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Tests for the Spin in the Decay of Particles of Arbitrary Spin*

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The polarization and angular distribution of a spin-$\frac{1}{2}$ particle produced in the decay of a fermion of arbitrary spin into two particles of spins 0 and $\frac{1}{2}$, and in the decay of a boson of arbitrary spin into two different spin-$\frac{1}{2}$ particles, one of which is a two-component neutrino, are calculated in terms of the general decay amplitudes. The results are applied to the discussion of tests for the spin of the decaying particle. Restrictions on the possible spins are found in terms of inequalities limiting the values of certain test functions applied to the experimental angular distributions and polarizations. The results for the boson decay are used in a discussion of the spin of the $K_{\mu 2}$ and of the possibility of lepton conservation in the $K_{\mu 2}$ and $\pi-\mu$ decays. A limit of 10% nonconservation of leptons is found if the K spin is 0. Application of the spin tests to the decay of the Λ^0 hyperon provides substantial evidence that the Λ^0 spin is $\frac{1}{2}$.

I. INTRODUCTION

IN a recent paper,[1] Lee and Yang have proposed several tests which may be used to obtain bounds on the spin of a fermion which decays with a violation of parity conservation into two particles, of spins $\frac{1}{2}$ and 0. The utility of their results, formulated in terms of inequalities limiting the values of certain test functions applied to the decay angular distribution, was demonstrated in a discussion of the spin of the Λ^0 hyperon. In the present paper, we investigate the polarization as well as the angular distribution of an outgoing spin-$\frac{1}{2}$ particle produced by (a) the decay of an integral-spin particle into two different spin-$\frac{1}{2}$ particles; (b) the decay of a half-integral-spin particle into two particles of spins $\frac{1}{2}$ and 0. The results are used in the formulation of a number of tests or consistency relations which must be satisfied for an observed angular distribution and polarization distribution to be consistent with spin J. More tests than those discussed by Lee and Yang[1] result when the polarization as well as the angular distribution of the emergent spin-$\frac{1}{2}$ particle is discussed. The tests are useful only in connection with the decay of polarized particles; observation of the polarization of a spin-$\frac{1}{2}$ particle resulting from the decay of unpolarized particles of arbitrary spin provides by itself no information on that spin. No nonrelativistic approximations are made for either of the decay products in the calculations.

Case (a) above is discussed in detail for the situation in which one of the decay products is a two-component neutrino. The results for the decay of unpolarized bosons are rather simple, and are applied to the discussion of lepton conservation in the $\pi-\mu$ and $K_{\mu 2}$ decays and the spin of the K meson. Spin 0 is suggested but cannot be proved. A limit of 10% violation of lepton conservation is found on the assumption that the π and K spins are both 0.

The tests relevant to case (b) are applied to the decay of the Λ^0 hyperon in an extension of the prior work of Lee and Yang.[1] Substantial evidence for spin $\frac{1}{2}$ is found.

II. DECAY OF A BOSON OF ARBITRARY SPIN

1. Results for Unpolarized Bosons

In this section we will consider the polarization of a spin-$\frac{1}{2}$ particle resulting from the decay at rest of an unpolarized boson of arbitrary spin. The decay products will be assumed to be two different spin-$\frac{1}{2}$ particles. The polarization of the first particle will be considered for two cases, in the first of which the particles will be taken as Dirac (four-component) particles. In the other case, the second or unobserved particle will be taken as a neutrino satisfying the tenets of the two-component theory. The decay interaction will be assumed to be invariant under rotations. Thus the total angular momentum J and the z component of the angular momentum M will be conserved in the decay, and the decay amplitudes will be independent of M. The direction of motion $\hat{p}_1$ of the first particle in the center-of-mass system will be denoted by angles θ and ϕ relative to an arbitrary coordinate system. Then the amplitude of the outgoing wave arising from the decay of a boson in an eigenstate characterized by J and M is given in the absence of parity conservation by

$$f^{J,M}(\theta,\phi,s_1,s_2)=\sum_{LS} a_{LS}\mathcal{Y}_{JMLS}(\theta,\phi,s_1,s_2). \quad (1.1)$$

Here L is the orbital angular momentum of the decay products in the center-of-mass system and S is the total spin angular momentum of the two particles, $S=0$, 1. The $\mathcal{Y}_{JMLS}$ are the usual angular momentum eigenfunctions, defined in terms of the spherical harmonics $Y_L^{M_L}(\theta,\phi)$, the single-particle spin functions $\chi_{\frac{1}{2}}^{m_1}(s_1)$ and $\chi_{\frac{1}{2}}^{m_2}(s_2)$ for spin $\frac{1}{2}$, and the symmetrized vector

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[1] T. D. Lee and C. N. Yang, Phys. Rev. 109, 1755 (1958).

coupling coefficients[2] (Wigner $3j$ symbols) by

$$\mathcal{Y}_{JMLS}(\theta,\phi,s_1,s_2) = (-1)^{S-L-M}[2j+1]^{\frac{1}{2}} \sum_{M_L M_S} \begin{pmatrix} L & S & J \\ M_L & M_S & -M \end{pmatrix} \times Y_L^{M_L}(\theta,\phi)\chi_S^{M_S}(s_1,s_2), \quad (1.2)$$

where

$$\chi_S^{M_S}(s_1,s_2) = (-1)^{-M_S}[2S+1]^{\frac{1}{2}} \times \sum_{m_1 m_2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & S \\ m_1 & m_2 & -M_S \end{pmatrix} \chi_{\frac{1}{2}}^{m_1}(s_1)\chi_{\frac{1}{2}}^{m_2}(s_2). \quad (1.3)$$

These relations fix the definition of the decay amplitudes $a_{L,S}$, which may differ by factors of $(-1)^{\mu_{LS}}$ from the present definition if the $\mathcal{Y}_{JMLS}$ are altered in phase.

The angular distribution of the directions of emission of particle (1) with respect to the chosen axes is given by

$$I(\theta,\phi)=(f^J,f^J), \quad (2.1)$$

where the expectation value (f^J,f^J) is to be taken with respect to the spin coordinates of the particles. Similarly, the product of the polarization of the particle and its angular distribution is defined by

$$\mathbf{P}(\theta,\phi)I(\theta,\phi)=(f^J,\boldsymbol{\sigma}_{\text{Pauli}}^{(1)}f^J). \quad (2.2)$$

Although the definitions of Eqs. (1) and (2) are nonrelativistic in form, the results of the calculations are relativistically correct for a decay at rest provided the polarization $\mathbf{P}(\theta,\phi)$ is interpreted as the expectation value in its rest system of twice the spin angular momentum of the product particle (1), $\mathbf{P}=\langle\boldsymbol{\sigma}^{(1)}\rangle_{\text{rest}}$. [For a decay in flight, $\mathbf{P}$ is connected by a rotation with the expectation value of $\boldsymbol{\sigma}^{(1)}$ in the rest system of the particle as determined from the laboratory.[3]]

For the decay of an unpolarized boson of spin J into two different particles of spin $\frac{1}{2}$, neither a two-component neutrino, it is found by straightforward calculation that

$$I(\theta,\phi)=\frac{1}{4\pi}\sum_{L,S}|a_{L,S}|^2, \quad (3.1)$$

and

$$\mathbf{P}(\theta,\phi)I(\theta,\phi) = \frac{1}{4\pi}\hat{p}[2J+1]^{\frac{1}{2}}\{2\,\mathrm{Re}\,a_{J,0}^*[J^{\frac{1}{2}}a_{J-1,1}-(J+1)^{\frac{1}{2}}a_{J+1,1}] - 2\,\mathrm{Re}\,a_{J,1}^*[(J+1)^{\frac{1}{2}}a_{J-1,1}+J^{\frac{1}{2}}a_{J+1,1}]\}. \quad (3.2)$$

The polarization of the emergent particle is seen to be along its direction of motion, $\hat{p}=\mathbf{p}_1/p_1$, and to vanish if parity is conserved in the decay. This longitudinal polarization depends on the spin J of the decaying particle, but depends also on the decay amplitudes a_{LS} in a combination different from that in which they enter the angular distribution. Thus in the absence of information from some other source about these amplitudes, a measurement of the longitudinal polarization of a spin-$\frac{1}{2}$ particle resulting from the decay of unpolarized bosons does not provide a test of the boson spin.

More stringent conditions are obtained if one of the spin-$\frac{1}{2}$ decay products is a neutrino satisfying the tenets of the two-component theory.[4] This requires that the spin of the neutrino be either parallel or antiparallel to its direction of motion: $\langle\boldsymbol{\sigma}_\nu\cdot\mathbf{p}_\nu/p_\nu\rangle=\pm1$.

The eigenvalue $+1$ for $\boldsymbol{\sigma}_\nu\cdot\hat{p}_\nu$ corresponds to right-hand circular polarization of the neutrino, $\nu=\nu_R$, while the eigenvalue -1 corresponds to left-hand circular polarization, and $\nu=\nu_L$. If, then, the amplitude of the emergent wave, Eq. (1), is re-expressed in terms of angular momentum eigenfunctions quantized along the direction $\hat{p}_1$, appropriate subsidiary conditions must be applied to the decay amplitudes a_{LS} in order to eliminate the unwanted spin function $\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_2')$ or $\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_2')$ accordingly as the neutrino in the boson decay is ν_R or ν_L. The details of this calculation as well as the angular distribution and polarizations for polarized bosons are contained in Sec. 3. For unpolarized bosons, the results are quite simple,

$\nu=\nu_R$:

$$I_R(\theta,\phi)=\frac{2}{4\pi}\{|a_{J,0}^R|^2+|a_{J,1}^R|^2\}, \quad (4.1)$$

$$\mathbf{P}I_R(\theta,\phi)=\frac{2}{4\pi}\{|a_{J,0}^R|^2-|a_{J,1}^R|^2\}\hat{p}_1; \quad (4.2)$$

$\nu=\nu_L$:

$$I_L(\theta,\phi)=\frac{2}{4\pi}\{|a_{J,0}^L|^2+|a_{J,1}^L|^2\}, \quad (5.1)$$

$$\mathbf{P}I_L(\theta,\phi)=-\frac{2}{4\pi}\{|a_{J,0}^L|^2-|a_{J,1}^L|^2\}\hat{p}_1. \quad (5.2)$$

The subsidiary conditions necessary in a two-component theory of the neutrino relate the amplitudes $a_{J+1,1}$ and $a_{J-1,1}$ to $a_{J,0}$ and $a_{J,1}$, as is obvious from a comparison of Eqs. (4) and (5) with Eqs. (3). For the special case $J=0$, the terms in Eqs. (4) and (5) involving $a_{J,1}$ vanish, and the decay is completely described for a two-component theory by a single amplitude.

2. Application to the $\pi-\mu-e$ and $K_{\mu2}$ Decays; Lepton Conservation and the Spin of the K Meson

The results of the preceding section may be applied at once to the $\pi-\mu-e$ and $K_{\mu2}$ decays. We shall con-

[2] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 3.

[3] H. P. Stapp, Phys. Rev. **103**, 425 (1956); also, R. Spitzer and H. P. Stapp, Phys. Rev. **109**, 540 (1958).

[4] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957); L. Landau, Nuclear Phys. **3**, 127 (1957); A. Salam, Nuovo cimento **5**, 299 (1957).

sider first the case of $\pi^+ \to \mu^+ + \nu$. The polarization of the μ^+ meson has been measured using the forward-backward asymmetry in the subsequent decay $\mu^+ \to e^+ + \nu_L + \nu_R$. For a positron angular distribution function of the form $(1 + a\hat{p}_\mu \cdot \hat{p}_e)$, Coffin *et al.*[5] obtain for a the value $a = -0.305 \pm 0.033$. The theoretical value of a[4,6] derived from the phenomenological theory of μ decay with two-component, left-handed neutrinos, and corrected for the minimum acceptance energy for positrons in the experiment, is given by $a = -0.43\xi P_\mu$ for μ^+ decay. Here P_μ is the longitudinal polarization of the μ meson, $P_\mu = \hat{p}_\mu \cdot \mathbf{P}_\mu$, and ξ is the parameter

$$\xi = 2\,\mathrm{Re}C_V{}^* C_A [|C_V|^2 + |C_A|^2]^{-1}.$$

It follows at once from the measured value of a that $\xi P_\mu = +0.72 \pm 0.10$. Utilizing Eqs. (4.2) and (5.2) for the case $J=0$, one finds that for a two-component neutrino theory, independently of the details of the $\pi^+ \to \mu^+ + \nu$ decay,[4]

$$P_\mu = -1 \quad \text{for} \quad \nu = \nu_L,$$
$$P_\mu = +1 \quad \text{for} \quad \nu = \nu_R.$$

Conservation of leptons requires that the neutrino in the π^+ decay be left-handed,[7] hence, that $P_\mu = -1$ and $\xi = -0.72 \pm 0.10$.

On the other hand, a limit on P_μ is obtained by giving $|\xi|$ its maximum value, unity. For a $V-A$ interaction, this is attained for $C_A = -C_V$, yielding $P_\mu = -0.72 \pm 0.10$. Thus if we assume the validity of the phenomenological theory of $\mu - e$ decay, and take for P_μ the experimental limit $-1 \leq P_\mu < -0.6$, it follows for a two-channel interaction leading to neutrinos of both helicities that $|a_{0,0}{}^R|^2 / |a_{0,0}{}^L|^2 < 0.25$. This figure represents a limit of $\sim 20\%$ on the possible violation of lepton conservation in the $\pi - \mu$ decay. The phenomenological theory cannot, however, be considered as completely adequate, the predicted value of the Michel parameter for the $\mu - e$ decay being $\rho = \frac{3}{4}$[4] while the observed value is $\rho = 0.68$.[8] Thus it is perhaps more reasonable to assume the validity of lepton conservation in the π^+ decay and a two-component neutrino theory, and to use the measured value of a to obtain ξ. This procedure can be checked by a direct measurement of the longitudinal polarization of the μ meson, which must have the value -1 for $\pi^+ \to \mu^+ + \nu_L$.

Similar arguments can be advanced at once in the case of the $K_{\mu 2}$ decay, $K^+ \to \mu^+ + \nu$. Lepton conservation again requires that the neutrino be left-handed,[7] $\nu = \nu_L$. For the parameter a describing the forward-backward asymmetry in the subsequent $\mu - e$ decay, Coombes *et al.*[9] obtain $a = -0.31 \pm 0.04$. Using for the parameter ξ characterizing the $\mu - e$ decay the result $\xi = -0.72 \pm 0.10$ obtained from the $\pi - \mu - e$ experiments on the assumption of lepton conservation in the $\pi - \mu$ decay, and correcting for the cutoff in the positron spectrum,[9] one obtains for the longitudinal polarization of the μ meson in the $K_{\mu 2}$ decay the value $\hat{p}_\mu \cdot \mathbf{P}_\mu = -1.02 \pm 0.22$. Hence it is likely that the μ meson has almost perfect longitudinal polarization, but a polarization with $\hat{p}_\mu \cdot \mathbf{P}_\mu$ as small in magnitude as 0.8 cannot be excluded. We shall assume a two-component theory for the neutrino, with the consequence that the polarization is rigorously longitudinal in the decay of unpolarized mesons, and use for $P_\mu = \hat{p}_\mu \cdot \mathbf{P}_\mu$ the experimental limit of -0.8. Lepton conservation in the $K_{\mu 2}$ decay $(K^+ \to \mu^+ + \nu_L)$ then requires that in Eq. (5.2), $|a_{J,1}{}^L|^2 / |a_{J,0}{}^L|^2 \lesssim 0.1$. The value $P_\mu = -1$ can arise only for $a_{J,1} = 0$, which would seem unlikely for a nonzero K meson spin. The present experimental results are consistent with (and suggestive of) $J=0$ and rigorous conservation of leptons. Higher spins cannot be ruled out by this test alone, but the decay must then involve only a small admixture of the $a_{J,1}{}^L$ amplitude relative to the $a_{J,0}{}^L$. On the other hand, with complete violation of lepton conservation, $K^+ \to \mu^+ + \nu_R$, the negative sign of P_μ requires the spin of the K to be at least one. For $J > 0$, measurements on the angular distribution and polarization of μ's from polarized K's can determine whether lepton-conserving or -nonconserving decay is dominant, and can be used to place rigorous limits on the spin; this will be discussed in the Sec. 4. Finally, if it is assumed, as seems likely, that the spin of the K is zero, the experimental limit on P_μ requires a branching ratio of less than 0.1 for decays into the ν_R (lepton-conservation violating) as opposed to the ν_L (lepton-conserving) channel.

3. Angular Distributions and Polarizations with Two-Component Neutrinos

(a) One Neutrino

The two-component theory of the neutrino[4] requires that the spin of ν_R be quantized parallel, and that of ν_L, antiparallel, to its direction of motion. Taking particle (2) as the neutrino, we shall therefore re-express Eq. (1) relative to a new coordinate system with the axis of quantization along the direction of $\hat{p}_1$, and apply appropriate subsidiary conditions to the amplitudes a_{LS} to

[5] Coffin, Garwin, Lederman, Weinrich, and Berley, Bull. Am Phys. Soc. Ser. II, 2, 204 (1957).

[6] Larsen, Lubkin, and Tausner, Phys. Rev. 107, 856 (1957). Note that the formulas are given for μ^- decay. For μ^+ decay, the sign of the asymmetry parameter is reversed. We use $C_A' = C_A$, $C_V' = C_V$, and $C_S' = C_S = C_T' = \cdots = 0$, corresponding to left-handed neutrinos.

[7] Goldhaber, Grodzins, and Sunyar, Phys. Rev. 109, 1015 (1958). This work establishes that the neutrino in e^- capture is left-handed, $p + e^- \to n + \nu_L$. This observation, together with the assumptions of lepton conservation and the μ-decay process $\mu^- \to e^- + \nu_L + \nu_R$, assures that if e^- is defined to be an antilepton, then ν_L and μ^- are also. Hence, e^+, μ^+, and ν_R are leptons, and lepton conservation requires that $\pi^+ \to \mu^+ + \nu_L$ and $K^+ \to \mu^+ + \nu_L$. These results are consistent with the positive helicity observed for the positrons in μ^+ decay [Culligan, Frank, Holt, Kluyver, and Massam, Nature 180, 751 (1957)] together with negative asymmetry parameter of the $\pi - \mu - e$ sequence.[4,5]

[8] L. Rosenson, Phys. Rev. 109, 958 (1958); K. Crowe, Bull. Am. Phys. Soc. Ser. II, 2, 206 (1957).

[9] Coombes, Clark, Galbraith, Lambertson, and Wenzel, Phys. Rev. 108, 1348 (1957).

eliminate in the new coordinate system the terms involving the unwanted spin function $\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_2')$ or $\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_2')$ accordingly as the neutrino is right- or left-handed. The angular momentum eigenfunctions in the old coordinate frame may be expressed in terms of the eigenfunctions quantized in the new frame and the elements $\mathfrak{D}$ of the rotation matrix[10] as

$$\mathcal{Y}_{JMLS}(\theta,\phi,s_1,s_2)=\sum_{M'}\mathfrak{D}_{M'M}{}^{(J)}(0,\theta,\phi)\mathcal{Y}_{JM'LS}(0,0,s_1',s_2'),\quad(6)$$

where we have noted that in the new system $\theta'=\phi'=0$. The functions $\mathcal{Y}_{JM'LS}(0,0,s_1',s_2')$ will be defined in terms of the spherical harmonics and spin functions quantized along the direction of $\hat{p}_1$ by the formulas of Eqs. (1.2) and (1.3). Substitution of (6) in Eq. (1) and the use of the definition of $\mathcal{Y}_{JM'LS}$ yields for the amplitude of the outgoing wave originating from the state characterized by J and M in the original coordinates,

$$f^{J,M}(\theta,\phi,s_1,s_2)=\sum_{LSM_S}(-1)^{S-L-M_S}a_{LS}\left[\frac{(2J+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}}\times\begin{pmatrix}L & S & J\\ 0 & M_S & -M_S\end{pmatrix}\mathfrak{D}_{M_SM}{}^{(J)}(0,\theta,\phi)\chi_S{}^{M_S}(s_1',s_2').\quad(7)$$

The entire angular dependence of the amplitude is now contained in the matrix elements $\mathfrak{D}$. If particle (2) is assumed to be a left-hand polarized neutrino, ν_L, the coefficient of $\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_2')$ must vanish when the spin function $\chi_S{}^{M_S}(s_1',s_2')$ is expanded in accordance with Eq. (1.3). Imposition of this condition yields a relation among the coefficients a_{LS} which must hold for all values of m_1 and M_S,

$$\sum_{L,S}(-1)^{S-L}a_{LS}\left[\frac{(2J+1)(2L+1)(2S+1)}{4\pi}\right]^{\frac{1}{2}}\times\begin{pmatrix}L & S & J\\ 0 & M_S & -M_S\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & S\\ m_1 & -\frac{1}{2} & -M_S\end{pmatrix}=0.\quad(8)$$

The nonvanishing part of $f^{J,M}$ is given correspondingly by

$$f^{J,M}(\theta,\phi,s_1,s_2)=\sum_{LSM_Sm_1}(-1)^{S-L}a_{LS}\left[\frac{(2J+1)(2L+1)(2S+1)}{4\pi}\right]^{\frac{1}{2}}\times\begin{pmatrix}L & S & J\\ 0 & M_S & -M_S\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & S\\ m_1 & \frac{1}{2} & -M_S\end{pmatrix}\times\mathfrak{D}_{M_SM}{}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{m_1}(s_1')\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_2').\quad(9)$$

When Eq. (8) is multiplied by $(-1)^J\mathfrak{D}_{M_SM}{}^{(J)}(0,\theta,\phi)\times\chi_{\frac{1}{2}}^{m_1}(s_1')\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_2')$, summed over m_1 and M_S, and added to Eq. (9), the terms involving the decay amplitudes $a_{J-1,1}$ and $a_{J+1,1}$ vanish because of the symmetry properties of the $3j$ symbols.[2] Thus we obtain finally, in that case in which particle (2) is a left-handed two-component neutrino, the rather simple result for the amplitude of the outgoing wave,

$$f_{(L)}{}^{J,M}(\theta,\phi,s_1,s_2)=-\left[\frac{2(2J+1)}{4\pi}\right]^{\frac{1}{2}}\{a_{J,0}\mathfrak{D}_{0,M}{}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_1')+a_{J,1}\mathfrak{D}_{1,M}{}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_1')\}\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_2').\quad(10.1)$$

Similarly, for particle (2) a right-hand polarized neutrino, we obtain

$$f_{(R)}{}^{J,M}(\theta,\phi,s_1,s_2)=\left[\frac{2(2J+1)}{4\pi}\right]^{\frac{1}{2}}\{a_{J,0}\mathfrak{D}_{0,M}{}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{\frac{1}{2}}(s_1')+a_{J,1}\mathfrak{D}_{-1,M}{}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_1')\}\chi_{\frac{1}{2}}^{-\frac{1}{2}}(s_2').\quad(10.2)$$

Calculations of angular distributions and polarizations are now straightforward. Relatively simple tests for the boson spin may be derived from the integrals over the azimuthal angle of $I(\theta,\phi)$ and the product $\mathbf{P}(\theta,\phi)I(\theta,\phi)$, as will be shown in the next section. We shall define the quantities $I_{\text{Av}}(\theta)$ and $\langle\hat{n}\cdot\mathbf{P}I\rangle_{\text{Av}}(\theta)$ by

$$I_{\text{Av}}(\theta)=\int_0^{2\pi}I(\theta,\phi)d\phi,\quad(11.1)$$

and

$$\langle\hat{n}\cdot\mathbf{P}I\rangle_{\text{Av}}(\theta)=\int_0^{2\pi}\hat{n}(\theta,\phi)\cdot\mathbf{P}(\theta,\phi)I(\theta,\phi)d\phi.\quad(11.2)$$

It is easily seen that no interference terms between states of different M occur in $I_{\text{Av}}(\theta)$, a substantial simplification over the most general results for $I(\theta,\phi)$. Direct calculation yields the results for particle (2), a left-handed neutrino,

$$\langle I_{(L)}\rangle_{\text{Av}}(\theta)=(2J+1)\sum_{j,M}(-1)^Mp_M(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\left\{\begin{pmatrix}J & J & j\\ 0 & 0 & 0\end{pmatrix}|a_{J,0}|^2-\begin{pmatrix}J & J & j\\ 1 & -1 & 0\end{pmatrix}|a_{J,1}|^2\right\}.\quad(12.1)$$

The constants p_M are defined by $p_M=|c_M|^2$, with the total amplitude f^J of the outgoing wave given by $f^J=\sum_M c_Mf^{J,M}$. Thus the constants p_M are subject to

[10] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 4.

the restrictions $0\leqslant p_M\leqslant 1$, and $\sum_M p_M=1$. The distribution function $I_{\text{Av}}(\theta)$ is, aside from a factor of 2π, precisely the result which would be obtained for $I(\theta,\phi)$ from an initial incoherent mixture of the eigenstates of J_z, the eigenstate with $\langle J_z\rangle=M$ appearing with the weight p_M.

Similarly, for the product of the angular distribution and the polarization one obtains the results (with $\nu=\nu_L$),

$$\langle\hat{p}\cdot\mathbf{P}I_{(L)}\rangle_{\text{Av}}(\theta)=-(2J+1)\sum_{j,M}(-1)^M p_M(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\left\{\begin{pmatrix}J & J & j\\ 0 & 0 & 0\end{pmatrix}|a_{J,0}|^2+\begin{pmatrix}J & J & j\\ 1 & -1 & 0\end{pmatrix}|a_{J,1}|^2\right\},\quad(12.2)$$

$$\langle\hat{q}\cdot\mathbf{P}I_{(L)}\rangle_{\text{Av}}(\theta)=-(2J+1)2\,\text{Im}\{a_{J,0}a_{J,1}^*\}\times\sum_{j,M}(-1)^M p_M[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ 0 & -1 & 1\end{pmatrix},\quad(12.3)$$

$$\langle\hat{r}\cdot\mathbf{P}I_{(L)}\rangle_{\text{Av}}(\theta)=-(2J+1)2\,\text{Re}\{a_{J,0}a_{J,1}^*\}\times\sum_{j,M}(-1)^M p_M[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ 0 & -1 & 1\end{pmatrix}.\quad(12.4)$$

Here $\hat{p}$, $\hat{q}$, and $\hat{r}$ are unit vectors defined by $\hat{p}=\mathbf{p}_1/p_1$, $\hat{q}=(\hat{k}\times\hat{p})/|\hat{k}\times\hat{p}|$, and $\hat{r}=(\hat{q}\times\hat{p})/|\hat{q}\times\hat{p}|$, where $\hat{k}$ is a unit vector along the axis of quantization. From these results we shall, in the next section, obtain some practical tests for the boson spin. It should perhaps be emphasized that the results, and the amplitudes of Eqs. (10), are independent of the detailed theory of the decay process, depending only on the conservation of angular momentum and on the asymptotic properties of the particles involved. However, if it is assumed that interactions between the decay products are negligible, as should certainly be true if one is a neutrino, and that the decay amplitudes a_{LS} may be treated as essentially first order in the decay interaction, then further statements can be made concerning the matters of time-reversal and charge-conjugation invariance.[11] Time-reversal invariance requires the amplitudes a_{LS} to be relatively real. Consequently, one has $\text{Im}\{a_{J,0}a_{J,1}^*\}\rightarrow 0$, and observation of a nonvanishing value of $\langle\hat{q}\cdot\mathbf{P}I(\theta)\rangle_{\text{Av}}$ is evidence for the violation of time-reversal invariance.

[11] See, for example, Lee, Oehme, and Yang, Phys. Rev. **106**, 340 (1957).

Using the CPT theorem, the operation of charge conjugation can be defined as PT to within a phase which is here irrelevant. Charge-conjugation invariance requires $a_{J,0}$ and $a_{J,1}$, and $a_{J-1,1}$ and $a_{J+1,1}$, to be relatively real, and the sets $\{a_{J,0},a_{J,1}\}$ and $\{a_{J+1,1},a_{J-1,1}\}$ to be relatively imaginary. In this case, the polarization given in Eq. (3.2) vanishes. Observation of a nonzero longitudinal polarization in the decay of a boson of arbitrary spin into two spin-$\frac{1}{2}$ particles therefore evinces a violation of both parity conservation and charge conjugation invariance in the decay. This violation follows automatically for the case discussed above, in which one of the particles is a two-component neutrino.[4]

(b) *Decay into Two Neutrinos*

A case of some theoretical interest is the decay of a spin-J boson into two two-component neutrinos. For the lepton conserving decay, with the "observed" neutrino right-handed, the coefficient $a_{J,0}$ in Eq. (10.1) must vanish. The angular distribution in the decay, integrated over the azimuthal angle, is readily seen to be

$$I_{\text{Av}}(\theta)=-(2J+1)|a_{J,1}|^2\times\sum_{j,M}(-1)^M p_M(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}.\quad(13.1)$$

The polarization, which is now defined as the expectation value of $\boldsymbol{\sigma}\cdot\mathbf{p}/p$, is given by

$$\mathbf{P}I(\theta,\phi)=I(\theta,\phi)\hat{p}\quad\text{for}\quad\nu_1=\nu_R,\quad\nu_2=\nu_L.\quad(13.2)$$

For $J=0$, $I(\theta,\phi)$ vanishes. Thus conservation of leptons and of angular momentum forbids the decay of a spin-0 particle into a ν_L, ν_R pair. This is apparent from elementary considerations, since for the pair $M_S=\pm1$ along the direction of $\hat{p}$. Thus we must have $M_L=\mp1$ in order to obtain $M=0$, but for $M_L\neq0$, the decay amplitude along $\hat{p}$ vanishes. This is true for all directions of $\hat{p}$, so $I(\theta,\phi)$ vanishes.

We may also consider the case of a complete violation of lepton conservation in the decay. The particles are in this case identical, and it is necessary to antisymmetrize the wave function. For the decay into two left-handed neutrinos, $a_{J,1}=0$ in Eq. (10.1) and we obtain

$$I_{\text{Av}}(\theta)=2(2J+1)|a_{J,0}|^2\times\sum_{j,M}(-1)^M p_M(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix},\quad(14.1)$$

while

$$\mathbf{P}I(\theta,\phi)=-I(\theta,\phi)\hat{p},\quad\nu_1=\nu_2=\nu_L,\quad(14.2)$$

where again $\mathbf{P}=\langle\boldsymbol{\sigma}_1\cdot\mathbf{p}_1/p_1\rangle$.

4. Tests for the Boson Spin

The results of Sec. 3 are quite general, but do not normally provide useful tests for the spin of the decaying boson because of the presence of the unknown weights p_M. However, a number of restrictive conditions may be formulated which may, if the decaying particle is polarized, serve to restrict the possible values of the spin. The simplest restrictions are obtained by integrating $I_{\rm Av}(\theta)$ and $\langle\hat{p}\cdot\mathbf{P}I\rangle_{\rm Av}(\theta)$ over the range of θ with the weight $\cos\theta$. For left-handed neutrinos, with an arbitrary polar axis

$$\langle I\cos\theta\rangle=\int I(\theta,\phi)\cos\theta d\Omega=\frac{2}{J(J+1)}|a_{J,1}|^2\langle J_z\rangle. \quad (15.1)$$

Similarly,

$$\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle=\frac{2}{J(J+1)}|a_{J,1}|^2\langle J_z\rangle. \quad (15.2)$$

Here $\langle J_z\rangle$ is the expectation value of component of the total angular momentum along the chosen axis. Since $|\langle J_z\rangle|\leqslant J$ and since $2|a_{J,1}|^2\leqslant\langle I\rangle=2|a_{J,0}|^2+2|a_{J,1}|^2$, we obtain at once the potentially useful restrictions

$$|\langle I\cos\theta\rangle|\leqslant\frac{1}{J+1}\langle I\rangle, \quad (16.1)$$

and

$$\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle=\langle I\cos\theta\rangle,\quad \nu=\nu_L. \quad (16.2)$$

For right-handed neutrinos, the second restriction is changed, requiring

$$\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle=-\langle I\cos\theta\rangle,\quad \nu=\nu_R. \quad (16.3)$$

The nonvanishing of any of these quantities requires a nonvanishing value of $\langle J_z\rangle$, and that the spin of the decaying boson be at least 1. Similar but less stringent restrictions may be obtained by considering the transverse components of $\mathbf{P}(\theta,\phi)I(\theta,\phi)$. The condition on $|\langle I\cos\theta\rangle|$ is a necessary condition for spin J to be consistent with a measured angular distribution and thus can in principle provide an upper bound on the boson spin. The relative sign of $\langle I\cos\theta\rangle$ and $\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle$ provides a direct determination of the helicity of the neutrino involved in the decay.

Restrictions on the spin of a more general type, similar to those discussed by Lee and Yang[1] in connection with the spin of the Λ^0, are obtained by considering the integrals over the total solid angle of the products of $I(\theta,\phi)$ and $\mathbf{P}(\theta,\phi)I(\theta,\phi)$ with certain test functions, defined as

$$S_{J,M,S}^-(\theta)=\frac{1}{2J+1}\sum_{j\,\rm odd}(2j+1)P_j(\cos\theta)(-1)^{M+S}\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ S & -S & 0\end{pmatrix}^{-1}, \quad (17.1)$$

and

$$S_{J,M,S}^+(\theta)=\frac{1}{2J+1}\sum_{j\,\rm even}(2j+1)P_j(\cos\theta)(-1)^{M+S}\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ S & -S & 0\end{pmatrix}^{-1}. \quad (17.2)$$

It then follows from Eqs. (12) that

$$\langle S_{J,M,1}^-I\rangle+\langle S_{J,M,1}^-\hat{p}\cdot\mathbf{P}I\rangle=2|a_{J,1}|^2(p_M-p_{-M}), \quad (18.1)$$

$$\langle S_{J,M,1}^+I\rangle+\langle S_{J,M,1}^+\hat{p}\cdot\mathbf{P}I\rangle=2|a_{J,1}|^2(p_M+p_{-M}), \quad (18.2)$$

$$\langle S_{J,M,0}^-I\rangle-\langle S_{J,M,0}^-\hat{p}\cdot\mathbf{P}I\rangle=0, \quad (18.3)$$

$$\langle S_{J,M,0}^+I\rangle-\langle S_{J,M,0}^+\hat{p}\cdot\mathbf{P}I\rangle=2|a_{J,0}|^2(p_M+p_{-M}). \quad (18.4)$$

Since the weights p_M are positive, $0\leqslant p_M\leqslant 1$, $\sum_M p_M=1$, we obtain from the above results the consistency relations, necessary for spin J to be consistent with an observed angular distribution and polarization,

$$|\langle S_{J,M,1}^-(I+\hat{p}\cdot\mathbf{P}I)\rangle|\leqslant\langle S_{J,M,1}^+(I+\hat{p}\cdot\mathbf{P}I)\rangle\leqslant\langle I\rangle, \quad (19.1)$$

and

$$|\langle S_{J,M,0}^+(I-\hat{p}\cdot\mathbf{P}I)\rangle|\leqslant\langle I\rangle,\quad M=-J,\cdots J. \quad (19.2)$$

We note also the simpler condition, applicable to the angular distribution alone,

$$2|\langle S_{J,M,1}^-I\rangle|\leqslant\langle I\rangle,\quad M=-J,\cdots J. \quad (19.3)$$

Consistency tests involving the transverse components of $\mathbf{P}(\theta,\phi)I(\theta,\phi)$ may be formulated through the use of the test function

$$U_{J,M}(\theta)=-\frac{1}{2J+1}\sum_{j\,\rm odd}(-1)^M[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ 0 & -1 & 1\end{pmatrix}^{-1}. \quad (20)$$

Thus

$$\langle U_{J,M}\hat{q}\cdot\mathbf{P}I\rangle=2\,\mathrm{Im}\{a_{J,0}a_{J,1}^*\}(p_M-p_{-M}), \quad (21.1)$$

and

$$\langle U_{J,M}\hat{r}\cdot\mathbf{P}I\rangle=2\,\mathrm{Re}\{a_{J,0}a_{J,1}^*\}(p_M-p_{-M}). \quad (21.2)$$

Finally, we have, therefore, among the consistency relations necessary for spin J,

$$|\langle U_{J,M}\hat{q}\cdot\mathbf{P}I\rangle|\leqslant\tfrac{1}{2}\langle I\rangle, \quad (22.1)$$

and

$$|\langle U_{J,M}\hat{r}\cdot\mathbf{P}I\rangle|\leqslant\tfrac{1}{2}\langle I\rangle,\quad M=-J,\cdots J. \quad (22.2)$$

The use of relations similar to these will be demonstrated in Sec. 3 of the second part of this paper in connection with a spin determination for the Λ^0. We shall only note here that if it is assumed that the spin of the K meson is greater than zero, then the K's produced in association with the Σ hyperons in the process $\pi+N\to\Sigma+K$ would be expected in general to be polarized. The $K_{\mu2}$ decay of the charged K mesons provides the angular distribution required above, while the subsequent $\mu-e$ decay provides a natural analyzer of the polarization of the μ meson. However, as discussed earlier, the present indications favor $J=0$ for the K meson.

III. DECAY OF A FERMION OF ARBITRARY SPIN

1. $J\to 0, \frac{1}{2}$; Angular Distributions and Polarizations

The amplitude of the asymptotic final-state wave function describing the decay of a fermion of spin J into a boson of spin 0 and a fermion of spin $\frac{1}{2}$ will be taken in the form given in Eq. (1). Re-expressing the angular momentum eigenfunctions $\mathcal{Y}_{JMLS}$ in terms of functions quantized along the direction of motion of the final fermion [particle (1)], one obtains for the amplitude arising from the eigenstate of J_z with $\langle J_z\rangle=M$ in the original coordinate system

$$f^{J,M}(\theta,\phi,s_1)=\sum_{LM_S}(-1)^{\frac{1}{2}-L-M_S}\left[\frac{(2J+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}}a_{L,\frac{1}{2}}\times\begin{pmatrix}L & \frac{1}{2} & J\\ 0 & M_S & -M_S\end{pmatrix}\mathfrak{D}_{M_SM}^{(J)}(0,\theta,\phi)\chi_{\frac{1}{2}}^{M_S}(s_1'). \quad (23)$$

Since the total spin angular momentum in the final state is always $\frac{1}{2}$, the subscript $\frac{1}{2}$ on the coefficients $a_{L,S}$ is superfluous and will henceforth be omitted. Straightforward calculation yields for the angular distribution and polarization in the decay of initially unpolarized particles [that is, for the decay of an incoherent mixture of the pure states (J,M), each appearing with the weight $1/(2J+1)$] the results

$$I(\theta,\phi)=(1/4\pi)\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}, \quad (24.1)$$

and

$$\mathbf{P}(\theta,\phi)I(\theta,\phi)=-(1/4\pi)2\,\mathrm{Re}\{a_{J+\frac{1}{2}}^*a_{J-\frac{1}{2}}\}\hat{p}. \quad (24.2)$$

The angular distribution is isotropic, as expected, while the only nonvanishing component of the polarization is that along the direction of motion of the emergent fermion. These results provide no test for the spin of the decaying fermion; to obtain such tests it is necessary to consider situations in which the initial particle is polarized. However, the existence of the longitudinal polarization of Eq. (24.2) is conclusive evidence of the violation of parity conservation in the decay.

When the initial state of the decaying system is a coherent superposition of states of different M, the detailed angular distribution of the decay products contains azimuthally dependent terms, and is in general too involved to be useful in studying the spin of the initial particle. Some simplifications can be effected by integrating the distribution over the azimuthal angle; the resulting distribution is a function solely of θ,

$$I_{\mathrm{Av}}(\theta)=\frac{2J+1}{2}\sum_{j,M}(-1)^{j-M+\frac{1}{2}}(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\alpha_j p_M, \quad (25.1)$$

where

$$\alpha_j=|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2, \quad j \text{ even} \quad (25.2)$$

$$\alpha_j=2\,\mathrm{Re}\{a_{J+\frac{1}{2}}^*a_{J-\frac{1}{2}}\}, \quad j \text{ odd}. \quad (25.3)$$

The constants p_M are again the weights with which the eigenstates of J_z characterized by $\langle J_z\rangle=M$ are present.

Similar results are easily obtained for the polarization of the emergent spin-$\frac{1}{2}$ particle. As was noted previously, the polarization $\mathbf{P}$ calculated from the nonrelativistic formula

$$\mathbf{P}(\theta,\phi)I(\theta,\phi)=(f^J,\boldsymbol{\sigma}_{\mathrm{Pauli}}f^J)$$

is correct relativistically if interpreted as the expectation value in its rest system of the twice spin angular momentum of the particle. For the integral over the azimuthal angle of the component of $\mathbf{P}I$ along the direction of motion of the product fermion, one obtains

$$\langle\hat{p}\cdot\mathbf{P}I\rangle_{\mathrm{Av}}(\theta)=-\frac{2J+1}{2}\sum_{j,M}(-1)^{j-M+\frac{1}{2}}(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\alpha_{j+1}p_M. \quad (26.1)$$

This result differs from that for $I_{\mathrm{Av}}(\theta)$ only by the change $\alpha_j\to-\alpha_{j+1}$. The transverse components of the product of the polarization and the angular distribution, integrated over the azimuthal angle, are

$$\langle\hat{q}\cdot\mathbf{P}I\rangle_{\mathrm{Av}}(\theta)=\frac{2J+1}{2}2\,\mathrm{Im}\{a_{J+\frac{1}{2}}^*a_{J-\frac{1}{2}}\}\sum_{j\text{ odd}}\sum_M(-1)^{-M-\frac{1}{2}}p_M\times[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}, \quad (26.2)$$

and

$$\langle\hat{r}\cdot\mathbf{P}I\rangle_{\mathrm{Av}}(\theta)=\frac{2J+1}{2}\{|a_{J-\frac{1}{2}}|^2-|a_{J+\frac{1}{2}}|^2\}\sum_{j\text{ odd}}\sum_M(-1)^{-M-\frac{1}{2}}p_M\times[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}, \quad (26.3)$$

where $\hat{p}=\mathbf{p}_1/p_1$, $\hat{q}=(\hat{k}\times\hat{p})/|\hat{k}\times\hat{p}|$, $\hat{r}=(\hat{q}\times\hat{p})/|\hat{q}\times\hat{p}|$, and $\hat{k}$ is a unit vector along the axis of quantization. Specialization of these formulas to the case $p_M=(2J+1)^{-1}$, $M=-J,\cdots J$, leads aside from a factor of 2π to the results of Eqs. (24) for the decay of unpolarized particles.

In cases in which final-state interactions are small and in which it is sufficient to treat the decay amplitudes a_L to first order in the decay interaction, time-reversal invariance requires that the amplitudes be relatively real.[11] Under such circumstances $\mathrm{Im}(a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}})\rightarrow 0$, and $\langle\hat{q}\cdot\mathbf{P}I\rangle_{\mathrm{Av}}(\theta)$ vanishes. On the other hand, charge-conjugation invariance under similar restrictions requires $a_{J-\frac{1}{2}}$ and $a_{J+\frac{1}{2}}$ to be relatively imaginary.[11,12] In this case $\mathrm{Re}\{a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}}\}\rightarrow 0$ and the odd-order terms in $I_{\mathrm{Av}}(\theta)$ vanish. Thus observation of contributions of odd-order Legendre polynomials to the azimuthally integrated decay angular distribution evinces a violation of both parity conservation and charge-conjugation invariance in the decay, while observation of a nonvanishing $\langle\hat{q}\cdot\mathbf{P}I\rangle_{\mathrm{Av}}$ in the absence of significant final-state interaction indicates a violation of parity conservation and time-reversal invariance.

2. Tests for the Fermion Spin

The foregoing results do not in general provide useful tests for the spin of the decaying fermion unless further restrictions can be placed on the decay amplitudes or on the weights p_M, all of which may be unknown in any particular case. One possible restriction which leads to angular distributions depending uniquely on J is that proposed by Adair.[13] We consider a production-decay sequence of the type $0,\frac{1}{2}\rightarrow 0, J; J\rightarrow 0,\frac{1}{2}$; e.g., of the type $\pi^-+p\rightarrow K^0+\Lambda^0$, $\Lambda^0\rightarrow\pi^-+p$. If the axis of quantization is taken along the direction of motion of the incident particle, and if the initial spin-$\frac{1}{2}$ particle is unpolarized, then the fermions produced at (or near[14]) $\theta=0°$ or $\theta=180°$ can occupy only the states with $M_L=0$, $M=\pm\frac{1}{2}$. The states with $M=+\frac{1}{2}$ and $M=-\frac{1}{2}$ are produced incoherently and with equal weights. Specializing to this case, one finds for the angular distribution

$$I(\theta,\phi)=\frac{2J+1}{4\pi}\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}\sum_{j\text{ even}}(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}^2. \quad (27)$$

The angular distribution depends solely upon the value of J. An exactly similar result is obtained for the product $\mathbf{P}(\theta,\phi)I(\theta,\phi)$, with the consequence that

$$\mathbf{P}(\theta,\phi)=-\hat{p}2\,\mathrm{Re}\{a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}}\}/[|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2]. \quad (28)$$

The polarization is therefore longitudinal and angle-independent, and does not depend upon J in a characteristic way. However, as in the decay of initially unpolarized particles, a measurement of $\mathbf{P}$ provides an unambiguous determination of the sign and magnitude of the ratio $2\,\mathrm{Re}\{a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}}\}\times[|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2]^{-1}$, and is thus relevant to the discussion of parity conservation in, and the theory of, the decay.

Relatively simple tests for the spin of the decaying particle may be obtained by considering weighted averages over $I(\theta,\phi)$ and $\mathbf{P}(\theta,\phi)I(\theta,\phi)$. Thus, relative to any polar axis,

$$\langle I\cos\theta\rangle=\int I(\theta,\phi)\cos\theta d\Omega$$
$$=-\frac{1}{2J(J+1)}\langle J_z\rangle 2\,\mathrm{Re}\{a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}}\}, \quad (29.1)$$

where $\langle J_z\rangle$ is the expectation value in the initial state of the component of $\mathbf{J}$ along the chosen axis. Since

$$|\langle J_z\rangle 2\,\mathrm{Re}\{a_{J+\frac{1}{2}}{}^*a_{J-\frac{1}{2}}\}|\leqslant J\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}=J\langle I\rangle, \quad (29.2)$$

Eq. (23.1) yields the inequality

$$|\langle I\cos\theta\rangle|\leqslant\frac{1}{2(J+1)}\langle I\rangle. \quad (29.3)$$

This result has been used by Lee and Yang[1] to obtain from its large decay angular asymmetry[15–17] an upper bound on the spin of the Λ^0. An analogous limit may be obtained by considering the average with weight $\cos\theta$ of the longitudinal component of $\mathbf{P}(\theta,\phi)I(\theta,\phi)$,

$$\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle=\frac{1}{2J(J+1)}\langle J_z\rangle\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}. \quad (30.1)$$

Hence,

$$|\langle\hat{p}\cdot\mathbf{P}I\cos\theta\rangle|\leqslant\frac{1}{2(J+1)}\langle I\rangle. \quad (30.2)$$

This second relation provides an upper bound for the spin which is independent of the question of parity conservation in the decay.

More general tests for the spin of the decaying particle may be obtained in terms of the test functions $T_{J,M}{}^{\pm}$

[12] Charge-conjugation invariance in the Λ^0 decay has been considered by R. Gatto, Phys. Rev. **108**, 1103 (1957), taking into account the large final-state interactions between the π^- meson and the proton.

[13] R. K. Adair, Phys. Rev. **100**, 1540 (1955).

[14] F. Eisler *et al.*, Nuovo cimento **7**, 222 (1958).

[15] F. S. Crawford *et al.*, Phys. Rev. **108**, 1102 (1957).

[16] F. Eisler *et al.*, Phys. Rev. **108**, 1353 (1957).

[17] L. B. Leipuner and R. K. Adair, Phys. Rev. **109**, 1358 (1958).

defined as

$$T_{J,M}^{+}(\theta)=\frac{2}{2J+1}\sum_{j\ \text{even}}(-1)^{M-\frac{1}{2}}(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}^{-1},\quad(31.1)$$

$$T_{J,M}^{-}(\theta)=\frac{2}{2J+1}\sum_{j\ \text{odd}}(-1)^{M+\frac{1}{2}}(2j+1)P_j(\cos\theta)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}^{-1}.\quad(31.2)$$

The $T_{J,M}^{\pm}$ are essentially the same as the even and odd parts of the test functions considered by Lee and Yang.[1] Integrating the test functions over θ with the weight $I_{Av}(\theta)$, we obtain the results

$$\langle T_{J,M}^{+}I\rangle=\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}(p_M+p_{-M}),\quad(32.1)$$

and

$$\langle T_{J,M}^{-}I\rangle=2\,\text{Re}\{a_{J+\frac{1}{2}}^{*}a_{J-\frac{1}{2}}\}(p_M-p_{-M}),\quad(32.2)$$

where

$$\langle T_{J,M}I\rangle=\int_0^{\pi}T_{J,M}(\theta)I_{Av}(\theta)\sin\theta d\theta=\int T_{JM}(\theta,\phi)I(\theta,\phi)d\Omega.\quad(32.3)$$

The observation that the weights p_M are restricted by the conditions $0\leqslant p_M\leqslant 1$ and $\sum_M p_M=1$ leads at once to the inequalities discussed by Lee and Yang,[1]

$$|\langle T_{J,M}^{-}I\rangle|\leqslant\langle T_{J,M}^{+}I\rangle,\quad M=-J,\cdots J.\quad(33)$$

Alternatively, we may define $T_{J,M}=T_{J,M}^{+}+T_{J,M}^{-}$ and write the inequalities in the more compact form

$$0\leqslant\langle T_{J,M}I\rangle,\quad M=-J,\cdots J.\quad(33')$$

It is necessary that these inequalities be satisfied in order that spin J be consistent with a given angular distribution. The utility of these results will become apparent in the next section in connection with the discussion of the spin of the Λ^0.

Further inequalities necessary for spin J can be formulated when the polarization as well as the angular distribution in a decay are measured. From Eqs. (26.1) and (29) we find that

$$\langle T_{J,M}^{+}\hat{p}\cdot\mathbf{P}I\rangle=-2\,\text{Re}\{a_{J+\frac{1}{2}}^{*}a_{J-\frac{1}{2}}\}(p_M+p_{-M}),\quad(34.1)$$

and

$$\langle T_{J,M}^{-}\hat{p}\cdot\mathbf{P}I\rangle=-\{|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2\}(p_M-p_{-M}).\quad(34.2)$$

Combining the results of Eqs. (32) and (34), we obtain the potentially useful inequalities

$$|\langle T_{J,M}^{+}\hat{p}\cdot\mathbf{P}I\rangle|\leqslant\langle T_{J,M}^{+}I\rangle,\quad(35.1)$$

$$|\langle T_{J,M}^{-}I\rangle|\leqslant|\langle T_{J,M}^{-}\hat{p}\cdot\mathbf{P}I\rangle|,\quad(35.2)$$

and

$$|\langle T_{J,M}^{-}I\rangle-\langle T_{J,M}^{+}\hat{p}\cdot\mathbf{P}I\rangle|\leqslant\langle T_{J,M}^{+}I\rangle-\langle T_{J,M}^{-}\hat{p}\cdot\mathbf{P}I\rangle\leqslant 2\langle I\rangle,\quad M=-J,\cdots J.\quad(35.3)$$

A third set of inequalities can be obtained which involve the transverse components of the polarization. Defining

$$V_{J,M}=\frac{2}{2J+1}(-1)^{M+\frac{1}{2}}\sum_{j\ \text{odd}}[4\pi(2j+1)]^{\frac{1}{2}}Y_j^1(\theta,0)\times\begin{pmatrix}J & J & j\\ M & -M & 0\end{pmatrix}\begin{pmatrix}J & J & j\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}^{-1},\quad(36)$$

one obtains the results

$$|\langle V_{J,M}\hat{q}\cdot\mathbf{P}I\rangle|=|2\,\text{Im}\{a_{J+\frac{1}{2}}^{*}a_{J-\frac{1}{2}}\}(p_M-p_{-M})|\leqslant\langle T_{J,M}^{+}I\rangle,\quad(37.1)$$

and

$$|\langle V_{J,M}\hat{r}\cdot\mathbf{P}I\rangle|=|(|a_{J-\frac{1}{2}}|^2-|a_{J+\frac{1}{2}}|^2)(p_M-p_{-M})|\leqslant\langle T_{J,M}^{+}I\rangle,\quad M=-J,\cdots J.\quad(37.2)$$

The inequalities of the preceding equations are valid whatever polar axis is chosen, so long only as the choice is independent of the details of the decay events considered. These inequalities provide conditions necessary for spin J to be consistent with an observed decay angular distribution and polarization. Unfortunately, the tests involving the polarization do not appear to be feasible at present relative to the decays of the Λ^0 and Σ hyperons. However, in the case of the decay $\Xi^-\rightarrow\Lambda^0+\pi^-$, the subsequent decay $\Lambda^0\rightarrow p+\pi^-$ provides a natural analyzer of the polarization of the Λ^0, and it is conceivable that the tests, especially those of Eqs. (30.2) and (35), may provide useful information of the Ξ^- spin.

3. Application of the Tests; Spin of the Λ^0 Hyperon

In this section we shall apply the tests of Eqs. (33) to the consideration of the spin of the Λ^0 hyperon. A discussion of the spin on the basis of the inequality of Eq. (29.3) and the large "up-down" asymmetry in the process $\Lambda^0\rightarrow\pi^-+p$ has been given by Lee and Yang,[1] who noted also that if $\langle IP_j(\cos\theta)\rangle$ is assumed to vanish for $j>1$, the further very restrictive inequality

$$|\langle I\cos\theta\rangle|\leqslant\frac{1}{6J}\langle I\rangle\quad(38)$$

followed from the test inequality (33) and the definition of the test functions $T_{JJ}^{\pm}$. On the basis of this test and the value $\frac{1}{3}(-0.52\pm0.10)$ of $\langle I\cos\theta\rangle/\langle I\rangle$ obtained by

Eisler *et al.*[16] from the combined data of several groups,[15–17] a spin greater that $\frac{1}{2}$ was shown to be unlikely for the Λ^0. The less restrictive test

$$|\langle I\cos\theta\rangle|\leqslant\frac{1}{2J+2}\langle I\rangle, \qquad (29.3')$$

which requires no assumptions about the magnitude of the averages $\langle IP_j(\cos\theta)\rangle$ for its derivation, is satisfied for spin $\frac{3}{2}$ as well as for spin $\frac{1}{2}$, and does conclusively eliminate spin $\frac{5}{2}$. However, spins greater than $\frac{1}{2}$ may be shown to be unlikely without special assumptions through the use of the test functions $T_{J,M}{}^{\pm}$ and the tests of Eqs. (33).

We have used the angular distribution of the proton in the decay $\Lambda^0\rightarrow\pi^-+p$ as determined by Eisler *et al.*[16] for Λ^0 hyperons produced at angles between 30° and 150° in the center-of-mass system in the process $\pi^-+p\rightarrow\Lambda^0+K^0$. The polar axis was taken perpendicular to the plane of production along the direction of $\hat{p}_{\pi\ \mathrm{inc}}\times\hat{p}_\Lambda$. Using the test functions defined in Eqs. (31) and the experimental angular distributions with $\langle I\rangle$ normalized to unity, we obtain the results

$$\begin{aligned}
J&=\tfrac{1}{2}, & M&=\tfrac{1}{2}: & \langle T_{\frac{1}{2},\frac{1}{2}}{}^{+}I\rangle&=1.00\pm0.07,\\
&&&& \langle T_{\frac{1}{2},\frac{1}{2}}{}^{-}I\rangle&=0.47\pm0.13;\\
J&=\tfrac{3}{2}, & M&=\tfrac{3}{2}: & \langle T_{\frac{3}{2},\frac{3}{2}}{}^{+}I\rangle&=0.40\pm0.08,\\
&&&& \langle T_{\frac{3}{2},\frac{3}{2}}{}^{-}I\rangle&=0.68\pm0.20;\\
J&=\tfrac{3}{2}, & M&=\tfrac{1}{2}: & \langle T_{\frac{3}{2},\frac{1}{2}}{}^{+}I\rangle&=0.60\pm0.08,\\
&&&& \langle T_{\frac{3}{2},\frac{1}{2}}{}^{-}I\rangle&=0.28\pm0.10;\\
J&=\tfrac{5}{2}, & M&=\tfrac{5}{2}: & \langle T_{\frac{5}{2},\frac{5}{2}}{}^{+}I\rangle&=0.10\pm0.07,\\
&&&& \langle T_{\frac{5}{2},\frac{5}{2}}{}^{-}I\rangle&=0.73\pm0.23.
\end{aligned} \qquad (39)$$

The test inequality $|\langle T_{J,M}{}^{-}I\rangle|\leqslant\langle T_{J,M}{}^{+}I\rangle$ which must be satisfied for spin J to be consistent with the observed angular distribution $I(\theta,\phi)$ is satisfied for $J=\frac{1}{2}$ and violated for $J=\frac{3}{2}$ and for $J=\frac{5}{2}$. Using the alternative form of the test inequalities given in Eq. (33′), one may state the inconsistency for spins $\frac{3}{2}$ and $\frac{5}{2}$ in the simple form

$$J=\tfrac{3}{2},\quad M=-\tfrac{3}{2}:\quad 0 \text{ is not } \leqslant-0.28\pm0.22, \qquad (40.1)$$

$$J=\tfrac{5}{2},\quad M=-\tfrac{5}{2}:\quad 0 \text{ is not } \leqslant-0.63\pm0.25. \qquad (40.2)$$

A spin of $\frac{5}{2}$ for the Λ^0 hyperon is definitely inconsistent with the present experimental results. A spin of $\frac{3}{2}$ appears to be unlikely, but cannot be excluded completely because of the rather large uncertainty in the value of $\langle T_{\frac{3}{2}-\frac{3}{2}}I\rangle$. However, it should be noted that the lower bound of zero for $\langle T_{\frac{3}{2}-\frac{3}{2}}I\rangle$ can be attained only under the stringent conditions that the ratio $|2\,\mathrm{Re}\,a_{J+\frac{1}{2}}{}^{*}a_{J-\frac{1}{2}}|/[|a_{J-\frac{1}{2}}|^2+|a_{J+\frac{1}{2}}|^2]$ has its maximum value, unity, and that one of the weights $p_{\frac{3}{2}}$ or $p_{-\frac{3}{2}}$ vanishes. Any deviation from these conditions decreases the possibility of ascribing the inconsistencies to experimental errors. Thus, for example, if the smaller of $p_{\frac{3}{2}}$ and $p_{-\frac{3}{2}}$ is assumed to be as great or greater than 15% of the larger in the energy range considered (910–1300 Mev), one obtains instead of (40.1) the more severe inconsistency 0 is not $\leqslant-0.4\pm0.2$.

The present tests, based on the data of Eisler *et al.*[16] on the angular distribution of the product particles in the decay of polarized Λ^0 hyperons, are consistent with $J_\Lambda=\frac{1}{2}$ and appear to exclude higher spins. Thus $J_\Lambda=\frac{1}{2}$ is indicated for the Λ^0 hyperon, in agreement with the results obtained by Eisler *et al.*[14] using the method of Adair.[13] It should perhaps be remarked that the tests proposed here provide in general only limiting conditions on the spin of the decaying particle, while the selection of production and decay events according to the criteria of Adair leads to decay angular distributions [Eq. (27)] which depend uniquely on J. Nevertheless, the present method should provide, with better statistics, more certain restrictions on the Λ^0 spin than result from the Adair analysis, which requires in practice assumptions concerning the production mechanism of the Λ^0 and the spin of the K^0 meson which are unnecessary in the present work.

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