Minimum Theorem for the Interaction Radius in Two-Body Collisions*

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(Received May 28, 1958)

A general theorem has been derived for the behavior of a two-body scattering process. It is shown that the phase shift analysis must incorporate at least $\bar{l}$ non-negligible phase shifts, where $\bar{l}$ is determined by the total and elastic cross sections in a simple fashion. For large $\bar{l}$, a minimum value for the interaction radius can be inferred from $\bar{l}$. This result is applied to several scattering experiments which were previously analyzed, and the minimum radius obtained turns out in each case to be quite close to the interaction radius as estimated in the more elaborate discussion.

I. INTRODUCTION

IN recent years, experimental investigations have uncovered an increasing number of elementary-particle interactions in which inelastic processes have a rather high probability. This paper will be concerned with the implications of this situation for the phase-shift analyses of the elastic scattering and, in particular, with the consequences of a theorem showing that in such a case there is an integer $\bar{l}$ determined by the values of the elastic and total cross sections such that a partial wave analysis cannot be made solely in terms of waves with l smaller than $\bar{l}$. Further conclusions will be based on the assumption that the basic interaction aside from any Coulomb effects (taken as negligible here) can be represented in terms of a complex potential of short range—or, more generally, that the probability of any interaction occurring is appreciable only within a small region. Then for $\bar{l}$ large enough, the above results yield a minimum value, $\bar{R}$, of the range of interaction. This conclusion finds application to a number of situations which will be discussed later in this paper.

II. MINIMUM THEOREM FOR SCATTERING PROCESSES

To derive our basic theorem, we first consider the expressions for the total and elastic cross sections for two nonidentical particles with spin-independent central forces. In the notation used by Rarita,[1] we have

$$\sigma_e=(\pi/k^2)\sum_{l=0}^{\infty}(L_l^2+M_l^2)/(2l+1), \tag{1}$$

$$\sigma_t=(2\pi/k^2)\sum_{l=0}^{\infty}L_l, \tag{2}$$

with L_l and M_l (both real) related to the usual (complex) phase shift δ_l according to

$$L_l+iM_l=(2l+1)(1-e^{2i\delta_l}). \tag{2a}$$

For a given σ_e and σ_t, Eqs. (1) and (2) place certain restrictions on the (L_l,M_l). We shall be interested in the case in which $L_l=M_l=0$ for $l>\bar{l}$ and shall seek to determine the minimum allowable value of $\bar{l}$. In order to do this, we first consider the purely formal problem in which $\bar{l}$ is given and seek the condition that (L_l,M_l) must satisfy for $l\leqslant\bar{l}$ in order that σ_e be a minimum for a given σ_t. The method of Lagrange multipliers leads to

$$M_l=0,\quad L_l/(2l+1)=L_0 \tag{3}$$

for $l\leqslant\bar{l}$. In terms of phase shifts, we have

$$\begin{aligned}&\delta_l=i\chi,\ \sigma_e\leqslant\sigma_t/2,\\&\delta_l=i\chi+\pi/2,\ \sigma_e>\sigma_t/2,\end{aligned} \tag{4}$$

for $l\leqslant\bar{l}$, where χ is a (real) constant; the relation to the L's follows from Eq. (2a). When Eq. (3) holds, the sum in Eq. (2) can be carried out to give a relation expressing L_0 in terms of $\bar{l}$ and σ_t,

$$L_0=k^2\sigma_t/[2\pi(\bar{l}+1)^2]. \tag{5}$$

The corresponding value of σ_e, which we shall designate σ_{em}, is the minimum consistent with the assumed values of σ_t and $\bar{l}$. Therefore, for given values of the latter two quantities, σ_e must satisfy

$$\sigma_e\geqslant\sigma_{em}=L_0\sigma_t/2=k^2\sigma_t^2/[4\pi(\bar{l}+1)^2], \tag{6}$$

which is a consequence of Eqs. (1), (3), and (5). From (6) it follows that for arbitrary σ_e and σ_t the summations in (1) and (2) can be cut off above $l=\bar{l}$ only if $\bar{l}$ satisfies

$$(\bar{l}+1)^2\geqslant k^2\sigma_t^2/[4\pi\sigma_e]. \tag{7}$$

This provides the restriction on $\bar{l}$ sought at the outset.

For the case in which $\bar{l}$ is large, it is possible to obtain a very good approximation for the minimum radius of interaction $\bar{R}$ consistent with a given (σ_e,σ_t). This follows because, as is seen from Eq. (3), the (L_l,M_l) corresponding to minimum $\bar{l}$ are, under the conditions stated, just those for a gray disk of radius $\bar{R}=(\bar{l}+1)/k$, which is, therefore, the minimum radius in question. It may be remarked that the differential elastic cross section $\sigma_{em}(\theta)$ corresponding to minimum $\bar{l}$ and $\bar{R}$ is identical in shape with the differential elastic cross

* These results were briefly reported at the 1957 Thanksgiving Meeting of the American Physical Society [W. Rarita, Bull. Am. Phys. Soc. Ser. II, 2, 354 (1957)].
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[1] W. Rarita, Phys. Rev. 104, 221 (1956).

section $\Sigma(\theta)$ for a black sphere, the relation being

$$\sigma_{em}(\theta)=L_0^2\Sigma(\theta). \quad (8)$$

Under the given conditions, this is nearly equivalent to

$$\sigma_{em}(\theta)=L_0^2\bar{R}^2J_1^2(k\bar{R}\sin\theta)/\sin^2\theta \quad (8a)$$

for small θ. If the WKB method is applicable to the calculation of the phase shifts, the expression for the minimum range of interaction can be slightly refined as $R_0=(\bar{l}+\frac{1}{2})/k=\bar{R}-1/(2k)$. The simplifying assumptions made as to the nature of the interacting particles and the restrictions placed on the type of interaction they undergo are quite unessential to the main result, which depends simply on the fact that Eqs. (1) and (2) are quadratic and linear respectively in the set (L_l,M_l). Indeed, this result applies, with possibly some minor changes, to all cases of two-particle interactions including those involving identical particles with noncentral forces, etc. As an example of the results obtained when the restrictions are relaxed, the case of two identical fermions may be mentioned. The only modification necessary is that, in Eqs. (5), (6), and (7), $(\bar{l}+1)^2$ must be replaced by $(\bar{l}+1)(\bar{l}+1\pm\frac{1}{2})$ where the plus and minus signs refer to odd and even states, respectively.

III. APPLICATION TO SCATTERING PROCESSES

We shall now give several applications of the minimum theorem of Sec. II to cases in which the inelastic cross section is an appreciable fraction of the total. The processes considered specifically are (a) π^--p at 1.30 Bev,[2,3] (b) p-p at 1 Bev,[4,5] (c) $\bar{p}$-p at 333 Mev,[6,7] (d) $\bar{p}$-C at 700 Mev,[8] and (e) n-Pb at 14 Mev[9]; the energies cited are all the laboratory energy of the bombarding particle. Table I gives the experimental values of σ_e and σ_t, the calculated value of $\bar{l}$ (to the nearest integer) and of $\bar{R}$ and R(theor), the value of the interaction radius as obtained by a different mode of analysis. Specifically, the value of R(theor) for the π^--p case is that obtained by Chretien *et al.*[3] by fitting the diffraction equation $J_1^2(kR\sin\theta)/\sin^2\theta$ to their differential cross section for elastic scattering. The p-p radius was obtained by Rarita who applied a diffraction analysis to the data.[1] The radius given for the $\bar{p}$-p case is essentially that following from the analysis of this interaction by Ball and Chew.[10] The radius given for the $\bar{p}$-C case was estimated by adapting the result of Glassgold[11] for 140-Mev antiprotons on nitrogen, taking R(theor) as the value of L/k for which $T_L=\frac{1}{2}$ and assuming the usual $A^{\frac{1}{3}}$ dependence. The radius given for n-Pb was obtained by Feshbach and Weisskopf.[9] It may be noted that in this last case $\bar{R}>R$. However, the value of R_0, which is defined so as to eliminate from the radius the contribution resulting from the fact that the wavelength is nonvanishing, is 7.8×10^{-13} cm in good agreement with R. The most notable feature of Table I is the remarkably close correspondence between the value of $\bar{R}$ resulting from the simple application of Eq. (7) and the value of R derived in more elaborate analyses. In general, one may expect the agreement to be best for large $\bar{l}$.

TABLE I. Estimation of interaction range.

Case	E (Mev)	σ_t (mb)	σ_e (mb)	R(theor) (10^{-13} cm)	$\bar{l}$	$\bar{R}$ (10^{-13} cm)	R_0 (10^{-13} cm)
(a) π^--p	1300	30	10	1.08	2	0.84	
(b) p-p	1000	48	22	0.89	3	0.91	
(c) $\bar{p}$-p	333	117	53	1.4	2	1.4	
(d) $\bar{p}$-C	700	660	220	3.8	23	3.9	
(e) n-Pb	14	5050	2760	7.8	6	8.6	7.8

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[7] The use of the results obtained by O. Chamberlain *et al.* [Phys. Rev. **108**, 1553 (1957)] for the total and annihilation cross sections in the $\bar{p}$-p reaction at 452 Mev would lead to a considerably larger value of $\bar{l}$ and $\bar{R}$, the latter being of the order of 2.5×10^{-13} cm. The newer result appears to be closer to theoretical expectations.
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[9] H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949). Their treatment, which refers to a sharply cutoff range of interaction, is best suited for comparison with the treatment given here; however, there are more elaborate optical model calculations [see, for instance, F. Bjorklund and S. Fernbach, Phys. Rev. **109**, 1295 (1958)]. Estimates based on newer measurements of the cross sections [Taylor, Lönsjo, and Bonner, Phys. Rev. **100**, 174 (1955)] lead to essentially the same result.
[10] J. S. Ball, and G. F. Chew, Phys. Rev. **109**, 1385 (1958).
[11] A. E. Glassgold, Phys. Rev. **110**, 220 (1958).