417 (1957).

[2] H. Lehmann, Nuovo cimento **2**, 347 (1954). See, also, M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[3] We remark that the only previous definition of lifetime within the general framework of conventional theory is that of R. E. Peierls, *Proceedings of Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press, Inc., London, 1954), p. 296 and B. Zumino, New York University Research Report *CX*-23, 1956 (unpublished). Both these authors surmise that the mean mass and mean life of unstable particles are determined by the real and imaginary parts of complex poles of the propagator. Since this function contains branch points, the "physical" sheet is defined by a cut along the real axis. It has been shown by Bogoliubov *et al.* [Bogoliubov, Mednedev, and Polivanov, "Problems of the theory of dispersion relations," mimeographed lecture notes, Institute for Advanced Study, Princeton, 1957 (unpublished)] that the single function $\Delta(p^2)$, which defines the various propagation functions, has no singularities anywhere in the physical sheet of the p^2 plane, except on the positive real axis. Peierls has surmised that poles which give the mean mass and lifetime of an unstable particle lie on the "unphysical" sheets.

[4] K. Nishijima, Phys. Rev. **111**, 995 (1958); W. Zimmermann (to be published).

2. DEFINITIONS

Consider a particle ϕ which decays into two stable particles χ, each of mass m. We shall assume that these particles can be described within the context of relativistic quantum theory. That is to say, we assume[5]

(a) the possibility of describing them by linear field operators $\phi(x)$ and $\chi(x)$;

(b) the transformation laws of these fields under the transformations of the inhomogeneous Lorentz group;

(c) the causality condition, which asserts the commutativity of field operators with space-like separation;

(d) the asymptotic condition *for the stable particles only.*

For simplicity consider ϕ to be a Bose particle.

We assert that the properties of the unstable particle ϕ are determined by a density matrix

$$\rho=\sum|p_\mu,2\chi\rangle\frac{\rho(p^2)}{p^2}\langle p_\mu,2\chi|, \tag{2.1}$$

where $|p_\mu,2\chi\rangle$ are a (normalized) set of states of energy momentum p_μ, which asymptotically are the two χ states into which the ϕ can decay. Since the χ particles satisfy the asymptotic condition, these states can be formed in the conventional way. Ideally $\rho(p^2)$ determines the probability of a Q value corresponding to $(p^2)^{\frac{1}{2}}$ being observed in the decay. This point is discussed further in Sec. 4.

The mean mass of the ϕ particle is defined as the mean value of the (mass)^2 operator for the distribution

$$M^2=\frac{\mathrm{Tr}[P^2\rho]}{\mathrm{Tr}[\rho]}=\int\rho(p^2)d^4p\Big/\int\frac{\rho(p^2)}{p^2}d^4p$$
$$=\int p^2\rho(p^2)dp^2\Big/\int\rho(p^2)dp^2. \tag{2.2}$$

The last equality holds since $\rho(p^2)$ vanishes unless the vectors p_μ are time-like. By definition,

$$\rho(p^2)=0 \quad \text{for} \quad p^2<4m^2. \tag{2.3}$$

Thus

$$M^2>4m^2.$$

The inverse lifetime is determined by the spread of the mass values about the mean.

$$\left(\frac{M\lambda}{2\pi}\right)^2=\frac{\frac{1}{2}\,\mathrm{Tr}[\{(P^2)^2-(M^2)^2\}\rho]}{\mathrm{Tr}[\rho]}$$
$$=\frac{1}{2}\int(p^2-M^2)^2\rho(p^2)dp^2\Big/\int\rho(p^2)dp^2. \tag{2.4}$$

[5] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955). A. S. Wightman, preprint of Lille Conference talk, 1957 (unpublished).

To relate ρ to the field operator ϕ, we make the basic assumption of our theory that

$$\theta(p_0)\rho(p^2)d^4p=(2\pi)^3\sum_{p,p+dp}|\langle 0|\phi(0)|p_\mu,2\chi\rangle|^2, \tag{2.5}$$

where

$$\theta(p_0)=1,\quad p_0>0$$
$$\qquad\;\; 0,\quad p_0<0.$$

Note that the summation is only over 2χ states, but includes all the variables required to define these states, in addition to the energy and momentum.

For an unstable fermion of spin one-half, the final states must carry an additional label, which denotes the spin direction and particle-antiparticle character of the final state. We denote this variable explicitly by α, β. Then the unstable particle is described by a density matrix

$$\rho=\sum|p_\mu,\alpha\rangle\frac{\rho_{\alpha\beta}(p_\mu)}{p^2}\langle p_\mu,\beta|. \tag{2.6}$$

We can write the spinor matrix

$$\rho_{\alpha\beta}(p_\mu)=[\rho(\not p)]_{\alpha\beta}, \tag{2.7}$$

where

$$\not p=i\gamma\cdot p. \tag{2.8}$$

The mass operator[6] is now $i\gamma\cdot P$ and the mean mass is thus determined by

$$M=\frac{\mathrm{Tr}[i\gamma\cdot P\rho]}{\mathrm{Tr}[\rho]}$$
$$=\int\mathrm{tr}[\not p\rho(\not p)]dp^2\Big/\int\mathrm{tr}[\rho(\not p)]dp^2, \tag{2.9}$$

where tr[ ] denotes the spinor trace. The inverse lifetime is again determined by the spread in mass values,

$$\left(\frac{\lambda}{2\pi}\right)^2=\frac{\frac{1}{2}\,\mathrm{Tr}[(P^2-M^2)\rho]}{\mathrm{Tr}[\rho]}$$
$$=\frac{1}{2}\int\mathrm{tr}[(\not p-M)^2\rho]dp^2\Big/\int\mathrm{tr}[\rho]dp^2. \tag{2.10}$$

If the particle is described by a local, causal, relativistic field $\psi(x)$, we postulate that

$$\theta(p_0)\rho_{\alpha\beta}(\not p)d^4p$$
$$=-(2\pi)^3\sum_{p,p+dp}\frac{\langle 0|\psi_\alpha(0)|p_\mu\chi_1\chi_2\rangle\langle p_\mu\chi_1\chi_2|\psi_\beta^*(0)|0\rangle}{p_0}.$$

That this expression has the required transformation properties can be seen by comparing it with the expression for the number density.[7]

[6] Abdus Salam, Nuclear Phys. **5**, 687 (1958).

[7] See, for example, A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955). The Bose density may be cast in a similar form,

$$\rho(p^2)=(2\pi)^3\sum_{p,p+dp}\langle 0|\phi(0)|p,2\chi\rangle\langle p,2\chi|i\dot\phi(0)|0\rangle/p_0.$$

3. RELATION TO CONVENTIONAL THEORY

To make the discussion explicit, we shall consider in this section a model particle ϕ whose only interaction is that which causes its decay into two χ particles. The χ particles have no other function except to produce the ϕ particle when they come together. The ϕ, once created, can only decay.

For simplicity let us suppose that a selection rule exists so that the field ϕ only has nonvanishing matrix elements between the vacuum and 2χ states. Then the ρ defined by (2.5) can be identified with the ρ of Lehmann.[2] If we try to set the theory up in Lagrangian form, we may expect from such a Lagrangian an equation of motion of the form

$$\Box\phi=\alpha\phi+F[\chi], \tag{3.1}$$

where F is some functional of the field χ referring to the stable χ particle of mass m. Then a theorem of Lehmann[2] states that for such an equation, irrespective of the actual form of F,

$$\alpha=\int p^2\rho(p^2)dp^2\bigg/\int\rho(p^2)dp^2. \tag{3.2}$$

This was just our definition of M^2, and so we have

$$(\Box-M^2)\phi=F[\chi]. \tag{3.3}$$

For $F[\chi]$ we make the usual assumption

$$F=g\chi^2. \tag{3.4}$$

In the limit $g\rightarrow 0$, we have

$$\rho=\delta(p^2-M^2). \tag{3.5}$$

Lehmann's theorem[2] states that the effect of the decay interaction for the model particle is to spread this δ function into a finite mass distribution about the same mean value. To obtain an approximate expression for the small g, we make use of the exact expression

$$\rho(p^2)=(1/2\pi)\Delta^{1\prime}(p^2)=(1/\pi)\,\mathrm{Im}\Delta_c{}'(p^2), \tag{3.6}$$

where

$$\Delta_c(p^2)=(p^2-M^2+i\epsilon)^{-1}. \tag{3.7}$$

According to Dyson,[8] $\rho(p^2)$ has the form of a "Cauchy" distribution with

$$\rho(p^2)=\frac{I(p^2)}{[p^2-M^2-R(p^2)]^2+I^2(p^2)}, \tag{3.8}$$

where

$$I(p^2)=g^2\theta(p^2-4m^2)(1+4m^2/p^2)^{\frac{1}{2}}/\pi, \tag{3.9}$$

if one approximates to order g^2. This expression has a mean value approximately at M^2, provided $M^2\gg 4m^2$ and g^2 is small. The second moment of a Cauchy distribution is mathematically infinite, but it is clear that with any reasonable definition the spread of the distribution is determined by $I(p^2)$. It has been suggested by Källén[9] that the correct form for $\rho(p^2)$ may be a series of the nature of an exponential, in which case (3.8) would be replaced by

$$\rho(p^2)\sim\frac{1}{I(p^2)}\exp\left[-\left(\frac{p^2-M^2}{I(p^2)}\right)^2\right]. \tag{3.10}$$

With this form the second moment would exist[10] and give a lifetime, according to (2.4), in agreement with the conventional calculation.

The fact that perturbation theory takes the zeroth approximation to ρ to be a δ function shows that the limit $g\rightarrow 0$ is certainly not a straightforward mathematical procedure, and it is clear that if the correct form for (3.8) is in the nature of (3.10) an expansion in powers of g^2 is not valid under any circumstances.

4. PHYSICAL INTERPRETATION OF ρ

To give a complete theory of unstable particles it would be necessary to consider not only the "static" properties such as mean mass and mean life, but also the dynamics of the particle interactions—its production and scattering. In principle, these two aspects cannot be separated since conceptually the end products of any complete interaction must be regarded as stable. (One can only discuss the production of unstable particles in the approximation in which they are regarded as stable.) To treat unstable particles on an exact basis, it is necessary to consider both the production and the decay process together. The density ρ through which we have defined the mean mass and mean life is related to the probability distribution of the Q values of the decay products in a production and decay chain, such as $\pi^-+p\rightarrow\kappa^0+\Sigma^0\rightarrow\kappa^0+\Lambda^0+\gamma$. (For this discussion we treat K as stable.) The experimenter observes the Q values of the $\Lambda+\gamma$. These show a sharp peak (due to "real" Σ's) and a diffuse spread (due to "virtual" Σ's, or direct Λ production with radiative γ's from the original charged particles). We shall assume that these parts of the Q spectrum can be clearly separated. Our ρ is closely related to the shape of the sharp peak (which has a width $\sim$10 kev due to the uncertainty principle, corresponding to the mean life $\sim 10^{-19}$ sec). In order to give a precise definition of ρ in terms of this shape, it is necessary to extract the dependence of the shape on the particular production process, which has been selected. This, together with the dynamic properties

[8] We have used the relations $\Delta_c{}'(p^2)=[p^2-M^2-\Sigma^*(p^2)]^{-1}$, and define $R(p^2)+iI(p^2)\equiv\Sigma^*(p^2)/\pi$.

[9] G. Källén, *Proceedings of the* CERN *Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956), Vol. 2, p. 187.

[10] If one introduces a variable s, canonical to the mass variable p^2, and defines $G(s)=\int\exp(ip^2s)\rho(p^2)ds$, then $\mathrm{Tr}[\rho]=G(0)$, $\mathrm{Tr}[P^2\rho]=-iG'(0)$, etc. Since $\rho(p^2)=0$ for $p^2<4m^2$, $G(s)$ is the boundary value of an analytic function in the upper-half $s(=s_1+is_2)$ plane. The existence of higher moments of $\rho(p^2)$ is connected with the analyticity of $G(s)$ at $s=0$. Note that the variable s is not the same as the explicit proper time variable $\tau=x^2$ which appears in the Fourier transform, $\Delta(\tau)=(2\pi)^{-4}\times\int\Delta(k)e^{ik\cdot x}d^4k$.

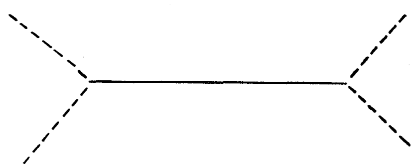

FIG. 1. The Feynman diagram of pure resonance scattering of two χ particles through the ϕ-particle resonance.

of the unstable-particle problem, will be considered in a separate paper.

If the lifetime of the particle (or bound state) is even shorter than the Σ^0 lifetime, the distinction between the sharp and diffuse parts of the Q spectrum become obscured. In this case the particle shows itself most clearly as a resonance. In the terminology of the previous section, we may consider the scattering of the decay products—the two χ particles. The total cross section is related by the optical theorem to the imaginary part of the forward scattering amplitude. If this may be assumed to be purely resonance scattering with the ϕ particle as intermediate state, the scattering amplitude is proportional to the ϕ-particle propagator (Fig. 1). The imaginary part is just ρ, apart from numerical factors. Thus within the severe limitations of the approximation, the total cross section provides a measure of ρ. Even in the case of a relatively long-lived particle the consideration of the resonance in the elastic scattering of the decay products may provide the clearest theoretical means to isolate ρ. Thus, conceptually, ρ for neutron decay would be given most clearly by the spread of the so-called "bound-state" term in the conventional $\pi+p$ scattering dispersion relations.

5. DISCUSSION

A more detailed analysis of the structure of unstable particles would require information on the higher moments of the distribution. As long as the particle is described by its first two moments only—the mean mass and mean life—the conventional treatment in terms of a complex mass provides an adequate phenomenological description.

To bring the discussion of the model particle nearer to reality, let us suppose that it has two interactions: a weak decay interaction into two particles of mass m_w, specified by a coupling constant g_w; and a strong interaction with particles of mass m_s, $(M^2<4m_s^2)$, and coupling g_s. The complete spectral function is $\rho(g_s,g_w)$. The forms of ρ in the various approximations of letting one or other, or both, of these interactions become zero are shown in Fig. 2. M_0 is the so-called "bare" mass and the forms of ρ follow directly from the discussion of Lehmann[2] and its extension given below.[11]

[11] It may be necessary to modify the definitions of mass and life time to conform more closely with experimental practice. One simple procedure is to limit the integrations in Eqs. (2.2) and (2.4) and set the upper limit $=4m_s^2$. A discussion for the realistic cases will be published elsewhere.

It will be remarked that from the point of view proposed here, stable and unstable particles are treated on a precisely equal footing. The basic quantities are the fields and their associated spectral functions. The notion of a particle is a qualitative one which is related to the peaks in the spectral function. If the width of the peak is of the order of the π-meson mass, the mean life is approximately 10^{-23} sec, but the mass spread of the "particle" may then be observed through the resonance in the scattering of the particles into which it decays. (The "33" resonance of the π-nucleon system is a

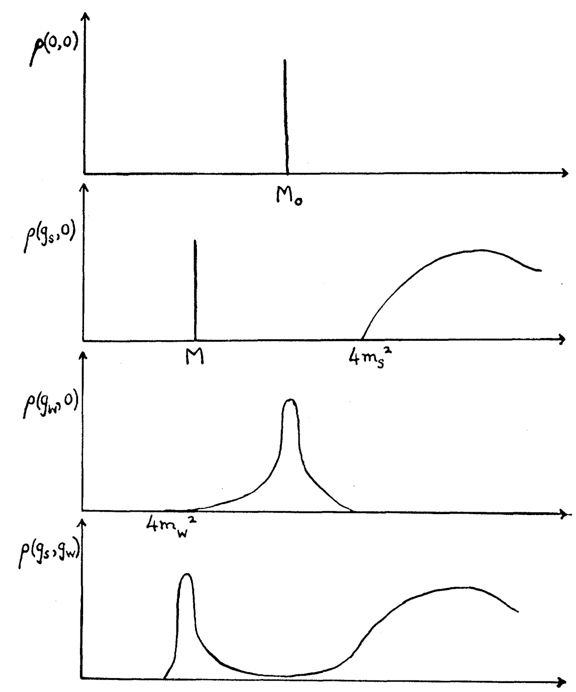

FIG. 2. A schematic drawing of the forms of the spectral function of a strongly produced (g_s) and weakly decaying (g_w) particle in various limits in which the different couplings are taken to be zero. The heavy lines denote δ functions. Note that any sharp peak in ρ may usefully be interpreted as an unstable particle, and in this way a single field may represent more than one unstable particle.

"particle" of this type.) For mean lives greater than 10^{-15} sec (corresponding to mass uncertainties of less than 1 ev) the mean life is experimentally the more accessible quantity, and the fundamental uncertainty in the mass is masked by experimental errors. The stable particles are those peaks in the spectral functions which can be approximated by δ functions. Postulating the appearance of a δ function in ρ, corresponding to each stable particle, is equivalent to the existence of asymptotic fields in the axiomatic approach.

From the present point of view the existence of several particles may then be subsumed in a single field, and the problem of elementary particle physics is shifted from the question of the number of elementary particles to the number of elementary fields. It would appear that the future task of fundamental theory would be to look for criteria which specify elementary fields.

ACKNOWLEDGMENTS

The authors would like to thank many people for stimulating discussions and some healthy scepticism, particularly Professor J. R. Oppenheimer, Professor R. E. Peierls, and members of the high-energy physics groups in Edinburgh and Imperial College, London.

PHYSICAL REVIEW VOLUME 112, NUMBER 1 OCTOBER 1, 1958

Conservation Laws in General Relativity as the Generators of Coordinate Transformations

PETER G. BERGMANN
Department of Physics, Syracuse University, Syracuse, New York
(Received June 9, 1958)

The components of the so-called canonical energy-stress pseudotensor in general relativity may be thought of as the generators of infinitesimal coordinate transformations corresponding to a rigid parallel displacement of the coordinate origin, just as in Lorentz-covariant theories. In this paper it is shown that the canonical expressions, as well as the expressions proposed by Landau and Lifshitz and the expressions for the angular momentum density, are all special cases of an infinity of conservation laws whose pseudovectors generate arbitrary curvilinear coordinate transformations. This approach enables us to construct the transform of every one of these conservation laws under an arbitrary (finite) coordinate transformation. Finally it is shown that every one of these conservation laws may be used to obtain a surface integral relationship that describes the motion of singularities in a general-relativistic theory. It is concluded that there is an infinite number of parameters that describes a singularity of the field, a fact that had previously been in doubt.

1. INTRODUCTION

THROUGHOUT mechanics and field theories, it is well known that the fundamental conservation laws are related to the universal invariance properties of physical laws, e.g., the conservation of linear momentum to the invariance with respect to displacement of the coordinate origin; the conservation of energy depends likewise on invariance with respect to the choice of the origin of the time scale (the instant $t=0$), and the conservation of angular momentum on the invariance with respect to orthogonal transformations in three-space. The structure of conservation laws in general relativity and in general-relativistic theories differs from that in nonrelativistic and in Lorentz-covariant theories because of the much wider scope of coordinate transformations in general relativity. It was discovered a long time ago that the so-called conservation laws of energy and linear momentum in general relativity,

$$\mathfrak{t}^{\rho}{}_{\mu,\rho}=0, \qquad \mathfrak{t}^{\rho}{}_{\mu}\equiv g_{\alpha\beta,\mu}\frac{\partial\mathfrak{L}}{\partial g_{\alpha\beta,\rho}}-\delta^{\rho}{}_{\mu}\mathfrak{L}, \tag{1.1}$$

which hold only insofar as the field equations of the theory are satisfied, are related to a set of identities, the so-called "strong" conservation laws,[1,2]

$$\mathfrak{T}^{\rho}{}_{\mu,\rho}\equiv 0, \qquad \mathfrak{T}^{\rho}{}_{\mu}=\mathfrak{U}^{[\rho\sigma]}{}_{\mu,\sigma}. \tag{1.2}$$

The quantities $\mathfrak{T}^{\rho}{}_{\mu}$ equal $\mathfrak{t}^{\rho}{}_{\mu}$ when the field equations are satisfied. The "superpotentials" $\mathfrak{U}^{[\rho\sigma]}{}_{\mu}$, which were first discovered by von Freud,[3] can also be constructed in general-relativistic theories that differ in detail from Einstein's 1916 theory.[4,5] The existence of the strong laws leads to the (partial) determination of the equations of motions of singularities by the surrounding field.

The canonical energy-stress components do not form a tensor density, nor even a geometric object. Being formed of the components of the metric tensor and its first derivatives, all components can be made to vanish simultaneously at any one world point, though not in a whole region. Moreover, the integrals over the energy-stress expressions that in Lorentz-covariant theories would be interpreted as the whole energy and as the whole linear momentum, respectively, transform as the components of a four-vector only with respect to a very restricted group of coordinate transformations. This elusive character of the energy-stress tensor has rendered the physical interpretation of the corresponding constants of the motion dubious.

This somewhat unsatisfactory situation has been complicated further by the discovery of another expression in general relativity which also obeys a set of equations of continuity, by Landau and Lifshitz.[6] Gold-

[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).
[2] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).
[3] P. von Freud, Ann. Math. **40**, 417 (1939).
[4] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
[5] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).
[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Publishing Company, Inc., Reading, 1951), p. 316 of the English translation.