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Radiative Corrections to Muon and Neutron Decay

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The corrections to muon decay due to electromagnetic interactions have been recalculated. Our results differ from those of Behrends, Finkelstein, and Sirlin, because those authors used an inconsistent method for handling the infrared divergences which arise separately in the real and virtual processes. The disagreement is especially significant near the end of the electron (positron) spectrum where our results indicate that the radiative correction to the Michel ρ parameter is approximately 1% larger than previously supposed, a result in the direction of increasing agreement between experiment and theory. With the radiative corrections to muon decay given here, the predicted value of the muon lifetime using the universal theory is $(2.27\pm0.04)\times10^{-6}$ sec. As a preliminary to studying the decay of particles with structure, the β decay of the neutron is examined. This leads to an increase in the Coulomb F factor independent of the nuclear charge and of amount approximately 2.6%. As a result the universal coupling constant obtained from the decay of O^{14} is decreased to $G=(1.37\pm0.02)\times10^{-49}$ erg cm^3 and increases the value of the muon lifetime to $(2.33\pm0.05)\times10^{-6}$ sec.

RECENT improvements in experimental techniques make it possible to observe the effects of radiative corrections on the electron (positron) spectrum in muon decay. These corrections have been calculated by Behrends, Finkelstein, and Sirlin[1] and their results have been used in interpreting the experiments, especially in determining the Michel ρ parameter. The fact that there appears to be a discrepancy between theory and experiment warrants careful scrutiny of this situation in order that any disagreement be clearly understood. Since the effect of electromagnetic interaction is proportional to e^2 the radiative corrections to the probability of decay will be of order of a few percent.

In charge retention order, the decay of the muon (1) into an electron (2) and two neutrinos (3) and (4) with electromagnetic coupling has the interaction Lagrangian[2]

$$\mathfrak{L}_{\text{int}}=2\sqrt{2}G(\bar{\psi}_1\gamma_\mu a\psi_2)(\bar{\psi}_3\gamma_\mu a\psi_4)+e(\bar{\psi}_1\gamma_\mu j_\mu\psi_1)+e(\bar{\psi}_2\psi_\mu j_\mu\psi_2)+\text{Herm. conj.},\qquad(1)$$

where

$$\gamma_\mu a=\gamma_\mu(1+i\gamma_5)/2$$

is the form of the universal beta interaction introduced by Feynman and Gell-Mann[3] and j_μ is the usual four-vector electromagnetic current. Applying perturbation theory, the decay process with electromagnetic interactions may be described to order e^2G^2 by the six diagrams of Fig. 1.

The corrections to the spectrum of the emitted electron arise partly from the virtual processes described by diagrams II through IV and partly by the process of inner bremsstrahlung described by diagrams V and VI. Using the form of the beta interaction given in Eq. (1), the probability of finding the electron with energy η in the interval η to $\eta+d\eta$ may be expressed as

$$P(\eta)d\eta=\frac{m_1{}^5G^2}{(2\pi)^3 48}\left\{4\eta^2(3-2\eta)\left[1+\frac{e^2}{2\pi}(\alpha_1+b_1)\right]\right\}d\eta,\qquad(2)$$

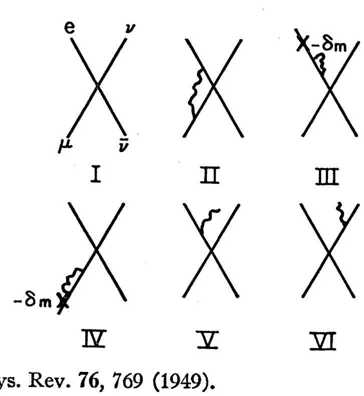
where we have used the notation of Behrends, Finkelstein, and Sirlin,[1] hereafter referred to as B.F.S.

The portion $1+(e^2/2\pi)\alpha_1$ arises from the coherent addition to the contributions of diagrams I through IV and the part b_1 arises similarly from diagrams V and VI.

We have limited our calculations to the case of vector and axial vector interaction. We are in complete agreement with B.F.S. on the contributions α_1 to the spectrum arising from virtual processes; however, the value obtained here for b_1 is different from that obtained by these authors. This point is discussed in more detail below.

Prevention of an infrared divergence arising in the integration over photon momenta in the virtual process is accomplished in a covariant manner by giving the photon a small mass $\lambda_{\min}$.[4] For the complete spectrum, the differential transition probability for inner bremsstrahlung integrated over real-photon momentum must be added to the contribution from virtual photons. An infrared divergence arises here also; and since the inner bremsstrahlung is part of a radiative correction,

FIG. 1. Feynman diagrams for muon decay with electromagnetic corrections.



[1] Behrends, Finkelstein, and Sirlin, Phys. Rev. **101**, 866 (1956).
[2] We choose $\hbar=c=1$.
[3] R. P. Feynman and Murray Gell-Mann, Phys. Rev. **109**, 193 (1958).
[4] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

TABLE I. The function $h(\eta)$.

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.94	0.98
$100\,h(\eta)$	24.68	9.69	5.54	3.43	2.01	+0.85	−0.23	−1.42	−3.06	−4.16	−6.45

the elimination of this divergence should be done in a manner consistent with the virtual process. This means that in calculating the inner bremsstrahlung we must also consider the photons to have a small mass $\lambda_{\min}$. The sum α_1+b_1 will, of course, not depend on $\lambda_{\min}$.

The differential transition probability of decay with the emission of a photon with mass $\lambda_{\min}$ has been calculated and is in agreement with that of B.F.S. Eq. (17b), provided we sum over the 3 directions of polarization of the photon of mass $\lambda_{\min}$ and neglect terms proportional to $\lambda_{\min}^2$. Since the photon has a mass $\lambda_{\min}$, to be rigorous we must sum over 3 directions of polarization compatible with $e\cdot K=0$, where e and K are the polarization and momentum four-vectors of the massive photon. In such a case, a sum of the form $\sum(M\cdot e)(M\cdot e)$ where the sum is over 3 directions of polarization may be expressed as

$$\sum_{\text{pol}}(M\cdot e)(M\cdot e)=-M^2+\frac{1}{\lambda_{\min}^2}(M\cdot K)^2. \qquad (3)$$

If $M\cdot e$ is the matrix element for inner bremsstrahlung, $M\cdot K$ is of order $\lambda_{\min}^2$ so that for small $\lambda_{\min}$ we may neglect the second term on the right of Eq. (3).[5] It is interesting to note the similarity between Eq. (3) and the propagator for a spin-one particle with mass $\lambda_{\min}$.

The quantity b_1 has been calculated by integrating Eq. (17b) of B.F.S. over all photon momenta consistent with conservation of energy and momentum. We obtain, in the limit of small mass m_2,

$$\tfrac{1}{2}\eta(3-2\eta)b_1=\eta(3-2\eta)X+\tfrac{1}{6}(\ln\eta+\omega-1)(1-\eta)\times[(5/\eta)+17-34\eta]+(5/6)\eta(1-\eta)^2, \qquad (4)$$

where

$$X=(\ln\eta+\omega-1)[2\ln(1-\eta+\Delta\eta/e)-\ln\eta+\omega-2\omega_<]+\sum_{k=1}^{\infty}\frac{\eta^k}{k^2}-\frac{\pi^2}{6}-\frac{1-\eta}{\eta}\ln(1-\eta+\Delta\eta/e).$$

As is required, b_1 is positive for $0<\eta<1$. The difference between our result and that of B.F.S. is contained in X above and the expression V in Eq. (25d) of B.F.S.[6,7]

The effect of replacing V by X is most important near the end of the spectrum where it tends to depress the results even further than those of B.F.S. We express the spectrum in a manner similar to Eq. (27a) of B.F.S., namely, as

$$P(\eta)d\eta=\frac{m_1^5G^2}{(2\pi)^3 48}\{4\eta^2(3-2\eta)[1+h(\eta)]\}d\eta. \qquad (5)$$

The function $100h(\eta)$ is tabulated in Table I and may be compared with the vector case in Table I of B.F.S.

The result stated by B.F.S. that the three h functions (scalar, vector, tensor) average to approximately the vector h will still be valid here since the b's for other interactions will have essentially the same correction.

Since we believe the interaction to be given by Eq. (1) and since the average h is approximately the vector h, the quantities Λ_1 and Λ_2 defined by B.F.S. are given for this interaction. These numbers are insensitive to the choice $\Delta\eta$ appearing in the logarithms of Eq. (4) as long as it is small compared to the experimental widths, so that we set $\Delta\eta=0$ for Λ_1 and Λ_2. We obtain

$\Lambda_1=0.0138$; B.F.S. gives $\Lambda_1=0.046$;

$\Lambda_2=-0.0220$; B.F.S. gives $\Lambda_2=0.016$.

These quantities are used in Eq. (28) of B.F.S. in order to find the best parameter for fitting the experimental results.[8] The interpretation of experiments using this equation must be reconsidered in light of the above-mentioned error.

As a further consequence there is a radiative correction to the lifetime of the μ meson. It amounts to an increase of 0.44% (B.F.S. found 3.5% decrease). With this correction the value predicted from the universal theory, using the value of $G=(1.41\pm0.01)\times10^{-49}$ erg cm^3 given by the beta decay of O^{14}, is $(2.27\pm0.04)\times10^{-6}$ sec, while the experimental value is $(2.22\pm0.02)\times10^{-6}$ sec.

In order that a precise value be assigned to the universal coupling constant G, some consideration should be given to those effects of electromagnetic origin which could modify the lifetime of O^{14}. Before entertaining the problem of the nucleus as a whole, we have taken as a simpler example the calculation of the

[5] To be completely consistent, one should add, in the virtual processes, the extra terms arising from using the propagator for a neutral spin-one particle with mass $\lambda_{\min}$. However, since the source of these "vector mesons" is conserved no additional contributions will arise.

[6] The integration of the differential transition probability over photon variables is done by using $K\cdot K=\lambda_{\min}^2$ and a lower limit of zero on the momentum. If instead we set $\omega^2=\mathbf{k}^2$ and integrate k with a lower limit $\lambda_{\min}$, the X would be replaced by V and we would obtain the results of B.F.S. But this latter procedure is not consistent with the method of handling the integrals used in calculating the contribution α_1 from virtual photons.

[7] A copying error made in Eq. (4) in the preprint of the present paper led us to believe incorrectly that B.F.S. had made a computational mistake. The author wishes to thank Dr. Sirlin for calling this to his attention.

[8] The effect on the ρ value of the radiative corrections given here as compared to those of B.F.S. has been determined by Dudziak [W. F. Dudziak, University of California Radiation Laboratory Report UCRL-8202 Suppl., 1958 (unpublished)]. The result obtained is to increase the experimental ρ value by 1%.

radiative corrections to the β decay of the neutron. Ordinarily these corrections would be expected to be quite small, of order e^2/π (1/4%); however, we have found that large coefficients of e^2/π arise which are of order $\ln(M/m)$, where M and m refer to the mass of the nucleon and electron, respectively. Also investigated is the question as to whether the inclusion of anomalous moments can give surprisingly large coefficients such as $(\mu_p-\mu_n)^2$, where μ_p and μ_n are the anomalous moments of the proton and neutron, respectively. It is found that this latter type of coefficient appears divided by large numerical factors that depress its effect. These problems are now discussed in more detail below.

First let us consider the problem in the approximation of neglecting the anomalous moments, in which case the electromagnetic corrections to the neutron β decay may be described by diagrams A, B, and C of Fig. 2.

If the limit of large mass M is taken, it would be expected that diagram A would give rise to the usual Coulomb correction factor F, to order e^2. However, in this limit, we find that in addition to a term $2\ln(\Lambda/M)$, Λ being the cutoff, diagram A has a contribution of the form $\frac{1}{2}\ln(M/m)$, where we have neglected constant and energy-dependent terms relative to $\ln(M/m)$. The mass and wave-function renormalization pictured by diagrams B and C have the effect of adding another $\frac{1}{4}\ln(M/m)$ and furthermore do not compensate for the $\ln(\Lambda/M)$ arising from A. If we neglect the constant and energy-dependent factors and estimate the size of the logarithmic terms by taking $\Lambda\sim M$, then the main effect of A, B, and C is to multiply the original spectrum by $(3e^2/2\pi)\ln(M/m)$, producing an increase in the overall rate and an electromagnetic renormalization of the β-coupling constant by 2.6%.[9]

The reason why diagram A in the limit of large M does not correspond to the usual Coulomb factor F to order e^2 may be explained as follows:

As is well known, the electromagnetic interaction may be divided into the instantaneous Coulomb potential and the transverse-wave part. For an infinitely heavy nucleon the transverse-wave parts, in diagrams of type D and E of Fig. 3, do not contribute and the whole

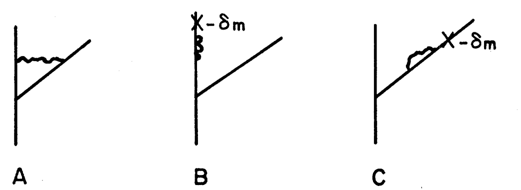


FIG. 2. Feynman diagrams involving virtual photons in neutron decay.

[9] Since the result is independent of transverse waves and comes from the limit of large M, the same amount of $\ln(M/m)$ should result if we treat the nucleus as a spin-zero particle and take for the "β" coupling $(p_1+p_2)_\mu\bar{\psi}_e\gamma_\mu\psi_\nu$, where p_1 and p_2 refer to the momentum of the "nucleus" of charge Z and charge $(Z-1)$, respectively. We have calculated this as a check and indeed find the same amount of $\ln(M/m)$ term as before.

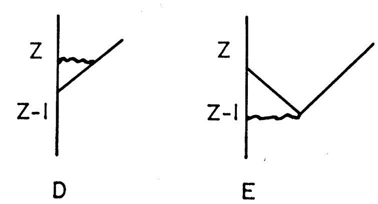


FIG. 3. Feynman diagrams showing Coulomb interaction accompanying β decay.

effect must come from the Coulomb interaction. If we use second-order perturbation theory for a nucleus of charge $Z-1$ undergoing a β decay to a nucleus of charge Z, the matrix $\mathfrak{M}$ to be evaluated between electron and neutrino states may be expressed, in the case of a heavy nucleus, as

$$\mathfrak{M}=4\pi e^2\int\frac{d^3K}{(2\pi)^3}\left[Z\frac{(\epsilon_K+\boldsymbol{\alpha}\cdot\mathbf{K}+\beta m)}{2(\epsilon-\epsilon_K)\epsilon_K}+(Z-1)\frac{(\epsilon_K-\boldsymbol{\alpha}\cdot\mathbf{K}-\beta m)}{2(\epsilon+\epsilon_K)\epsilon_K}\right]\frac{1}{(\mathbf{p}-\mathbf{K})^2},$$

where ϵ and ϵ_K are the energy of the electron in the final and intermediate state, respectively.

The two terms of $\mathfrak{M}$ are equally logarithmically divergent but of opposite sign; hence, if it were not for the fact that $(Z-1)$ instead of Z multiplies the negative-energy contribution, $\mathfrak{M}$ would be finite and lead to the F function to order Ze^2. But we see that the usual F function neglects the fact that it is a charge $(Z-1)$ that contributes to diagram E and a charge Z that contributes to diagram D. Therefore, when $\langle e|\mathfrak{M}|\nu\rangle$ is calculated, there appears a logarithmic divergence independent of the nuclear charge Z. In addition, there will be a logarithmically divergent term from the effect of transverse waves in the electron self-energy, diagram C, as mentioned above.

Besides all the logarithmic factors discussed there are energy-dependent corrections from diagram A and from the process of inner bremsstrahlung. These have been calculated in complete detail and we find that in addition to the usual Coulomb factor they make an average modification of the probability of decay by about $\frac{1}{2}$%. Since this is presently below the limit of experimental resolution, we do not include these terms here.

The second possibility of large coefficients comes from including the anomalous moments of the proton and neutron. If we attempt to compute the radiative corrections by treating the nucleons as bare point particles with anomalous moments, we find that the electromagnetic effects introduce quadratic as well as logarithmic divergences. It is necessary, in order that finite results be obtained, to cut off the integrals over virtual photon momenta by including the nucleon form factor. There are two problems that arise in this connection.

First, even with form factors the complete end result would not be finite as there remains the logarithmic divergence from the wave function renormalization of the electron (diagram C). (However, if only the energy-dependent correction is desired this is of no significance since the result is just to provide an electromagnetic renormalization of the β-coupling constant.) Second, the only form factor that is known is for the interaction between electron and nucleon, but the complete electromagnetic corrections would require the form factor for the interaction of the neutron moment with the proton charge and moment. This latter form factor is, of course, unknown. However, a reasonable calculational procedure would be to include the effect of the moments by using only the Stanford form factor.[10]

For the purpose of estimating the kind of corrections introduced by the moments as well as the order of magnitude, we have considered the problem without form factors and using a cutoff Λ. The moments give rise to four additional diagrams, two from the interaction of the electron with the neutron and proton moments, and two from the interaction of the neutron with the proton charge and moment. The last type, the neutron-proton moment interaction, is quadratically divergent, whereas the other three types are logarithmically divergent.

We find that the main effect, when Λ is large compared to M, is that the unperturbed β interaction $\gamma_\mu a$ evaluated between proton and neutron states becomes

$$\gamma_\mu a+\frac{3}{32}\frac{e^2}{\pi}\mu_n\mu_p\frac{\Lambda^2}{M^2}\gamma_\mu\bar{a}.$$

This leads to an interference term in the energy spectrum and the anisotropy from polarized neutrons, of the form

$$-\frac{3e^2G^2}{32\pi}\frac{\Lambda^2}{M^2}\mu_n\mu_p\left(\frac{1}{2\pi^3}\right)[\epsilon q-\mathbf{p}\cdot\mathbf{q}+q_z\epsilon-qp_z] +q\epsilon d\epsilon d(\cos\theta_p)d(\cos\theta_q),$$

where we have taken the neutron as polarized in the z direction and where q, $\mathbf{q}$ are the energy and momentum of the neutrino. The interaction $\gamma_\mu\bar{a}$ is the only new type of operator which arises when the anomalous moments are included.

Since we have calculated the corrections assuming Λ large compared to M, it would certainly not be legitimate to use the results of the Stanford experiments which indicate that Λ is of order M. However we have taken $\Lambda\sim M$ in order that a rough estimate be given for the magnitude of the moment corrections. In that case the effect is approximately a $\frac{1}{8}\%$ change in the probability of decay and the isotropy.

We summarize the results of the neutron problem as follows. There is an increase in the Coulomb F factor independent of the nuclear charge Z and electron energy of amount $(2.6\pm0.5)\%$, the uncertainty being due to the energy-dependent parts, arising from diagram A and from inner bremsstrahlung, and contributions from the anomalous moment interactions. The result decreases the universal coupling constant obtained from O^{14} to $G=(1.37\pm0.02)\times10^{-49}$ erg cm^3 and increases the value of the predicted value of the muon lifetime from the value given above to $(2.33\pm0.05)\times10^{-6}$ sec, while the experimental value is $(2.22\pm0.02)\times10^{-6}$ sec. The disagreement between experiment and theory appears to be outside of the limit of experimental error and might be regarded as an indication of the lack of universality even by the strangeness-conserving part of the vector interaction. However, it is very difficult to understand the mechanism for such a slight deviation from universality; that is, if universality is to be broken at all why should it be by such a small amount? One possibility in the direction of universality is that the overlap in the O^{14} matrix element might be as low as 95%. However, MacDonald[11] has examined this problem theoretically and finds that the deviation from perfect overlap should be less than one percent.

ACKNOWLEDGMENTS

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[10] A further difficulty is that the nucleons appearing in diagrams A and B are not free, as compared to the nucleons in the Stanford experiments which are free. Hence the experimental form factors are not necessarily relevant.

[11] W. M. MacDonald, University of Maryland Technical Report No. 94, 1957 (unpublished).