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Nuclear Spin-Electron-Neutrino Correlation in Forbidden β Decay*

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The distribution function correlating the electron and neutrino momenta with the spin orientation of the parent or daughter nucleus is calculated for forbidden transitions with Coulomb effects included. Attention is focused on the β-decay interaction law form and behavior under time reversal. The near equivalence of the VA and STP forms is noted and a generalized substitution law relating the VA, STP, and $VASTP$ forms is derived. A number of experiments distinguishing between the VA and STP forms are discussed in detail. Lastly, experiments testing time-reversal invariance are discussed with special emphasis placed on those terms in the distribution function which are not due to Fierz-like interferences and are not obscured by Coulomb effects.

I. INTRODUCTION

To specify completely a four-body process in which all four bodies may have nonzero spin, one should specify the linear momentum and polarization tensor of each of the four bodies. Because of conservation laws following from invariance under proper Lorentz transformations, it is actually sufficient in the center-of-momentum reference frame to specify any two momenta and any three polarization tensors. Thus the most general distribution function describing β decay should have these five quantities as arguments. In practice, only experiments involving at most three of these quantities have ever been done. Various such three-argument distribution functions (independent of any assumptions about invariance under space inversion, charge conjugation, or time reversal) have been published in the recent past.[1]

The purpose of this paper is to add one more set of such three-argument distribution functions to the literature, namely the functions correlating the two linear momenta with a nuclear polarization tensor. This has been given by Jackson, Treiman, and Wyld[1] for allowed transitions; we consider forbidden (in particular first forbidden) transitions. Forbidden transitions involve in general many unknown nuclear matrix elements making the analysis of experiments ambiguous. However, for some judiciously chosen experimental arrangements the analysis can be considerably simplified. The interest in this particular distribution function is twofold:

(1) Observation of certain terms in this distribution function can answer the question whether the β-decay interaction law is VA or STP.

(2) Observation of other terms can answer the question whether β decay is invariant under time reversal or not, provided the interaction law is mainly VA or mainly STP (as is most likely).

We note that other three-argument distribution functions are usually not suitable for testing both questions (1) and (2). Thus the distribution function correlating neutrino momentum with the electron's momentum and polarization is suitable for testing question (1); however, it cannot test question (2) if the interaction law is pure VA or pure STP. On the other hand, a measurement of the beta-gamma angular correlation for decays from oriented nuclei will test question (2) but not question (1).[2]

II. FORMULAS

We take for the β-decay interaction Hamiltonian the expression given by Lee and Yang.[3] The calculations are done explicitly for the process in which an electron and an antineutrino are emitted. To obtain the results for positron emission (or K capture) one should make the following substitutions in all formulas:

$$\begin{aligned}(C_V,C_T,C_S',C_P',C_A')&\rightarrow(C_V,C_T,C_S',C_P',C_A')^*,\\(C_V',C_T',C_S,C_P,C_A)&\rightarrow-(C_V',C_T',C_S,C_P,C_A)^*.\end{aligned}\tag{1}$$

For proton emission, in addition $Z\rightarrow -Z$.

We choose as the two independent linear momenta the electron momentum $\mathbf{p}$ and the antineutrino momentum $\mathbf{q}$. These symbols are used throughout this paper to denote *unit* vectors in the specified directions. The nuclear polarization tensor may be chosen to be either that of the initial or of the final nucleus. In the latter case one might in an actual experiment observe instead the circular polarization of a subsequent γ quantum.

For the distribution in electron and antineutrino momenta for decays from oriented nuclei, we find (the

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[1] Jackson, Treiman, and Wyld, Nuclear Phys. 4, 206 (1957); M. E. Ebel and G. Feldman, Nuclear Phys. 4, 213 (1957); R. B. Curtis and R. R. Lewis, Phys. Rev. 107, 543 and 1381 (1957); M. Morita and R. S. Morita, Phys. Rev. 107, 1316 (1957); Berestetsky, Ioffe, Rudik, and Ter-Martirosyan, Nuclear Phys. 5, 464 (1958); A. Z. Dolginov, Nuclear Phys. 5, 512 (1958); S. B. Treiman, Phys. Rev. 110, 448 (1958); Frauenfelder, Jackson, and Wyld, Phys. Rev. 110, 551 (1958); R. R. Lewis and R. B. Curtis, Phys. Rev. 110, 910 (1958); Morita, Morita, and Yamada, Phys. Rev. 111, 237 (1958). The above list is not claimed to be complete.

[2] These statements are exactly valid for allowed, and approximately valid for first forbidden, transitions.

[3] T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956).

TABLE I. The factor $h_k(I_0,M_0)$ defined by Eq. (A-13).

k	$h_k(I_0,M_0)$
0	1
1	$\sqrt{3}\times\dfrac{M_0}{[(I_0+1)I_0]^{1/2}}$
2	$(\sqrt{5})\times\dfrac{3M_0^2-I_0(I_0+1)}{[(2I_0+3)(I_0+1)I_0(2I_0-1)]^{1/2}}$
3	$(\sqrt{7})\times\dfrac{M_0[5M_0^2-3I_0(I_0+1)+1]}{[(I_0+2)(2I_0+3)(I_0+1)I_0(2I_0-1)(I_0-1)]^{1/2}}$
4	$\frac{3}{2}\times\dfrac{5M_0^2[7M_0^2-6I_0(I_0+1)+5]+3(I_0+2)(I_0+1)I_0(I_0-1)}{[(2I_0+5)(I_0+2)(2I_0+3)(I_0+1)I_0(2I_0-1)(I_0-1)(2I_0-3)]^{1/2}}$

TABLE II. The factor $Q_k(k_e,k_\nu)$ defined by Eq. (A-15).

k	k_e	k_ν	$Q_k(k_e,k_\nu)$
0	0	0	1
0	1	1	$-\sqrt{3}\mathbf{p}\cdot\mathbf{q}$
0	2	2	$\frac{1}{2}(5)^{1/2}[3(\mathbf{p}\cdot\mathbf{q})^2-1]$
1	1	0	$\mathbf{p}\cdot\mathbf{j}$
1	0	1	$\mathbf{q}\cdot\mathbf{j}$
1	1	1	$(\frac{3}{2})^{1/2}i\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}$
1	2	1	$-(1/\sqrt{2})(3\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}-\mathbf{q})\cdot\mathbf{j}$
1	1	2	$-(1/\sqrt{2})(3\mathbf{q}\cdot\mathbf{p}\,\mathbf{q}-\mathbf{p})\cdot\mathbf{j}$
1	2	2	$(15/2)^{1/2}i\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\times\mathbf{q}\cdot\mathbf{j}$
2	2	0	$\frac{1}{2}[3(\mathbf{p}\cdot\mathbf{j})^2-1]$
2	0	2	$\frac{1}{2}[3(\mathbf{q}\cdot\mathbf{j})^2-1]$
2	1	1	$-(\frac{3}{10})^{1/2}(3\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}-\mathbf{p}\cdot\mathbf{q})$
2	2	1	$(3/\sqrt{2})i\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}$
2	1	2	$(3/\sqrt{2})i\mathbf{q}\cdot\mathbf{j}\,\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}$
2	3	1	$-(3/\sqrt{5})\{\frac{1}{2}\mathbf{p}\cdot\mathbf{q}[5(\mathbf{p}\cdot\mathbf{j})^2-1]-\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}\}$
2	1	3	$-(3/\sqrt{5})\{\frac{1}{2}\mathbf{p}\cdot\mathbf{q}[5(\mathbf{q}\cdot\mathbf{j})^2-1]-\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}\}$
2	2	2	$(5/14)^{1/2}[3\mathbf{p}\cdot\mathbf{j}(\mathbf{p}\times\mathbf{q})\cdot(\mathbf{j}\times\mathbf{q})+3\mathbf{q}\cdot\mathbf{j}(\mathbf{q}\times\mathbf{p})\cdot(\mathbf{j}\times\mathbf{p})+3\mathbf{p}\cdot\mathbf{q}(\mathbf{p}\times\mathbf{j})\cdot(\mathbf{q}\times\mathbf{j})-2]$
3	3	0	$\frac{1}{2}\mathbf{p}\cdot\mathbf{j}[5(\mathbf{p}\cdot\mathbf{j})^2-3]$
3	0	3	$\frac{1}{2}\mathbf{q}\cdot\mathbf{j}[5(\mathbf{q}\cdot\mathbf{j})^2-3]$
3	2	1	$(3/\sqrt{7})\{\frac{1}{2}[3(\mathbf{p}\cdot\mathbf{j})^2-1]\mathbf{q}+\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\times(\mathbf{p}\times\mathbf{j})\}\cdot\mathbf{j}$
3	1	2	$(3/\sqrt{7})\{\frac{1}{2}[3(\mathbf{q}\cdot\mathbf{j})^2-1]\mathbf{p}+\mathbf{q}\cdot\mathbf{j}\,\mathbf{p}\times(\mathbf{q}\times\mathbf{j})\}\cdot\mathbf{j}$
3	3	1	$\frac{3}{4}i\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}[5(\mathbf{p}\cdot\mathbf{j})^2-1]$
3	1	3	$\frac{3}{4}i\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}[5(\mathbf{q}\cdot\mathbf{j})^2-1]$
3	2	2	$(3/2)(5/14)^{1/2}i\mathbf{p}\times\mathbf{q}\cdot\mathbf{j}(5\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}-\mathbf{p}\cdot\mathbf{q})$
4	3	1	$(4\sqrt{3})^{-1}\{5\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}[7(\mathbf{p}\cdot\mathbf{j})^2-3]-3\mathbf{p}\cdot\mathbf{q}[5(\mathbf{p}\cdot\mathbf{j})^2-1]\}$
4	1	3	$(4\sqrt{3})^{-1}\{5\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}[7(\mathbf{q}\cdot\mathbf{j})^2-3]-3\mathbf{p}\cdot\mathbf{q}[5(\mathbf{q}\cdot\mathbf{j})^2-1]\}$
4	2	2	$\frac{1}{4}(5/14)^{1/2}\{35(\mathbf{p}\cdot\mathbf{j})^2(\mathbf{q}\cdot\mathbf{j})^2-20\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\cdot\mathbf{j}\,\mathbf{q}\cdot\mathbf{j}-5(\mathbf{p}\cdot\mathbf{j})^2-5(\mathbf{q}\cdot\mathbf{j})^2+2(\mathbf{p}\cdot\mathbf{q})^2+1\}$

TABLE III. The parameters $b_k(0,0;\,k_e,k_\nu)$. These parameters contribute to $\Delta J=0$ transitions.

k	k_e	k_ν	$b_k(0,0;\,k_e,k_\nu)$
0	0	0	$\frac{1}{4}\,\mathrm{Re}\{\alpha_{55}K_{11}+2[\frac{1}{3}qR\alpha_{55}+\alpha_{52}]M_{1-1}+[\frac{1}{9}(qR)^2\alpha_{55}+\frac{2}{3}qR\alpha_{52}+\alpha_{22}]K_{-1-1}\}$
0	1	1	$\frac{1}{4}(\frac{1}{3})^{\frac{1}{2}}\,\mathrm{Re}\{\alpha_{55}M_{11}+2[\frac{1}{3}qR\alpha_{55}+\alpha_{52}]K_{1-1}+[\frac{1}{9}(qR)^2\alpha_{55}+\frac{2}{3}qR\alpha_{52}+\alpha_{22}]M_{-1-1}\}$

TABLE IV. The parameters $b_k(0,1;\,k_e,k_\nu)$ and $b_k(1,0;\,k_e,k_\nu)$. These parameters contribute to $\Delta J=0$ (except $0\to0$) transitions.

k	k_e	k_ν	$b_k(0,1;\,k_e,k_\nu)+b_k(1,0;\,k_e,k_\nu)$
1	1	0	$-\frac{1}{6}\,\mathrm{Re}\{[\beta_{54}+\beta_{51}]M_{11}+[2\beta_{51}-\beta_{54}]M_{1-2}+[\frac{1}{3}qR(\beta_{51}+\beta_{15}+\beta_{45}-\beta_{54})+\beta_{12}+\beta_{42}+\beta_{56}]K_{1-1}$
			$+[\frac{1}{3}qR(2\beta_{51}-\beta_{54})+2\beta_{21}-\beta_{24}]K_{-1-2}+[\frac{1}{9}(qR)^2(\beta_{51}-\beta_{54})+\frac{1}{3}qR(\beta_{21}-\beta_{24}+\beta_{56})+\beta_{26}]M_{-1-1}\}$
1	0	1	$-\frac{1}{6}\,\mathrm{Re}\{[\beta_{54}+\beta_{51}]K_{11}+[\frac{1}{3}qR(3\beta_{51}+\beta_{15}+\beta_{45})+\beta_{12}+\beta_{42}+\beta_{56}]M_{1-1}+\frac{1}{3}[(qR)^2\beta_{51}+qR(3\beta_{21}+\beta_{56})+3\beta_{26}]K_{-1-1}\}$
*1	1	1	$\frac{1}{6}(\frac{1}{6})^{\frac{1}{2}}i\,\mathrm{Im}\{[2(\alpha_{54}+\alpha_{51})]M_{11}+[\alpha_{54}-2\alpha_{51}]M_{1-2}+[\frac{1}{3}qR(2\alpha_{45}+2\alpha_{15}-3\alpha_{54})+2(\alpha_{12}+\alpha_{42}+\alpha_{56})]K_{1-1}$
			$+[\frac{1}{3}qR(\alpha_{54}-2\alpha_{51})+\alpha_{24}-2\alpha_{21}]K_{-1-2}-\frac{1}{3}[(qR)^2\alpha_{54}+qR(-2\alpha_{56}+3\alpha_{24})-6\alpha_{26}]M_{-1-1}\}$
1	2	1	$-\frac{1}{6}(1/\sqrt{2})\,\mathrm{Re}\{[\beta_{54}-2\beta_{51}]K_{1-2}+[\frac{1}{3}qR(\beta_{54}-2\beta_{51})+\beta_{24}-2\beta_{21}]M_{-1-2}\}$
1	1	2	$\frac{1}{6}(1/\sqrt{2})\,\mathrm{Re}\{[\frac{1}{3}qR(\beta_{54}+2\beta_{51})]K_{1-1}+[\frac{1}{9}(qR)^2(\beta_{54}+2\beta_{51})+\frac{1}{3}qR(\beta_{24}+2\beta_{21})]M_{-1-1}\}$

TABLE V. The parameters $b_k(0,2;\,k_e,k_\nu)$ and $b_k(2,0;\,k_e,k_\nu)$. These parameters contribute to $\Delta J=0$ (except $0\to0$, $\frac{1}{2}\to\frac{1}{2}$) transitions.

k	k_e	k_ν	$b_k(0,2;\,k_e,k_\nu)+b_k(2,0;\,k_e,k_\nu)$
2	2	0	$-\frac{1}{10}(\frac{1}{6})^{\frac{1}{2}}\,\mathrm{Re}\{3\alpha_{53}K_{1-2}+[qR\alpha_{53}+3\alpha_{23}]M_{-1-2}\}$
2	0	2	$-\frac{1}{10}(\frac{1}{6})^{\frac{1}{2}}\,\mathrm{Re}\{qR\alpha_{53}M_{1-1}+\frac{1}{3}qR[qR\alpha_{53}+3\alpha_{23}]K_{-1-1}\}$
2	1	1	$-(1/12)(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}\{\alpha_{53}[qRK_{1-1}+3M_{1-2}]+\frac{1}{3}[qR\alpha_{53}+3\alpha_{23}][qRM_{-1-1}+3K_{-1-2}]\}$
*2	2	1	$\frac{1}{10}(\frac{1}{3})^{\frac{1}{2}}i\,\mathrm{Im}\{\frac{3}{2}\beta_{53}K_{1-2}+\frac{1}{2}[qR\beta_{53}+3\beta_{23}]M_{-1-2}\}$
*2	1	2	$\frac{1}{10}(\frac{1}{3})^{\frac{1}{2}}i\,\mathrm{Im}\{qR\frac{1}{2}\beta_{53}K_{1-1}+\frac{1}{6}[qR\beta_{53}+3\beta_{23}]M_{-1-1}\}$

details of this calculation can be found in Appendix A)

$$W(I_0|\mathbf{p},\mathbf{q})d\Omega_e d\Omega_\nu=4F(Z,E)d\Omega_e d\Omega_\nu \times\sum_{kLL'}(-)^{k+L+L'}h_k(I_0,M_0)F_k(L,L',I_f,I_0) \times\sum_{k_e k_\nu}b_k(L,L';\,k_e,k_\nu)Q_k(k_e,k_\nu),\quad(2)$$

the absolute transition rate being given by

$$N(I_0|\mathbf{p},\mathbf{q},E)dp\,d\Omega_e d\Omega_\nu =(2\pi)^{-5}W(I_0|\mathbf{p},\mathbf{q})p^2q^2dp\,d\Omega_e d\Omega_\nu.\quad(3)$$

Here I_0 and I_f are the nuclear spins before and after β decay; $F(Z,E)$ is the Fermi function[4] with $E=$ the total energy of the electron. The functions h_k, F_k, and Q_k are of geometrical nature whereas b_k is the characteristic parameter of the β process, independent of geometry. It is assumed that the initial nucleus is in a definite substate with magnetic quantum number M_0, the entire dependence on M_0 being contained in the factor $h_k(I_0,M_0)$. In general a weighted average over $h_k(I_0,M_0)$ with respect to M_0 should be taken. The entire dependence on the electron's and antineutrino's angular variables is similarly contained in $Q_k(k_e,k_\nu)$. The exact definitions of $h_k(I_0,M_0)$ and $Q_k(k_e,k_\nu)$ are given in Appendix A, Eqs. (A-13) and (A-15), and they are tabulated in Tables I and II for values of arguments of interest for allowed and first forbidden transitions. $F_k(L,L',I_f,I_0)$ depends on the multipole orders L and L' of the β radiation as well as on the initial and final spin values I_0 and I_f. It is symmetric in L and L' but not in I_f and I_0. It has been tabulated[5] for most values of the arguments of interest. Its exact definition is given by Eq. (A-14), Appendix A.

Similarly, the distribution in electron and antineutrino momenta in correlation with circular polarization of a subsequent γ quantum is found to be

$$W(\mathbf{p}_\gamma,\tau_\gamma|\mathbf{p},\mathbf{q})d\Omega_e d\Omega_\nu=4F(Z,E)(2I_f+1)^{-1}d\Omega_e d\Omega_\nu \times\sum_{k\lambda\lambda'LL'}(-\tau_\gamma)^k\delta_\lambda\delta_{\lambda'}F_k(\lambda,\lambda',I_{ff},I_f)F_k(L,L',I_0,I_f) \times\sum_{k_e k_\nu}b_k(L,L';\,k_e,k_\nu)Q_k(k_e,k_\nu),\quad(4)$$

the absolute transition rate being given again by Eq. (3) with $W(I_0|\mathbf{p},\mathbf{q})$ replaced by $W(\mathbf{p}_\gamma,\tau_\gamma|\mathbf{p},\mathbf{q})$. Here I_{ff} is the nuclear spin after the γ transition, $\mathbf{p}_\gamma$ is a unit vector in the direction of the γ quantum momentum and τ_γ is $+1$ for right- and -1 for left-handed circular polarization. The γ may be a mixture of multipoles λ and λ' with amplitudes δ_λ and $\delta_{\lambda'}$. The other symbols are the same as in Eq. (2).

[4] See, e.g., J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 682.

[5] Alder, Stech, and Winther, Phys. Rev. **107**, 728 (1957) and references cited therein.

TABLE VI. The parameters $b_k(1,1;k_e,k_\nu)$. These parameters contribute to $\Delta J=0$ (except $0\to 0$) and $\Delta J=1$ transitions.

k	k_e	k_ν	$b_k(1,1;k_e,k_\nu)$
0	0	0	$\frac{1}{4}\,\mathrm{Re}\{[\alpha_{44}+2\alpha_{41}+\alpha_{11}]K_{11}-2[\frac{1}{3}qR(\alpha_{44}-\alpha_{41}+\alpha_{14}-\alpha_{11})-\alpha_{46}-\alpha_{16}]M_{1-1}+[\frac{1}{2}\alpha_{44}-2\alpha_{41}+2\alpha_{11}]K_{-2-2}+[\frac{1}{3}(qR)^2(\frac{1}{2}\alpha_{44}+\alpha_{11})-\frac{2}{3}qR(\alpha_{46}-\alpha_{16})+\alpha_{66}]K_{-1-1}\}$
0	1	1	$\frac{1}{12}(\frac{1}{3})^{\frac{1}{2}}\,\mathrm{Re}\{[\alpha_{44}+2\alpha_{41}+\alpha_{11}]M_{11}-2[qR(\alpha_{44}+\alpha_{41}+\alpha_{14}+\alpha_{11})-\alpha_{46}-\alpha_{16}]K_{1-1}+4[\alpha_{44}-2\alpha_{41}+\alpha_{14}-2\alpha_{11}]M_{1-2}-2[qR(\frac{1}{2}\alpha_{44}-\alpha_{41}-\alpha_{14}+2\alpha_{11})+2\alpha_{64}-4\alpha_{61}]K_{-1-2}-[\frac{1}{2}\alpha_{44}-2\alpha_{41}+2\alpha_{11}]M_{-2-2}+[(qR)^2(\frac{1}{2}\alpha_{44}+\alpha_{11})-2qR(\alpha_{46}+\alpha_{16})+\alpha_{66}]M_{-1-1}\}$
0	2	2	$-\frac{1}{12}(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}\{[\alpha_{44}-2\alpha_{41}+2\alpha_{14}-4\alpha_{11}]qRM_{-1-2}\}$
1	1	0	$-\frac{1}{6}\,\mathrm{Re}\{[\beta_{44}+2\beta_{41}+\beta_{11}]M_{11}-2[\frac{1}{3}qR(\beta_{44}-\beta_{41}+\beta_{14}-\beta_{11})-\beta_{46}-\beta_{16}]K_{1-1}+[\beta_{44}-2\beta_{41}+\beta_{14}-2\beta_{11}]M_{1-2}-[\frac{1}{3}qR(\beta_{44}-2\beta_{41}-\beta_{14}+2\beta_{11})-\beta_{64}+2\beta_{61}]K_{-1-2}+[\frac{1}{4}\beta_{44}-\beta_{41}+\beta_{11}]M_{-2-2}+[\frac{1}{3}(qR)^2(\frac{1}{4}\beta_{44}-\beta_{41})-\frac{2}{3}qR(\beta_{46}-\beta_{16})+\beta_{66}]M_{-1-1}\}$
1	0	1	$\frac{1}{6}\,\mathrm{Re}\{[\beta_{44}+2\beta_{41}+\beta_{11}]K_{11}-[\frac{1}{3}qR(\beta_{44}+2\beta_{41}+\beta_{14}+2\beta_{11})-2\beta_{46}-2\beta_{16}]M_{1-1}-\frac{2}{3}qR(\beta_{44}-\beta_{41}+\beta_{14}-\beta_{11})K_{1-1}-(\frac{1}{4}\beta_{44}-\beta_{41}+\beta_{11})K_{-2-2}+[\frac{1}{4}(qR)^2\beta_{44}-qR\beta_{46}+\beta_{66}]K_{-1-1}\}$
*1	1	1	$-\frac{1}{2}(\frac{1}{6})^{\frac{1}{2}}i\,\mathrm{Im}\{[\alpha_{44}-2\alpha_{41}+\alpha_{14}-2\alpha_{11}]M_{1-2}+\frac{1}{3}qR(\alpha_{44}+2\alpha_{41}+\alpha_{14}+2\alpha_{11})K_{1-1}-[\frac{1}{3}qR(\frac{1}{2}\alpha_{44}-\alpha_{41}-2\alpha_{14}+4\alpha_{11})-\alpha_{64}+2\alpha_{61}]K_{-1-2}-[\frac{1}{9}(qR)^2(\alpha_{44}+2\alpha_{41}-\alpha_{14}-2\alpha_{11})-\frac{1}{3}qR(\alpha_{64}+2\alpha_{61})]M_{-1-1}\}$
1	2	1	$-\frac{1}{6}(1/\sqrt{2})\,\mathrm{Re}\{[\beta_{44}-2\beta_{41}+\beta_{14}-2\beta_{11}]K_{1-2}-[qR(\frac{1}{2}\beta_{44}-\beta_{41})-\beta_{64}+2\beta_{61}]M_{-1-2}+[\frac{1}{2}\beta_{44}-2\beta_{41}+2\beta_{11}]K_{-2-2}\}$
1	1	2	$-\frac{1}{6}(1/\sqrt{2})\,\mathrm{Re}\{\frac{1}{3}qR\{(\beta_{44}+2\beta_{41}+\beta_{14}+2\beta_{11})K_{1-1}+(\frac{1}{2}\beta_{44}-\beta_{41}+\beta_{14}-2\beta_{11})K_{-1-2}-[qR(\frac{1}{2}\beta_{44}+\beta_{41})-\beta_{64}-2\beta_{61}]M_{-1-1}\}\}$
*1	2	2	$\frac{1}{2}(\frac{3}{10})^{\frac{1}{2}}i\,\mathrm{Im}\{\frac{1}{3}qR(\frac{1}{2}\alpha_{44}-\alpha_{41}+\alpha_{14}-2\alpha_{11})M_{-1-2}\}$
2	2	0	$\frac{1}{2}\,\mathrm{Re}\{[\alpha_{44}-2\alpha_{41}+\alpha_{14}-2\alpha_{11}]K_{1-2}-[\frac{1}{3}qR(\alpha_{44}-2\alpha_{41}-\alpha_{14}+2\alpha_{11})-\alpha_{64}+2\alpha_{61}]M_{-1-2}-(\frac{1}{4}\alpha_{44}-\alpha_{41}+\alpha_{11})K_{-2-2}\}$
2	0	2	$\frac{1}{2}\,\mathrm{Re}\{\frac{1}{3}qR\{[\alpha_{44}+2\alpha_{41}+\alpha_{14}+2\alpha_{11}]M_{1-1}+[(qR)(\frac{1}{4}\alpha_{44}-\alpha_{11})-\alpha_{64}-2\alpha_{61}]K_{-1-1}\}\}$
2	1	1	$-\frac{1}{3}(\frac{5}{6})^{\frac{1}{2}}\,\mathrm{Re}\{[\alpha_{44}+2\alpha_{41}+\alpha_{11}]M_{11}-[qR(\frac{1}{2}\alpha_{44}-\alpha_{41}+\frac{1}{2}\alpha_{14}-\alpha_{11})-2\alpha_{46}-2\alpha_{16}]K_{1-1}-[\frac{1}{2}\alpha_{44}-\alpha_{41}+\frac{1}{2}\alpha_{14}-\alpha_{11}]M_{1-2}-[qR(\frac{1}{4}\alpha_{44}-\frac{1}{2}\alpha_{41}+\alpha_{14}-2\alpha_{11})+\frac{1}{2}\alpha_{64}-\alpha_{61}]K_{-1-2}-\frac{1}{5}[\frac{1}{4}\alpha_{44}-\alpha_{41}+\alpha_{11}]M_{-2-2}+[\frac{1}{5}(qR)^2(\frac{1}{4}\alpha_{44}-\frac{3}{2}\alpha_{41}+\alpha_{11})-qR(\frac{1}{2}\alpha_{46}-\alpha_{16})+\alpha_{66}]M_{-1-1}\}$
*2	2	1	$\frac{1}{2}(1/\sqrt{2})i\,\mathrm{Im}\{[\beta_{44}-2\beta_{41}+\beta_{14}-2\beta_{11}]K_{1-2}-[qR(\frac{1}{2}\beta_{44}-\beta_{41})-\beta_{64}+2\beta_{61}]M_{-1-2}\}$
*2	1	2	$-\frac{1}{2}(1/\sqrt{2})i\,\mathrm{Im}\{\frac{1}{3}qR\{[\beta_{44}+2\beta_{41}+\beta_{14}+2\beta_{11}]K_{1-1}-[\frac{1}{2}\beta_{44}-\beta_{41}+\beta_{14}-2\beta_{11}]K_{-1-2}-[\frac{1}{3}qR(\beta_{44}+2\beta_{41}-\beta_{14}-2\beta_{11})-\beta_{64}-2\beta_{61}]M_{-1-1}\}\}$
2	3	1	$\frac{3}{2}(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}\{[\frac{1}{4}\alpha_{44}-\alpha_{41}+\alpha_{11}]M_{-2-2}\}$
2	1	3	$\frac{3}{2}(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}\{\frac{1}{9}(qR)^2[\frac{1}{4}\alpha_{44}+\alpha_{41}+\alpha_{11}]M_{-1-1}\}$
2	2	2	$-\frac{1}{5}(7)^{\frac{1}{2}}\,\mathrm{Re}\{\frac{1}{3}qR[\frac{1}{2}\alpha_{44}+\alpha_{41}-\alpha_{14}-2\alpha_{11}]M_{-1-2}\}$

III. PARAMETERS $b_k(L,L';k_e,k_\nu)$

The dependence on the physical (as opposed to geometrical) structure of the β process is contained in the parameters $b_k(L,L';k_e,k_\nu)$ and the main contributions of this paper lies in the calculation of these b_k. They are given for a transition of arbitrary order of forbiddenness by Eq. (A-16), Appendix A. For first forbidden transitions they are displayed explicitly in Tables III–VIII under the assumption that the β-decay interaction law is VA. If the interaction law is of the most general $VASTP$ form, the parameters b_k will have additional terms due to STP as well as due to VA-STP interferences. In Appendix B a substitution law is formulated which allows one to obtain the contributions due to STP and VA-STP interferences once the VA contributions have been calculated.

The following are the physical quantities that enter into the makeup of the parameters b_k: the β-decay coupling constants, the nuclear reduced matrix elements, the electron's radial functions and Coulomb phase shifts, and the antineutrino's radial functions. In the tables the dependence on the coupling constants and the nuclear reduced matrix elements is given in terms of the combinations $\alpha_{\epsilon\eta}$ and $\beta_{\epsilon\eta}$, the dependence on the electron's radial functions and Coulomb phase shifts is given in terms of the functions $K_{\kappa\kappa'}$ and $M_{\kappa\kappa'}$, and the dependence on the antineutrino's radial functions is exhibited explicitly.

The combinations $\alpha_{\epsilon\eta}$ and $\beta_{\epsilon\eta}$ are defined by

$$\alpha_{\epsilon\eta}=\Gamma_\epsilon\Gamma_\eta^*+\Gamma_\epsilon'\Gamma_\eta'^*,\quad \beta_{\epsilon\eta}=\Gamma_\epsilon\Gamma_\eta'^*+\Gamma_\epsilon'\Gamma_\eta^*,$$
$$\Gamma_\epsilon=C_\epsilon\int\mathcal{O}_\epsilon^{\mathrm{Cart.}},\qquad \Gamma_\epsilon'=C_\epsilon'\int\mathcal{O}_\epsilon^{\mathrm{Cart.}}. \tag{5}$$

We note that $\alpha_{\eta\epsilon}=\alpha_{\epsilon\eta}^*$, $\beta_{\eta\epsilon}=\beta_{\epsilon\eta}^*$; hence $\alpha_{\epsilon\eta}$ and $\beta_{\epsilon\eta}$ are real if $C_\epsilon=C_\eta$ [6] and in addition $\alpha_{\epsilon\epsilon}$ is non-negative. The $\int\mathcal{O}_\epsilon^{\mathrm{Cart.}}$ are the nuclear reduced matrix elements given in the familiar Cartesian notation. The operators $\mathcal{O}_\epsilon^{\mathrm{Cart.}}$ are exhibited explicitly for first forbidden transitions in Table IX. This table also gives the relation between C_ϵ, C_ϵ' and the coupling constants in the notation of Lee and Yang.[3] The presence in the b_k of terms containing $\beta_{\epsilon\eta}$ indicates violation of space inversion invariance

[6] We assume, for simplicity, $\mathrm{Im}(\int\mathcal{O}^{\mathrm{Cart.}})=0$, which implies that the nuclear wave functions are eigenfunctions of a Hamiltonian invariant under time reversal.

TABLE VII. The parameters $b_k(1,2;k_e,k_\nu)$ and $b_k(2,1;k_e,k_\nu)$. These parameters contribute to $\Delta J=1$ (except $0\leftrightarrow 1$) transitions.

k	k_e	k_ν	$b_k(1,2;k_e,k_\nu)+b_k(2,1;k_e,k_\nu)$
1	1	0	$\frac{1}{4}(10/3)^{\frac{1}{2}}\operatorname{Re}\{[\beta_{43}+\beta_{13}]M_{1-2}-[\frac{1}{3}qR(\beta_{43}-\beta_{13})-\beta_{63}]K_{-1-2}-\frac{1}{5}[\frac{1}{2}\beta_{43}-\beta_{13}]M_{-2-2}+\frac{1}{9}(qR)^2[\frac{1}{2}\beta_{43}+\beta_{13}]M_{-1-1}\}$
1	0	1	$\frac{1}{4}(10/3)^{\frac{1}{2}}\operatorname{Re}\{\frac{1}{3}qR[\beta_{43}+\beta_{13}]M_{1-1}+[\frac{1}{2}\beta_{43}-\beta_{13}]K_{-2-2}-\frac{1}{3}qR[\frac{1}{5}qR(\frac{3}{2}\beta_{43}-2\beta_{13})-\beta_{63}]K_{-1-1}\}$
*1	1	1	$-(1/12)(5)^{\frac{1}{2}}i\operatorname{Im}\{(\alpha_{43}+\alpha_{13})(M_{1-2}+\frac{1}{3}qRK_{1-1})-[\frac{1}{3}qR(\frac{3}{2}\alpha_{43}-\alpha_{31}+\frac{1}{2}\alpha_{34})-\alpha_{63}]K_{-1-2}$
			$-\frac{1}{3}qR[\frac{1}{5}qR(\alpha_{43}-3\alpha_{13})-\alpha_{63}]M_{-1-1}+\frac{2}{5}[\alpha_{43}-2\alpha_{13}]M_{-2-2}\}$
1	2	1	$\frac{1}{4}(15)^{-\frac{1}{2}}\operatorname{Re}\{(\beta_{43}+\beta_{13})K_{1-2}-[\frac{1}{3}qR((9/2)\beta_{43}+6\beta_{13}+\beta_{31}-\frac{1}{2}\beta_{34})-\beta_{63}]M_{-1-2}+[\beta_{43}-2\beta_{13}]K_{-2-2}\}$
1	1	2	$\frac{1}{4}(15)^{-\frac{1}{2}}\operatorname{Re}\{\frac{1}{3}qR\{(\beta_{43}+\beta_{13})K_{1-1}-(\frac{1}{2}\beta_{43}+\beta_{13}+7\beta_{31}-\frac{7}{2}\beta_{34})K_{-1-2}-[\frac{1}{3}qR(2\beta_{43}+\beta_{13})-\beta_{63}]M_{-1-1}\}\}$
*1	2	2	$(1/20)i\operatorname{Im}\{qR(\frac{1}{2}\alpha_{43}+\alpha_{13}-\alpha_{31}+\frac{1}{2}\alpha_{34})M_{-1-2}\}$
2	2	0	$-\frac{1}{2}(\frac{3}{10})^{\frac{1}{2}}\operatorname{Re}\{(\alpha_{43}+\alpha_{13})K_{1-2}-[\frac{1}{3}qR(\alpha_{43}-\alpha_{13})-\alpha_{63}]M_{-1-2}+(\frac{1}{2}\alpha_{43}-\alpha_{13})K_{-2-2}\}$
2	0	2	$\frac{1}{2}(\frac{3}{10})^{\frac{1}{2}}\operatorname{Re}\{\frac{1}{3}qR[(\alpha_{43}+\alpha_{13})M_{1-1}-(\frac{1}{2}qR\alpha_{43}-\alpha_{63})K_{-1-1}]\}$
2	1	1	$\frac{1}{4}\operatorname{Re}\{(\alpha_{43}+\alpha_{13})(M_{1-2}-\frac{1}{3}qRK_{1-1})-[\frac{1}{3}qR(\frac{3}{2}\alpha_{43}-\alpha_{31}+\frac{1}{2}\alpha_{34})-\alpha_{63}]K_{-1-2}$
			$+(\frac{1}{5}\alpha_{43}-\frac{2}{5}\alpha_{13})M_{-2-2}+\frac{1}{3}qR[\frac{1}{5}qR(2\alpha_{43}-\alpha_{13})-\alpha_{63}]M_{-1-1}\}$
*2	2	1	$\frac{1}{4}(15)^{-\frac{1}{2}}i\operatorname{Im}\{(\beta_{43}+\beta_{13})K_{1-2}+[qR(\frac{1}{2}\beta_{43}+2\beta_{13}+\beta_{31}-\frac{1}{2}\beta_{34})+\beta_{63}]M_{-1-2}+(2\beta_{43}-4\beta_{13})K_{-2-2}\}$
*2	1	2	$-\frac{1}{4}(15)^{-\frac{1}{2}}i\operatorname{Im}\{\frac{1}{3}qR\{(\beta_{43}+\beta_{13})K_{1-1}-(\frac{3}{2}\beta_{43}-3\beta_{13}+5\beta_{31}-\frac{5}{2}\beta_{34})K_{-1-2}-[qR(\beta_{43}+\beta_{13})-\beta_{63}]M_{-1-1}\}\}$
2	3	1	$-\frac{1}{10}(\frac{3}{8})^{\frac{1}{2}}\operatorname{Re}\{(\alpha_{43}-2\alpha_{13})M_{-2-2}\}$
2	1	3	$-\frac{1}{10}(\frac{3}{8})^{\frac{1}{2}}\operatorname{Re}\{\frac{1}{9}(qR)^2(\alpha_{43}+2\alpha_{13})M_{-1-1}\}$
2	2	2	$-(21)^{\frac{1}{2}}(40)^{-1}\operatorname{Re}\{\frac{1}{3}qR(\alpha_{43}+2\alpha_{13}-2\alpha_{31}+\alpha_{34})M_{-1-2}\}$
3	3	0	$\frac{3}{4}(\frac{3}{10})^{\frac{1}{2}}\operatorname{Re}\{(\beta_{43}-2\beta_{13})M_{-2-2}\}$
3	0	3	$-\frac{3}{4}(\frac{3}{10})^{\frac{1}{2}}\operatorname{Re}\{\frac{1}{9}(qR)^2(\beta_{43}+2\beta_{13})K_{-1-1}\}$
3	2	1	$-(7/30)^{\frac{1}{2}}\operatorname{Re}\{[\beta_{43}+\beta_{13}]K_{1-2}-[\frac{1}{2}qR(\frac{1}{2}\beta_{43}-\beta_{13}-\beta_{31}+\frac{1}{2}\beta_{34})-\beta_{63}]M_{-1-2}-(\frac{1}{4}\beta_{43}-\frac{1}{2}\beta_{13})K_{-2-2}\}$
3	1	2	$-(7/30)^{\frac{1}{2}}\operatorname{Re}\{\frac{1}{3}qR\{[\beta_{43}+\beta_{13}]K_{1-1}-[\frac{1}{2}qR(\frac{1}{2}\beta_{43}-\beta_{13})-\beta_{63}]M_{-1-1}+\frac{1}{4}[3\beta_{43}+6\beta_{13}+2\beta_{31}-\beta_{34}]K_{-1-2}\}\}$
*3	3	1	$-(3/40)^{\frac{1}{2}}i\operatorname{Im}\{(\alpha_{43}-2\alpha_{13})M_{-2-2}\}$
*3	1	3	$(3/40)^{\frac{1}{2}}i\operatorname{Im}\{\frac{1}{9}(qR)^2(\alpha_{43}+2\alpha_{13})M_{-1-1}\}$
*3	2	2	$\frac{1}{10}(21)^{\frac{1}{2}}i\operatorname{Im}\{\frac{1}{3}qR(\frac{1}{2}\alpha_{43}+\alpha_{13}-\alpha_{31}+\frac{1}{2}\alpha_{34})M_{-1-2}\}$

in the β process, the presence of terms containing $\operatorname{Re}(\beta_{\epsilon\eta})$ or $\operatorname{Im}(\alpha_{\epsilon\eta})$ indicates violation of charge conjugation invariance, and the presence of terms containing $\operatorname{Im}(\alpha_{\epsilon\eta})$ or $\operatorname{Im}(\beta_{\epsilon\eta})$ indicates violation of time reversal invariance.

The functions $K_{\kappa\kappa'}$ and $M_{\kappa\kappa'}$ are obtained by taking the upper sign in the defining Eq. (6):

$$\begin{pmatrix} K_{\kappa\kappa'} \\ L_{\kappa\kappa'} \end{pmatrix} = \{g_\kappa g_{\kappa'} e^{i[\Delta(\kappa)-\Delta(\kappa')]} \pm S(-\kappa)S(-\kappa') \times f_{-\kappa}f_{-\kappa'}e^{i[\Delta(-\kappa)-\Delta(-\kappa')]}\}/F(Z,E),$$
$$\begin{pmatrix} M_{\kappa\kappa'} \\ N_{\kappa\kappa'} \end{pmatrix} = \{S(-\kappa')g_\kappa f_{-\kappa'}e^{i[\Delta(\kappa)-\Delta(-\kappa')]} \pm S(-\kappa)f_{-\kappa}g_{\kappa'}e^{i[\Delta(-\kappa)-\Delta(\kappa')]}\}/F(Z,E). \tag{6}$$

If the β-decay interaction law is VA, the functions $L_{\kappa\kappa'}$ and $N_{\kappa\kappa'}$, defined by the lower signs in Eq. (6), do not appear. The notation in Eq. (6) is standard: $S(\kappa)=$sign of κ, $\Delta(\kappa)$ is the Coulomb phase shift, and g_κ and f_κ are the radial functions of the electron (see Appendix A). Since g_κ, f_κ are real we have $K_{\kappa'\kappa}=K_{\kappa\kappa'}{}^*$, $L_{\kappa'\kappa}=L_{\kappa\kappa'}{}^*$, $M_{\kappa'\kappa}=M_{\kappa\kappa'}{}^*$, and $N_{\kappa'\kappa}=-N_{\kappa\kappa'}{}^*$. In particular $K_{\kappa\kappa}$, $L_{\kappa\kappa}$, $M_{\kappa\kappa}$, and $iN_{\kappa\kappa}$ are real, as well as $M_{\kappa-\kappa}$ and $N_{\kappa-\kappa}$. The relation of these functions to other combinations appearing in the literature, as well as their values under certain approximations, are given in Appendix C.

The following properties of the parameters b_k are useful to note:

$$b_k(L,L';k_e,k_\nu)=(-)^{k+k_e+k_\nu}b_k{}^*(L',L;k_e,k_\nu), \tag{7}$$

$$b_k(L,L';k_e,0)=\delta(k,k_e)b_k(L,L'), \tag{8}$$

where $b_k(L,L')$ are the parameters tabulated in reference 5 (except for the cases $L=2$, $L'\neq 2$, which those authors omit). Each of Tables III–VIII gives the parameters b_k for a given pair of values of the multipole orders L and L'; within each table the parameters are further classified according to the values of k, k_e, and k_ν. Thus for unique transitions one needs only Table VIII for which $L=L'=2$; for $0\rightarrow 0$ (yes) transitions one needs only Table III for which $L=L'=0$, etc.

IV. FORM OF INTERACTION LAW

It follows from the substitution law formulated in Appendix B and summarized in Table X that the VA and STP forms of the β-decay interaction law may, in principle, be distinguished whenever a measurement is performed which selects from the Tables III–VIII terms proportional to

(a) $\alpha_{\epsilon\eta}M_{\kappa\kappa'}$ and $\alpha_{\epsilon\eta}K_{\tau\tau'}$ (except that for $\epsilon=\eta$ it is sufficient to observe just $\alpha_{\epsilon\epsilon}M_{\kappa\kappa'}$), or

(b) $\beta_{\epsilon\eta}M_{\kappa\kappa'}$ and $\beta_{\epsilon\eta}K_{\tau\tau'}$.

TABLE VIII. The parameters $b_k(2,2;k_e,k_\nu)$. These parameters contribute to $\Delta J=0$ (except $0\to0$, $\frac{1}{2}\to\frac{1}{2}$), $\Delta J=1$ (except $0\to1$) and $\Delta J=2$ transitions.

k	k_e	k_ν	$b_k(2,2;k_e,k_\nu)$
0	0	0	$\frac{3}{16}\,\mathrm{Re}\{\alpha_{33}[K_{-2-2}+\frac{1}{9}(qR)^2K_{-1-1}]\}$
0	1	1	$(\sqrt{3}/80)\,\mathrm{Re}\{\alpha_{33}[M_{-2-2}+(10/3)qRK_{-1-2}+\frac{1}{9}(qR)^2M_{-1-1}]\}$
0	2	2	$-(40\sqrt{5})^{-1}\,\mathrm{Re}(\alpha_{33}qRM_{-1-2})$
1	1	0	$-\frac{3}{16}\,\mathrm{Re}\{\beta_{33}[\frac{3}{5}M_{-2-2}+\frac{1}{9}(qR)^2M_{-1-1}]\}$
1	0	1	$\frac{3}{16}\,\mathrm{Re}\{\beta_{33}[K_{-2-2}+(15)^{-1}(qR)^2K_{-1-1}]\}$
*1	1	1	$\frac{1}{16}(6)^{\frac{1}{2}}i\,\mathrm{Im}(\alpha_{33}qRK_{-1-2})$
1	2	1	$(3/40)(1/\sqrt{2})\,\mathrm{Re}\{\beta_{33}[K_{-2-2}+qRM_{-1-2}]\}$
1	1	2	$-(3/40)(1/\sqrt{2})\,\mathrm{Re}\{\beta_{33}qR[K_{-1-2}+\frac{1}{9}qRM_{-1-1}]\}$
*1	2	2	$(40)^{-1}(\frac{3}{10})^{\frac{1}{2}}i\,\mathrm{Im}(\alpha_{33}qRM_{-1-2})$
2	2	0	$(21/80)\,\mathrm{Re}(\alpha_{33}K_{-2-2})$
2	0	2	$(21/80)\,\mathrm{Re}[\alpha_{33}\frac{1}{9}(qR)^2K_{-1-1}]$
2	1	1	$(7/40)(\frac{3}{10})^{\frac{1}{2}}\,\mathrm{Re}\{\alpha_{33}[M_{-2-2}+(5/3)qRK_{-1-2}+\frac{1}{9}(qR)^2M_{-1-1}]\}$
*2	2	1	$(7/40)(1/\sqrt{2})i\,\mathrm{Im}(\beta_{33}qRM_{-1-2})$
*2	1	2	$-(7/40)(1/\sqrt{2})i\,\mathrm{Im}(\beta_{33}qRK_{-1-2})$
2	3	1	$(9/80)(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}(\alpha_{33}M_{-2-2})$
2	1	3	$(80)^{-1}(\frac{1}{5})^{\frac{1}{2}}\,\mathrm{Re}[\alpha_{33}(qR)^2M_{-1-1}]$
2	2	2	$-(40)^{-1}(7/10)^{\frac{1}{2}}\,\mathrm{Re}(\alpha_{33}qRM_{-1-2})$
3	3	0	$-(9/80)\,\mathrm{Re}(\beta_{33}M_{-2-2})$
3	0	3	$(80)^{-1}\,\mathrm{Re}[\beta_{33}(qR)^2K_{-1-1}]$
3	2	1	$(3/80)(7)^{\frac{1}{2}}\,\mathrm{Re}\{\beta_{33}[K_{-2-2}-\frac{2}{3}qRM_{-1-2}]\}$
3	1	2	$-(7^{\frac{1}{2}}/80)\,\mathrm{Re}\{\beta_{33}qR[\frac{1}{3}qRM_{-1-1}-2K_{-1-2}]\}$
*3	2	2	$(20)^{-1}(7/10)^{\frac{1}{2}}i\,\mathrm{Im}(\alpha_{33}qRM_{-1-2})$
4	3	1	$-(9/80)\sqrt{3}\,\mathrm{Re}(\alpha_{33}M_{-2-2})$
4	1	3	$-(\sqrt{3}/80)\,\mathrm{Re}[\alpha_{33}(qR)^2M_{-1-1}]$
4	2	2	$-(3/20)(7/10)^{\frac{1}{2}}\,\mathrm{Re}(\alpha_{33}qRM_{-1-2})$

Observation of both a term proportional to M and a term proportional to K is needed to establish the phase of $\alpha_{\epsilon\eta}$ or $\beta_{\epsilon\eta}$. This phase depends on the unknown (in the absence of a theory of nuclear structure) relative phases of the nuclear matrix elements and, in the case of $\beta_{\epsilon\eta}$, also on the relative phase of C_ϵ and C_η'. The exception noted in (a) follows from the fact that $\alpha_{\epsilon\epsilon}$ is always non-negative.

As mentioned in the introduction, forbidden transitions involve in general a multitude of nuclear matrix elements. In this section we shall discuss in some detail those cases for which the analysis can be simplified. The general case can be obtained from Eq. (A-16) and from the Tables III–VIII.

TABLE IX. The tensor operators (in spherical and Cartesian notation) and coupling constants in the β-decay interaction Hamiltonian.

ϵ	$\mathcal{O}_\epsilon^{LM}$	C_ϵ	C_ϵ'	$\bar{C}_\epsilon$	$\bar{C}_\epsilon'$	$\mathcal{O}_\epsilon^{\mathrm{Cart.}}$	N_ϵ
1	i^LY_{LM}	C_V	C_V'	C_S	C_S'	$i\mathbf{r}/r$	$\sqrt{3}$
2	$i^L\gamma_5Y_{LM}$	$-C_A$	$-C_A'$	C_P	C_P'	γ_5	1
3	$i^{L-1}\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L-1}{}^M$	$-C_A$	$-C_A'$	$-C_T$	$-C_T'$	iB_{ij}/r	$(\frac{3}{4})^{\frac{1}{2}}$
4	$i^L\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L}{}^M$	$-C_A$	$-C_A'$	$-C_T$	$-C_T'$	$\mathbf{r}\times\boldsymbol{\sigma}/r$	$(\frac{3}{2})^{\frac{1}{2}}$
5	$i^{L+1}\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L+1}{}^M$	$-C_A$	$-C_A'$	$-C_T$	$-C_T'$	$(1/i)\mathbf{r}\cdot\boldsymbol{\sigma}/r$	1
6	$i^{L-1}\gamma_5\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L-1}{}^M$	C_V	C_V'	$-C_T$	$-C_T'$	$\gamma_5\boldsymbol{\sigma}$	1
7	$i^L\gamma_5\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L}{}^M$	C_V	C_V'	$-C_T$	$-C_T'$		
8	$i^{L+1}\gamma_5\boldsymbol{\sigma}\boldsymbol{\Phi}_{L,L+1}{}^M$	C_V	C_V'	$-C_T$	$-C_T'$		

1. 0→0 (yes) Transitions

We briefly consider this well-known case because of its simplicity although it does not properly belong in this paper since no nuclear polarization is involved. We see from Table III that $b_0(0,0;0,0)$ (which measures the simplest of all properties of β decay, namely, the spectrum shape) can distinguish between VA and STP. Specifically,

$$
\begin{aligned}
b_0(0,0;0,0)=\tfrac{1}{4}\left|\textstyle\int i\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2
&\times\{\xi^2(\alpha_{AA}+\alpha_{TT})+\tfrac{2}{3}\xi[(p^2/E+q-3x)\alpha_{AA}\\
&+(p^2/E-q)\alpha_{TT}-3(\xi^2/M)(\mathrm{Re}\,\alpha_{TP})]\}\\
&-\tfrac{1}{2}\left|\textstyle\int i\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2\{(\xi^2/E)(\mathrm{Re}\,\alpha_{AT})\\
&\quad+(\xi/E)[x(\mathrm{Re}\,\alpha_{AT})-(\xi^2/M)(\mathrm{Re}\,\alpha_{AP})]\}. \qquad (9)
\end{aligned}
$$

Equation (9) was obtained by evaluating the Coulomb wave functions in the approximation of Appendix C

TABLE X. The substitution law relating $\rho(VA)$, $\rho(STP)$, and $\rho(VA\text{-}STP)$.

$\rho(VA)$	$\rho(STP)$	$\rho(VA\text{-}STP)$
$\Gamma_\epsilon\Gamma_\eta^* K_{\kappa\kappa'}$	$\bar{\Gamma}_\epsilon\bar{\Gamma}_\eta^* K_{\kappa\kappa'}$	$(\bar{\Gamma}_\epsilon\Gamma_\eta^*+\Gamma_\epsilon\bar{\Gamma}_\eta^*)L_{\kappa\kappa'}$
$\Gamma_\epsilon\Gamma_\eta^* L_{\kappa\kappa'}$	$\bar{\Gamma}_\epsilon\bar{\Gamma}_\eta^* L_{\kappa\kappa'}$	$(\bar{\Gamma}_\epsilon\Gamma_\eta^*+\Gamma_\epsilon\bar{\Gamma}_\eta^*)K_{\kappa\kappa'}$
$\Gamma_\epsilon\Gamma_\eta^* M_{\kappa\kappa'}$	$-\bar{\Gamma}_\epsilon\bar{\Gamma}_\eta^* M_{\kappa\kappa'}$	$(\bar{\Gamma}_\epsilon\Gamma_\eta^*-\Gamma_\epsilon\bar{\Gamma}_\eta^*)N_{\kappa\kappa'}$
$\Gamma_\epsilon\Gamma_\eta^* N_{\kappa\kappa'}$	$-\bar{\Gamma}_\epsilon\bar{\Gamma}_\eta^* N_{\kappa\kappa'}$	$(\bar{\Gamma}_\epsilon\bar{\Gamma}_\eta^*-\Gamma_\epsilon\bar{\Gamma}_\eta^*)M_{\kappa\kappa'}$

and keeping only terms of order ξ^2 and ξ, where $\xi=(\alpha Z)/(2R)$ (R=nuclear radius). The α_{AA}, etc., differ from the α_{55} [Eq. (5)], etc., by the reduced nuclear matrix elements (i.e., $\alpha_{AA}=C_AC_A^*+C_A'C_A'^*$, etc.). p and q are the magnitudes of the electron and antineutrino momenta, respectively, and $x=(\int\gamma_5)/(\int i\boldsymbol{\sigma}\cdot\mathbf{r})$. (We use units such that $\hbar=c=m=1$.)

For heavy nuclei $\xi\gg 1$ and so the major contribution to Eq. (9) comes from the ξ^2 terms which do not distinguish between VA and STP. One must therefore look at the ξ terms which are, unfortunately, a small correction. Since for heavy nuclei $(\xi^2/M)\simeq 1$, where M=nucleon mass, the contribution from interferences with the pseudoscalar covariant is of order ξ and we have included it.[7] The second curly bracket in Eq. (9) contains contributions due to VA-STP interferences and displays the $1/E$ dependence characteristic of Fierz terms. If one assumes the Fierz terms to be absent, the dominant ξ^2 terms as well as the T-P interference will give rise to an allowed spectrum shape and a measurement of the deviation from allowed can decide between the A and T covariants. The lighter the nucleus, the more favorable the situation.

The remaining parameter in Table III, $b_0(0,0;1,1)$, measures the electron-neutrino angular correlation. Here it is the leading ξ^2 term that distinguishes between A and T. This leading term is

$$b_0(0,0;1,1) \cong -(\tfrac{1}{4}\sqrt{\tfrac{1}{3}})|\textstyle\int i\boldsymbol{\sigma}\cdot\mathbf{r}|^2\xi^2(p/E)(\alpha_{AA}-\alpha_{TT}). \quad (10)$$

To this order there are no Fierz-like terms, nor are there any contributions from the $\int\gamma_5$ nuclear matrix element. If we write the electron-neutrino angular correlation in the standard form

$$1+\lambda(v/c)\cos\theta_{e\nu}, \quad (11)$$

then $\lambda=1$ for A and $\lambda=-1$ for T (only ξ^2 terms included).

2. Unique Transitions

Another case not obscured by the nuclear matrix elements is that of unique transitions for which the spin change equals $n+1$ where n is the order of forbiddenness. Since k is equal to the rank of the nuclear polarization tensor, if no nuclear polarization is measured only the b_k for which $k=0$ can contribute. For this case we have in Table VIII three parameters: $b_0(2,2;0,0)$, $b_0(2,2;1,1)$ and $b_0(2,2;2,2)$. The first of these measures the spectrum shape; in contrast to the $0\to 0$ (yes) case it does not distinguish between VA and STP. The second and third measure the electron-neutrino angular correlation and do distinguish between VA and STP. Specifically,

$$\begin{aligned}\sum_{k_ek_\nu} b_0(2,2;k_e,k_\nu)Q_0(k_e,k_\nu) =& (1/240)|\textstyle\int iB_{ij}|^2\{5(p^2+q^2)(\alpha_{AA}+\alpha_{TT})-(p/E)\\ &\times[(p^2+10qE+q^2)\alpha_{AA}-(p^2-10qE+q^2)\alpha_{TT}]\cos\theta_{e\nu}+(p/E)pq(\alpha_{AA}-\alpha_{TT})(3\cos^2\theta_{e\nu}-1)\}\\ &+(1/240)|\textstyle\int iB_{ij}|^2(1/E)\{10(p^2+q^2)(\mathrm{Re}\,\alpha_{AT})-[20pq(\mathrm{Re}\,\alpha_{AT})+\alpha Z(p^2+2q^2)(\mathrm{Im}\,\alpha_{AT})]\cos\theta_{e\nu}\\ &+\tfrac{3}{2}\alpha Zpq(\mathrm{Im}\,\alpha_{AT})(3\cos^2\theta_{e\nu}-1)\}, \quad (12)\end{aligned}$$

where the second curly bracket contains the Fierz-like terms, the Coulomb functions have been evaluated in the approximation of Appendix C and the $Q_0(k_e,k_\nu)$ have been taken from Table II.

When nuclear polarization is measured (either by having the initial nucleus oriented or by observing the circular polarization of a subsequent gamma[8]) one also has contributions from the b_k with $k\neq 0$. The electron angular distribution relative to the nuclear polarization is measured by $b_1(2,2;1,0)$, $b_2(2,2;2,0)$ and $b_3(2,2;3,0)$; the corresponding antineutrino angular distribution is measured by $b_1(2,2;0,1)$, $b_2(2,2;0,2)$, and $b_3(2,2;0,3)$. Using Table VIII we find (in the approximation of Appendix C)

$$b_1(2,2;1,0)Q_1(1,0)=-(1/48)|\textstyle\int iB_{ij}|^2[(\tfrac{3}{5}p^2+q^2)(p/E)(\beta_{AA}-\beta_{TT})+(\tfrac{3}{5}p^2+2q^2)(\alpha Z/E)(\mathrm{Im}\,\beta_{AT})]\mathbf{p}\cdot\mathbf{j}, \quad (13e)$$

$$b_1(2,2;0,1)Q_1(0,1)=(1/48)|\textstyle\int iB_{ij}|^2(\tfrac{3}{5}q^2+p^2)[\beta_{AA}+\beta_{TT}+(2/E)(\mathrm{Re}\,\beta_{AT})]\mathbf{q}\cdot\mathbf{j}, \quad (13\nu)$$

$$b_2(2,2;2,0)Q_2(2,0)=(7/240)|\textstyle\int iB_{ij}|^2p^2[\alpha_{AA}+\alpha_{TT}+(2/E)(\mathrm{Re}\,\alpha_{AT})][\tfrac{3}{2}(\mathbf{p}\cdot\mathbf{j})^2-\tfrac{1}{2}], \quad (14e)$$

$$b_2(2,2;0,2)Q_2(0,2)=(7/240)|\textstyle\int iB_{ij}|^2q^2[\alpha_{AA}+\alpha_{TT}+(2/E)(\mathrm{Re}\,\alpha_{AT})][\tfrac{3}{2}(\mathbf{q}\cdot\mathbf{j})^2-\tfrac{1}{2}], \quad (14\nu)$$

$$b_3(2,2;3,0)Q_3(3,0)=-(1/160)|\textstyle\int iB_{ij}|^2p^2[(p/E)(\beta_{AA}-\beta_{TT})+(\alpha Z/E)(\mathrm{Im}\,\beta_{AT})][5(\mathbf{p}\cdot\mathbf{j})^3-3\mathbf{p}\cdot\mathbf{j}], \quad (15e)$$

$$b_3(2,2;0,3)Q_3(0,3)=(1/160)|\textstyle\int iB_{ij}|^2q^2[\beta_{AA}+\beta_{TT}+(2/E)(\mathrm{Re}\,\beta_{AT})][5(\mathbf{q}\cdot\mathbf{j})^3-3\mathbf{q}\cdot\mathbf{j}]. \quad (15\nu)$$

[7] See discussion at the end of Appendix B.

[8] It is, of course, understood that for even k all of our formulas apply equally well if instead directional correlation with the gamma is measured (or the initial nucleus is aligned and not oriented).

Here $\mathbf{j}$ is a unit (pseudo) vector in the direction of the nuclear polarization axis. Thus the combined information obtainable from, for example, an experiment of the type performed by Ambler *et al.*[9] [Eqs. (13e)–(15e)] with that obtainable from an experiment of the type performed by Goldhaber *et al.*[10] [Eqs. (13ν)–(15ν)] can decide between the A and T covariants.

Finally there remain to be discussed in Table VIII the parameters b_k with all three of the arguments k, k_e, and k_ν different from zero. It is these b_k that describe a truly three-argument distribution function since they will contribute only if one measures simultaneously the electron momentum, the antineutrino momentum, and a nuclear polarization tensor. An experiment measuring these b_k is accordingly more difficult to perform than the "simple" experiments measuring any of the previously discussed b_k's. For β^+-emitting (or K-capturing) nuclei several of the "simple" experiments have been carried out and they indicate that the interaction law is VA.[10,11] Although there is no reason to believe that β^- emitters behave differently from β^+ emitters, as yet no "simple" experiments on β^- emitters giving unambiguous results have been performed.[12] Accordingly, we complete this section by listing (in the approximation of Appendix C) the remaining $b_k(2,2;k_e,k_\nu)Q_k(k_e,k_\nu)$ involved in a first forbidden unique transition with $k=1$:

$$b_1(2,2;1,1)Q_1(1,1)=\tfrac{1}{16}|\textstyle\int iB_{ij}|^2(q/E)[\alpha Z(\tfrac{3}{4}E^2+\tfrac{1}{4})(\alpha_{AA}+\alpha_{TT})-2p(\mathrm{Im}\,\alpha_{AT})]\mathbf{p}\times\mathbf{q}\cdot\mathbf{j}, \tag{16}$$

$$b_1(2,2;2,1)Q_1(2,1)=-(1/80)|\textstyle\int iB_{ij}|^2[(\tfrac{1}{3}+q/E)p^2\beta_{AA}+(\tfrac{1}{3}-q/E)p^2\beta_{TT} \\ +\tfrac{2}{3}(p^2/E)(\mathrm{Re}\,\beta_{AT})+\tfrac{1}{2}qp(\alpha Z/E)(\mathrm{Im}\,\beta_{AT})](3\mathbf{p}\cdot\mathbf{q}\;\mathbf{p}\cdot\mathbf{j}-\mathbf{q}\cdot\mathbf{j}), \tag{17}$$

$$b_1(2,2;1,2)Q_1(1,2)=(1/80)|\textstyle\int iB_{ij}|^2[(1+\tfrac{1}{3}q/E)qp\beta_{AA}+(1-\tfrac{1}{3}q/E)qp\beta_{TT} \\ +2(qp/E)(\mathrm{Re}\,\beta_{AT})+\tfrac{2}{3}q^2(\alpha Z/E)(\mathrm{Im}\,\beta_{AT})](3\mathbf{q}\cdot\mathbf{p}\;\mathbf{q}\cdot\mathbf{j}-\mathbf{p}\cdot\mathbf{j}), \tag{18}$$

$$b_1(2,2;2,2)Q_1(2,2)=(3/320)|\textstyle\int iB_{ij}|^2qp\alpha Z(\alpha_{AA}-\alpha_{TT})\mathbf{q}\cdot\mathbf{p}\;\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}. \tag{19}$$

It is clear that the above b_k distinguish between the A and T covariants. The structure of the remaining b_kQ_k with $k>1$, $k_e\neq 0$, $k_\nu\neq 0$ is not essentially different and we omit them here. It should be noted that although in Table VIII entries for $k\leqslant 4$ are listed, in any particular case not all these k values need appear. The value of k is limited not only by the order of forbiddenness n [by the relation $k\leqslant 2(n+1)$] but also by the nuclear spin I (by the relation $k\leqslant 2I$). Here $I=I_0$ [Eq. (2)] or $I=I_f$ [Eq. (4)]. Thus, if one deals with a nucleus for which $I=\frac{1}{2}$, all the entries in Table VIII with $k>1$ may be ignored. Finally, in the case of Eq. (4), k is also limited by the multipole order of the gamma radiation (by the relation $|\lambda-\lambda'|\leqslant k\leqslant\lambda+\lambda'$).

In order to apply our results to an experiment of the type performed by Goldhaber *et al.*[9] (however with a β-emitting, not K-capturing, nucleus), one must integrate over the electron and antineutrino variables subject to the constraint that the recoil momentum be held fixed. Since $\mathbf{q}\times\mathbf{p}$ is perpendicular to the recoil momentum, terms proportional to $\mathbf{q}\times\mathbf{p}\cdot\mathbf{j}$ will vanish and the only remaining dependence of the b_k on αZ will be in the terms proportional to $\mathrm{Im}\,\alpha_{AT}$ or $\mathrm{Im}\,\beta_{AT}$. We shall ignore these terms as very small (if not zero) because (a) they are proportional to αZ, (b) they require time-reversal-invariance violation, and (c) they require the interaction law to be a mixture of VA and STP. Thus all Coulomb effects are contained in the Fermi function $F(Z,E)$. It is shown in Appendix D that if one sets $F(Z,E)=1$ the necessary integrations can be performed analytically, with the curious result that if the interaction law is T the gamma will be circularly polarized but if the interaction law is A the circular polarization will be exactly zero (for unique transitions only T and A are relevant). This result is valid for any unique transition independent of order of forbiddenness. In practice, since the Fermi function differs from unity, the above result is only approximately true; nevertheless the observation of a large circular polarization in light nuclei would necessarily imply that the interaction law is T. In any case the sign of the circular polarization, in light or heavy nuclei, can distinguish between A and T (see next section).

3. Coulomb Transitions

In this section we consider transitions that are dominated by Coulomb effects. All forbidden transitions involving heavy nuclei fall into this category except unique transitions. Let us take as representative a $0\rightarrow 1$ (yes) [or $1\rightarrow 0$ (yes)] transition for which only the entries in Table VI are needed. A glance at this table shows that even if the interaction law is pure VA (or STP), six different products of nuclear matrix elements appear. Thus an analysis of these transitions is virtually impossible unless some additional approximations, beyond those of Appendix C, are made. We shall assume that it is sufficient to keep for each nuclear matrix element only the terms of highest order in ξ;

[9] Ambler, Hayward, Hoppes, Hudson, and Wu, Phys. Rev. **106**, 1361 (1957).

[10] Goldhaber, Grodzins, and Sunyar, Phys. Rev. **109**, 1015 (1958).

[11] Hermannsfeldt, Maxson, Stähelin, and Allen, Phys. Rev. **107**, 641 (1957).

[12] With the exception of the spectrum shape measurement of Pr[144]. D. A. Bromley in *Proceedings of the Rehovoth Conference on Nuclear Structure* (North-Holland Publishing Company, Amsterdam, 1958), p. 457.

for heavy nuclei the error involved in this approximation is only a few percent.[13] Furthermore we shall ignore entirely contributions due to VA-STP interferences (Fierz terms).

We define, in analogy to Eq. (5), the following combinations of coupling constants and nuclear reduced matrix elements:

$$\begin{aligned}\alpha_{(VA)(VA)}&=\Gamma_{(VA)}\Gamma_{(VA)}^*+\Gamma_{(VA)}'\Gamma_{(VA)}'^*;\\ \beta_{(VA)(VA)}&=\Gamma_{(VA)}\Gamma_{(VA)}'^*+\Gamma_{(VA)}'\Gamma_{(VA)}^*,\\ \alpha_{(ST)(ST)}&=\Gamma_{(ST)}\Gamma_{(ST)}^*+\Gamma_{(ST)}'\Gamma_{(ST)}'^*;\\ \beta_{(ST)(ST)}&=\Gamma_{(ST)}\Gamma_{(ST)}'^*+\Gamma_{(ST)}'\Gamma_{(ST)}^*,\end{aligned}\tag{20}$$

where

$$\begin{aligned}\Gamma_{(VA)}&=\xi(C_A\textstyle\int\boldsymbol{\sigma}\times\mathbf{r}+C_V\int i\mathbf{r})+C_V\int\gamma_5\boldsymbol{\sigma},\\ \Gamma_{(ST)}&=\xi(C_T\textstyle\int\boldsymbol{\sigma}\times\mathbf{r}+C_S\int i\mathbf{r})+C_T\int\beta\gamma_5\boldsymbol{\sigma},\end{aligned}\tag{21}$$

and $\Gamma_{(VA)}'$, $\Gamma_{(ST)}'$ are obtained by replacing in Eq. (21) every C by C'. If one takes in Table VI $k=0$, one obtains the spectrum shape

$$b_0(1,1;0,0)=\tfrac{1}{4}(\alpha_{(VA)(VA)}+\alpha_{(ST)(ST)}),\tag{22}$$

and the electron-neutrino angular correlation

$$b_0(1,1;1,1)=(\tfrac{1}{12}\sqrt{\tfrac{1}{3}})(p/E)(\alpha_{(VA)(VA)}-\alpha_{(ST)(ST)}).\tag{23}$$

Thus the spectrum shape is insensitive to the form of the interaction law whereas the electron-neutrino angular correlation [when expressed in the standard form—see Eq. (11)] gives $\lambda=-\frac{1}{3}$ for VA, $\lambda=+\frac{1}{3}$ for ST.

For $k=1$ and 2 we have

$$\begin{aligned}\sum_{k_ek_\nu} b_1(1,1;k_e,k_\nu)Q_1(k_e,k_\nu)&\\ =\tfrac{1}{6}[\mathbf{q}\cdot\mathbf{j}(\beta_{(VA)(VA)}+\beta_{(ST)(ST)})&\\ -\mathbf{p}\cdot\mathbf{j}(p/E)(\beta_{(VA)(VA)}-\beta_{(ST)(ST)})],&\quad(24)\\ \sum_{k_ek_\nu} b_2(1,1;k_e,k_\nu)Q_2(k_e,k_\nu)&\\ =\tfrac{1}{6}(3\mathbf{p}\cdot\mathbf{j}\mathbf{q}\cdot\mathbf{j}-\mathbf{p}\cdot\mathbf{q})(p/E)&\\ \times(\alpha_{(VA)(VA)}-\alpha_{(ST)(ST)}).&\quad(25)\end{aligned}$$

Clearly the $k=2$ term distinguishes between VA and STP; it measures the electron-neutrino angular distribution either from aligned nuclei [Eq. (2)] or in correlation with the direction of a subsequent gamma [Eq. (4)]. The two parts of the $k=1$ term measure separately the antineutrino and electron angular distributions either from oriented nuclei [Eq. (2)] or in correlation with the circular polarization of a subsequent gamma [Eq. (4)]. To apply our results to an experiment on a β^- emitter of the type performed by Goldhaber *et al.*[9] we again assume that the Fermi function may be replaced by unity. Then, as in Appendix D, we get circular polarization for ST but not for VA. Now in the present case it is a very poor approximation to set $F(Z,E)=1$, and in fact the circular polarization can be very nearly as large for VA as for ST. However, because of the preceding argument we can now determine the sign of the circular polarization. For β^- emitters the Fermi function emphasizes the $\mathbf{q}\cdot\mathbf{j}$ term over the $\mathbf{p}\cdot\mathbf{j}$ term in Eq. (24) and if we ignore the $\mathbf{p}\cdot\mathbf{j}$ term Eq. (24) goes over into the corresponding equation for K capture except for an over-all minus sign [see Eq. (1)]. Hence, if the interaction law is VA we should get the opposite sign, and if it is ST the same sign, as the actual sign observed in the K-capture[10] experiment.[14] For β^+ emitters the situation is unfavorable because the Fermi function emphasizes instead the $\mathbf{p}\cdot\mathbf{j}$ term over the $\mathbf{q}\cdot\mathbf{j}$ term, with the result that circular polarization of the same sign is predicted for both VA and ST.

V. TIME REVERSAL

As stated before, noninvariance under time reversal manifests itself in the presence of terms containing Im $\alpha_{\epsilon\eta}$ or Im $\beta_{\epsilon\eta}$. There are many such terms in the Tables III–VIII and various experiments can be designed to measure them. However, because the Coulomb functions are complex, an experiment designed to measure Im $\alpha_{\epsilon\eta}$ or Im $\beta_{\epsilon\eta}$ will usually also measure contributions from Re $\alpha_{\epsilon\eta}$ or Re $\beta_{\epsilon\eta}$, and unless the latter can be regarded as small the results of such an experiment will be ambiguous. An examination of the Coulomb functions in the approximation of Appendix C reveals an important difference between those involved in pure VA (or pure STP) terms (the functions $K_{\kappa\kappa'}$, $M_{\kappa\kappa'}$) and those involved in Fierz-like terms (the functions $L_{\kappa\kappa'}$, $N_{\kappa\kappa'}$). For the former the imaginary parts are at most of order αZ compared with the real parts. For the latter this is not necessarily the case.[15] Thus, if the interaction law is pure VA (or pure STP) one can make the (unwanted) contributions from Re $\alpha_{\epsilon\eta}$, Re $\beta_{\epsilon\eta}$ to be of order αZ compared with the contributions from Im $\alpha_{\epsilon\eta}$, Im $\beta_{\epsilon\eta}$, by measuring only the entries marked in Tables III–VIII with an asterisk. In this section we consider just those terms (for simplicity we approximate the Coulomb functions as in Appendix C, and write out the VA contributions only).

The entries marked with an asterisk are those for which $k+k_e+k_\nu$ is an odd integer. The corresponding $Q_k(k_e,k_\nu)$ are the only ones to change sign upon reversal of direction of all three vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{j}$. Thus the experimental arrangement that will pick out the asterisk-marked terms consists in measuring the difference in counting rates between a given orientation of

[13] Unless an accidental cancellation occurs among these leading terms, as for example, in RaE; the present analysis does not apply in such a case.

[14] We assume here that either $C_\epsilon'=C_\epsilon$ and the interaction law is VA or $C_\epsilon'=-C_\epsilon$ and the interaction law is STP.

[15] This statement remains correct for the contributions from the P covariant even though all four functions, K, L, M, and N contribute to STP terms as well as to Fierz-like terms.

p, **q**, **j**, and its reverse. We note that the effect will be largest if **p**, **q**, **j** are taken mutually perpendicular and it vanishes if **p**, **q**, **j** are coplanar.

One sees immediately that $0\to0$ (yes) transitions, as well as unique transitions, cannot test time reversal invariance. In fact, this is true regardless of any assumptions about the Coulomb functions and follows from the observation that $\mathrm{Im}\,\alpha_{\epsilon\eta}=\mathrm{Im}\,\beta_{\epsilon\eta}=0$ if $C_\epsilon=C_\eta$. In other words, if the interaction law is such that there are no interferences between the five covariants V, A, S, T, and P it is meaningless to ask questions about time reversal invariance. Since $0\to0$ (yes) transitions are caused by A, T, and P and unique transitions by A and T, the assumption of pure VA automatically eliminates all interferences between different covariants.

The other first forbidden transitions, i.e., $\Delta J=0$ (except $0\to0$) and $\Delta J=1$ with parity change, can test time reversal invariance. We take as representative the case $0\to1$ (yes) or $1\to0$ (yes). For this case we need the asterisk-marked b_k from Table VI only. They are

$$\begin{aligned}b_1(1,1;1,1)=&-i(\tfrac{1}{6}\sqrt{\tfrac{1}{6}})(p/E)(\mathrm{Im}\,\alpha_{AV})\{[\xi(q-3E)-p^2-q^2]\textstyle\int i\mathbf{r}-(q+E)\int\gamma_5\}\int\boldsymbol{\sigma}\boldsymbol{\sigma}\times\mathbf{r}\\&-i(\tfrac{1}{12}\sqrt{\tfrac{1}{6}})(\alpha Z/E)\{\tfrac{1}{2}\alpha_{AA}[\xi(3E^2+1)-\tfrac{1}{2}q(E^2+3)+p^2E](\textstyle\int\boldsymbol{\sigma}\times\mathbf{r})^2\\&+\tfrac{1}{2}(\mathrm{Re}\,\alpha_{AV})[(3E^2+1)(\textstyle\int\gamma_5\boldsymbol{\sigma}-\xi\int i\mathbf{r})+p^2(3q-E)\int i\mathbf{r}]\int\boldsymbol{\sigma}\times\mathbf{r}\\&-\alpha_{VV}[(3E^2+1)(\textstyle\int\gamma_5\boldsymbol{\sigma}+\xi\int i\mathbf{r})+2q(E^2+1)\int i\mathbf{r}+p^2E\int i\mathbf{r}]\int i\mathbf{r}\},\end{aligned}\tag{26}$$

$$b_1(1,1;2,2)=-i(\tfrac{1}{9}\sqrt{\tfrac{3}{10}})(qp^2/E)(\mathrm{Im}\,\alpha_{AV})\textstyle\int i\mathbf{r}\int\boldsymbol{\sigma}\times\mathbf{r}+i(\tfrac{1}{12}\sqrt{\tfrac{3}{10}})qp\alpha Z[\tfrac{1}{4}\alpha_{AA}(\int\boldsymbol{\sigma}\times\mathbf{r})^2-\alpha_{VV}(\int i\mathbf{r})^2],\tag{27}$$

$$\begin{aligned}b_2(1,1;2,1)=&-i(\tfrac{1}{6}\sqrt{\tfrac{1}{2}})(p^2/E)(\mathrm{Im}\,\beta_{AV})[(3\xi+E-q)\textstyle\int i\mathbf{r}+\int\gamma_5\boldsymbol{\sigma}]\int\boldsymbol{\sigma}\times\mathbf{r}\\&+i(\tfrac{1}{8}\sqrt{\tfrac{1}{2}})(p/E)\alpha Z\{\beta_{AA}(\xi E-\tfrac{1}{2}qE+\tfrac{1}{3}p^2](\textstyle\int\boldsymbol{\sigma}\times\mathbf{r})^2\\&+(\mathrm{Re}\,\beta_{AV})[E(\textstyle\int\gamma_5\boldsymbol{\sigma}-\xi\int i\mathbf{r})+(Eq-\tfrac{1}{3}p^2)\int i\mathbf{r}]\int\boldsymbol{\sigma}\times\mathbf{r}-2\beta_{VV}[E(\int\gamma_5\boldsymbol{\sigma}+\xi\int i\mathbf{r})+\tfrac{1}{3}p^2\int i\mathbf{r}]\int i\mathbf{r}\},\end{aligned}\tag{28}$$

$$\begin{aligned}b_2(1,1;1,2)=&-i(\tfrac{1}{6}\sqrt{\tfrac{1}{2}})(qp/E)(\mathrm{Im}\,\beta_{AV})[(\xi+E-q)\textstyle\int i\mathbf{r}-\int\gamma_5\boldsymbol{\sigma}]\int\boldsymbol{\sigma}\times\mathbf{r}\\&+i(\tfrac{1}{6}\sqrt{\tfrac{1}{2}})(q/E)\alpha Z[\tfrac{1}{8}\beta_{AA}(3+E^2)(\textstyle\int\boldsymbol{\sigma}\times\mathbf{r})^2+(\mathrm{Re}\,\beta_{AV})\int\boldsymbol{\sigma}\times\mathbf{r}\int i\mathbf{r}-\tfrac{1}{2}\beta_{VV}p^2(\int i\mathbf{r})^2].\end{aligned}\tag{29}$$

If neither the initial nor final spin is zero we have additional terms from Table VII. Their structure is similar to Eqs. (26)–(29). They involve interferences with the nuclear matrix element $\int iB_{ij}$ of unique transitions and may be ignored whenever $\int iB_{ij}$ is expected to be small. If the transition is $\Delta J=0$ (yes) (except $0\to0$), we have in addition to Eqs. (26)–(29) a single contribution from Table IV which can give large effects as well as contributions from interferences with $\int iB_{ij}$ (Table V).

It is seen that no detailed quantitative statements can be made because of the large number of nuclear matrix elements involved. However, in all cases the time-reversal testing terms are at least by a factor $(\alpha Z)^{-1}$ larger than the unwanted terms. Hence observation, in light nuclei, of a reasonably large effect would imply violation of time reversal invariance.

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APPENDIX A

We write the β-decay interaction Hamiltonian in a multipole expansion[16] as follows:

$$H\equiv H(VA)+H(STP),\tag{A-1}$$

$$H(VA)=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\sum_{\epsilon=1}^{8}(-)^{L+M}(\psi_p{}^*\mathcal{O}_\epsilon{}^{L,-M}\psi_n)\times[\psi_e{}^*\mathcal{O}_\epsilon{}^{LM}(C_\epsilon+C_\epsilon{}'\gamma_5)\psi_\nu],\tag{A-2}$$

$$H(STP)=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\sum_{\epsilon=1}^{8}(-)^{L+M}(\psi_p{}^*\beta\mathcal{O}_\epsilon{}^{L,-M}\psi_n)\times[\psi_e{}^*\beta\mathcal{O}_\epsilon{}^{LM}(\bar{C}_\epsilon+\bar{C}_\epsilon{}'\gamma_5)\psi_\nu].\tag{A-3}$$

Not all values of ϵ contribute to transitions of given order of forbiddenness. In fact, in standard β theory $\epsilon=7$ and 8 are always neglected as small corrections.[17] The operators $\mathcal{O}^{LM}$ are listed in the second column of Table IX in the notation of reference 16. The next four columns of Table IX give the relation between the coupling constants in our notation and in that of Lee and Yang.[3] In the last two columns the operators $\mathcal{O}^{LM}$ involved in first forbidden transitions are given in the Cartesian notation according to the following definition:

$$N_\epsilon\textstyle\int\mathcal{O}_\epsilon{}^{\text{Cart.}}\equiv(4\pi)^{\frac{1}{2}}(2I_0+1)^{-\frac{1}{2}}\langle I_f\|\mathcal{O}_\epsilon{}^L\|I_0\rangle.\tag{A-4}$$

The spherical wave expansion of the electron wave function is given by[5]

$$|\mathbf{p}\tau\rangle=(4\pi)^{\frac{1}{2}}\sum_{\kappa m}[(2l+1)(2j+1)]^{\frac{1}{2}}(-)^{j+\tau}\times V(l\tfrac{1}{2}j;0\tau-\tau)e^{-i\Delta(\kappa)}\mathfrak{D}_{m\tau}{}^j(\mathbf{p})|\kappa m\rangle,\tag{A-5}$$

[16] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953), and reference 5.

[17] See, however, M. Gell-Mann, Phys. Rev. **111**, 363 (1958) and B. C. Carlson and G. B. Henton, Bull. Am. Phys. Soc. Ser. II, **2**, 358 (1957).

where

$$|\kappa m\rangle = i^l \begin{pmatrix} g_\kappa \chi_\kappa{}^m \\ i f_\kappa \chi_{-\kappa}{}^m \end{pmatrix},$$

$$\chi_\kappa{}^m = \sum_{m'+m''=m} (-)^{j+m}(2j+1)^{\frac{1}{2}} \times V(l\tfrac{1}{2}j;\, m''m'-m)\chi_{\frac{1}{2}}{}^{m'} Y_l{}^{m''},$$

$$\Delta(\kappa) = \tfrac{1}{2}\arg\left(\frac{-\kappa+i\Lambda}{\gamma_\kappa+i\Lambda E}\right) - \arg\Gamma(\gamma_\kappa+i\Lambda E) + \tfrac{1}{2}\pi[l(-\kappa)-\gamma_\kappa],$$

$$\Lambda \equiv \alpha Z/p, \quad \gamma_\kappa \equiv [\kappa^2-\alpha^2 Z^2]^{\frac{1}{2}}.$$

Here $l \equiv l(\kappa)$, τ is the spin component along $\mathbf{p}$, and $\mathfrak{D}_{m\tau}{}^j(\mathbf{p})$[18] describes the rotation from a fixed coordinate system to one whose z axis is along $\mathbf{p}$; $\chi_{\frac{1}{2}}{}^{m'}$ and $Y_l{}^{m''}$ are the usual two-component spinor and a sperical harmonic; V is the vector addition coefficient in the notation of Racah.[19] Finally g_κ and f_κ, the electron radial function, are (up to a phase) the same as defined by Rose[20] normalized in a box of unit volume. The wave function of the antineutrino is similar, with $\Delta(\kappa_\nu)=0$.

Any information desired about the β process may be obtained from the density matrix ρ:

$$\rho \equiv \rho(VA) + \rho(STP) + \rho(VA\text{-}STP), \qquad \text{(A-6)}$$

$$\rho(VA) = \sum_{\tau_\nu} \langle I_f M_f p\tau | H(VA) | I_0 M_0 q\tau_\nu\rangle \times \langle I_f M_f' p\tau' | H(VA) | I_0 M_0 q\tau_\nu\rangle^*, \qquad \text{(A-7)}$$

and $\rho(STP)$ is obtained by replacing in Eq. (A-7) both of the $H(VA)$ by $H(STP)$ whereas the interferences $\rho(VA\text{-}STP)$ are the sum of the two terms obtained by replacing first the first $H(VA)$ and then the second $H(VA)$ by $H(STP)$. Explicitly, $\rho(VA)$ is given by

$$\begin{aligned}\rho(VA) = (4\pi)^2 \sum (-)^{L+M+M_f-M_f'+1+l_\nu'+\mu-j'-\tau}(2k+1)(2k_e+1)(2k_\nu+1)\\ \times[(2l+1)(2l_\nu+1)(2j+1)(2j_\nu+1)(2l'+1)(2l_\nu'+1)(2j'+1)(2j_\nu'+1)]^{\frac{1}{2}} W(j_\nu k_\nu \tfrac{1}{2} l_\nu';\, j_\nu' l_\nu)\\ \times V(k_\nu l_\nu l_\nu';\, 000) V(l\tfrac{1}{2}j;\, 0\tau-\tau) V(l'\tfrac{1}{2}j';\, 0\tau'-\tau') V(jj'k_e;\, \tau-\tau's)\\ \times V(k_e k k_\nu;\, \mu\mu''-\mu') V(L'Lk;\, -M'M-\mu'') V(I_f I_0 L;\, -M_f M_0-M) V(I_f I_0 L';\, -M_f' M_0-M')\\ \times\begin{Bmatrix} k_e & k & k_\nu \\ j & L & j_\nu \\ j' & L' & j_\nu' \end{Bmatrix} \mathfrak{D}_{\mu s}{}^{k_e}(\mathbf{p}) \mathfrak{D}_{\mu' 0}{}^{k_\nu *}(-\mathbf{q}) B_\epsilon{}^L(\kappa\kappa_\nu) B_{\epsilon'}{}^{L'*}(\kappa'\kappa_\nu') e^{i\Delta(\kappa)-i\Delta(\kappa')}, \qquad \text{(A-8)}\end{aligned}$$

where the summation is over L, M, ϵ, κ, κ_ν, k_e, k_ν, k, L', M', ϵ', κ', κ_ν', μ, μ', μ'', and s, and

$$B_\epsilon{}^L(\kappa\kappa_\nu) \equiv \langle I_f \| \mathcal{O}_\epsilon{}^L \| I_0\rangle \langle\kappa\| \mathcal{O}_\epsilon{}^L (C_\epsilon + C_\epsilon'\gamma_5) \| \kappa_\nu\rangle, \qquad \text{(A-9)}$$

all reduced matrix elements being defined according to

$$\langle J'M' | \mathcal{O}^{JM} | J''M''\rangle = (-)^{J'+M'} V(J'J''J;\, -M'M''M) \times \langle J' \| \mathcal{O}^J \| J''\rangle. \qquad \text{(A-10)}$$

W is the Racah coefficient[19] and the 9-j symbol is given by

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = \sum_j (2j+1) W(j_{11} j_{21} j_{33} j_{32};\, j_{31} j) \times W(j_{12} j_{32} j_{23} j_{21};\, j_{22} j) \times W(j_{23} j_{33} j_{12} j_{11};\, j_{13} j). \qquad \text{(A-11)}$$

The phases of the lepton wave functions have been so chosen that $\langle\kappa\|\mathcal{O}_\epsilon{}^L\|\kappa_\nu\rangle$ and $\langle\kappa\|\mathcal{O}_\epsilon{}^L\gamma_5\|\kappa_\nu\rangle$ are real. Similarly, an appropriate choice of phases will make $\langle I_f\|\mathcal{O}_\epsilon{}^L\|I_0\rangle$ real if the nuclear wave functions are eigenstates of a Hamiltonian invariant under time reversal.

The distribution function correlating the electron momentum, the antineutrino momentum and the spin orientation of the parent nucleus is given by

$$\begin{aligned} W(I_0|\mathbf{p},\mathbf{q}) &= \sum_{\tau\tau' M_f M_f'} \delta(\tau\tau')\delta(M_f M_f')\rho \\ &= 4F(Z,E) \sum_{LL'k_e k_\nu k} (-)^{L+L'+k} \times h_k(I_0,M_0) F_k(L,L',I_f,I_0) \times b_k(L,L';\, k_e,k_\nu) Q_k(k_e,k_\nu), \qquad \text{(A-12)}\end{aligned}$$

where the z axis of the coordinate system has been taken along the orientation axis and

$$h_k(I_0,M_0) = (2I_0+1)^{\frac{1}{2}}(2k+1)^{\frac{1}{2}}(-)^{I_0+M_0} V(I_0 k I_0;\, M_0 0-M_0), \qquad \text{(A-13)}$$

$$F_k(L,L',I_f,I_0) = [(2I_0+1)(2k+1)(2L+1)(2L'+1)]^{\frac{1}{2}}(-)^{L+1} W(kL'I_0 I_f;\, LI_0) V(LL'k;\, 1-10), \qquad \text{(A-14)}$$

$$Q_k(k_e,k_\nu) = (2k_e+1)^{\frac{1}{2}}(2k_\nu+1)^{\frac{1}{2}} \sum_s (-)^{k_e+s} V(k_e k k_\nu;\, -s0s) \mathfrak{D}_{s0}{}^{k_e}(\mathbf{p}) \mathfrak{D}_{s0}{}^{k_\nu *}(-\mathbf{q}), \qquad \text{(A-15)}$$

[18] See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. IV.

[19] Giulio Racah, Phys. Rev. **62**, 438 (1942).

[20] M. E. Rose, Phys. Rev. **51**, 484 (1937). For finite nuclear size effects see, e.g., M. E. Rose and D. K. Holmes, Phys. Rev. **83**, 190 (1951).

$$b_k(L,L';k_e,k_\nu)=4\pi^2[F(Z,E)(2I_0+1)(2L+1)^{\frac{1}{2}}(2L'+1)^{\frac{1}{2}}V(LL'k;1-10)]^{-1}$$
$$\times(-)^{k+L+j+j_\nu}[(2k_e+1)(2k_\nu+1)(2j+1)(2j_\nu+1)(2l+1)(2l_\nu+1)(2j'+1)$$
$$\times(2j_\nu'+1)(2l'+1)(2l_\nu'+1)]^{\frac{1}{2}}V(k_e l l';000)V(k_\nu l_\nu l_\nu';000)W(jj'll';k_e\tfrac{1}{2})W(j_\nu j_\nu' l_\nu l_\nu';k_\nu\tfrac{1}{2})$$
$$\times\begin{Bmatrix} k_e & k_\nu & k \\ j & j_\nu & L \\ j' & j_\nu' & L' \end{Bmatrix} B_\epsilon{}^L(\kappa\kappa_\nu)B_{\epsilon'}{}^{L'*}(\kappa'\kappa_\nu')e^{i\Delta(\kappa)-i\Delta(\kappa')}. \quad \text{(A-16)}$$

When either L or L' equals zero, $V(LL'k;1-10)$ vanishes; the b_k in Tables III–VIII were computed for this case as if $V(0L'k;1-10)=V(L0k;1-10)=(-)^{k+1}$.

The same parameters b_k are involved when a correlation with the spin orientation of the daughter nucleus is observed. In practice this means a correlation with a subsequent gamma ray. Alder, Stech, and Winther[5] give for the density matrix ρ_γ of the gamma transition

$$\rho_\gamma=\sum_{k\lambda\lambda'}\left(\frac{2k+1}{2I_f+1}\right)^{\frac{1}{2}}(\tau_\gamma)^k(-)^{I_f-M_f}$$
$$\times V(I_fI_fk;M_f-M_f'0)F_k(\lambda,\lambda',I_{ff},I_f)\delta_\lambda\delta_{\lambda'}, \quad \text{(A-17)}$$

with the z axis of the coordinate system taken along the direction of propagation of the gamma. Then the distribution function correlating the electron momentum, the antineutrino momentum, and the direction and circular polarization of the gamma ray is given by

$$W(\mathbf{p}_\gamma,\tau_\gamma|\mathbf{p},\mathbf{q})=(2I_0+1)^{-1}\sum_{M_0M_fM_f'\tau\tau'}\delta(\tau\tau')\rho\rho_\gamma$$
$$=4F(Z,E)\sum_{LL'\lambda\lambda'k_ek_\nu k}(2I_f+1)^{-1}$$
$$\times(-\tau_\gamma)^k\delta_\lambda\delta_{\lambda'}F_k(\lambda,\lambda',I_{ff},I_f)$$
$$\times F_k(L,L',I_0,I_f)b_k(L,L';k_e,k_\nu)$$
$$\times Q_k(k_e,k_\nu). \quad \text{(A-18)}$$

APPENDIX B

In Appendix A a formula was given for $\rho(VA)$. It is clear that $\rho(STP)$ and $\rho(VA\text{-}STP)$ differ from $\rho(VA)$ only in the reduced matrix elements $B_\epsilon{}^L(\kappa\kappa_\nu)B_{\epsilon'}{}^{L'*}(\kappa'\kappa_\nu')$. For example, for $\rho(STP)$ these reduced matrix elements must be replaced by $\bar{B}_\epsilon{}^L(\kappa\kappa_\nu)\bar{B}_{\epsilon'}{}^{L'*}(\kappa'\kappa_\nu')$, where

$$\bar{B}_\epsilon{}^L(\kappa\kappa_\nu)\equiv\langle I_f\|\beta\mathcal{O}_\epsilon{}^L\|I_0\rangle\langle\kappa\|\beta\mathcal{O}_\epsilon{}^L(\bar{C}_\epsilon+\bar{C}_\epsilon'\gamma_5)\|\kappa_\nu\rangle. \quad \text{(B-1)}$$

The nuclear matrix elements are in general unknown, and the only thing that can be said about them without resorting to models is that, if β commutes with $\mathcal{O}_\epsilon{}^L$, then it is an excellent approximation to ignore it in $\langle I_f\|\beta\mathcal{O}_\epsilon{}^L\|I_0\rangle$ (this is the case for $\epsilon=1$, 3, 4, and 5). The lepton matrix elements, on the other hand, can be evaluated explicitly. One finds

$$\langle\kappa\|\mathcal{O}_\epsilon{}^L\|\kappa_\nu\rangle=X_\epsilon{}^L(\kappa,\kappa_\nu)g_\kappa f_{\kappa_\nu}$$
$$+X_\epsilon{}^L(-\kappa,-\kappa_\nu)f_\kappa g_{\kappa_\nu}, \quad \text{(B-2)}$$

$$\langle\kappa\|\mathcal{O}_\epsilon{}^L\gamma_5\|\kappa_\nu\rangle=Y_\epsilon{}^L(\kappa,\kappa_\nu)g_\kappa g_{\kappa_\nu}$$
$$-Y_\epsilon{}^L(-\kappa,-\kappa_\nu)f_\kappa f_{\kappa_\nu}, \quad \text{(B-3)}$$

where $X_\epsilon{}^L(\kappa,\kappa_\nu)$, $Y_\epsilon{}^L(\kappa,\kappa_\nu)$ are independent of the radial functions f and g. Since g_{κ_ν} and $f_{-\kappa_\nu}$ are equal to each other (up to a sign) and the coefficient of $B_\epsilon{}^L(\kappa\kappa_\nu)B_{\epsilon'}{}^{L'*}(\kappa'\kappa_\nu')\exp i[\Delta(\kappa)-\Delta(\kappa')]$ in the expression for $\rho(VA)$ [Eq. (A-8)] is invariant (up to a sign) under the substitution $(\kappa,\kappa_\nu,\kappa',\kappa_\nu')\to-(\kappa,\kappa_\nu,\kappa',\kappa_\nu')$, it follows that the dependence of $\rho(VA)$ on the electron radial functions can be expressed in terms of the four functions K, L, M, and N defined by Eq. (6).

Since $\beta|\mathbf{p}\tau\rangle$ is the same as $|\mathbf{p}\tau\rangle$ with f_κ replaced by $-f_\kappa$ [see Eq. (A-5)], we can formulate a substitution law relating $\rho(VA)$, $\rho(STP)$ and $\rho(VA\text{-}STP)$ as follows: whenever $\rho(VA)$ contains a combination of coupling constants, nuclear matrix elements, and electron radial functions given in column one of Table X, $\rho(STP)$ contains the corresponding entry given in column two, and $\rho(VA\text{-}STP)$ contains the corresponding entry given in column three. Furthermore, the same substitution law holds if either one Γ or both in column one is replaced by Γ' (with corresponding replacements in columns two and three). Γ_ϵ and Γ_ϵ' are defined by Eq. (5); the corresponding definition for $\bar{\Gamma}_\epsilon$ and $\bar{\Gamma}_\epsilon'$ is given by

$$\bar{\Gamma}_\epsilon=\bar{C}_\epsilon\int\beta\mathcal{O}_\epsilon{}^{\text{Cart.}};\quad\bar{\Gamma}_\epsilon'=\bar{C}_\epsilon'\int\beta\mathcal{O}_\epsilon{}^{\text{Cart.}}. \quad \text{(B-4)}$$

Several special cases of this substitution law have been given in the literature.[21] Thus, if a given experiment involves, for specified values of ϵ and η, only the entries in the first row of Table X, *or* only the entries in the third row, then it will not distinguish between a pure VA or pure STP interaction law for β decay.[22] This, for example, is the case in experiments involving allowed β transitions in which the neutrino momentum (i.e., recoil) is not observed.[23] This particular result, as well as some more general ones relating $\rho(VA)$ and $\rho(STP)$, follows much simpler from the observation:

[21] R. R. Lewis and R. B. Curtis, Phys. Rev. **110**, 910 (1958) and references cited therein.
[22] See Sec. IV for a more rigorous statement.
[23] It should perhaps be emphasized that in forbidden transitions observation of the neutrino momentum is not necessarily required to distinguish between VA and STP; see, e.g., 0→0 (yes) transitions, spectrum shape.

$\beta\psi_\nu(\mathbf{q})=\psi_\nu(-\mathbf{q})$. The advantage of the present formulation consists in permitting one to relate in a simple manner the interference $\rho(VA\text{-}STP)$ to the pure $\rho(VA)$ and $\rho(STP)$ in addition to relating $\rho(VA)$ to $\rho(STP)$.

The substitution law, as formulated so far, would be rigorously correct if both $\mathcal{O}_\epsilon^{\text{Cart.}}$ and $\beta\mathcal{O}_\epsilon^{\text{Cart.}}$ could be treated in the same manner, i.e., as adjustable parameters independent of the lepton variables. This is true to a high degree of accuracy except in first forbidden transitions with $\epsilon=2$.[24] Although the axial vector contribution $\int\gamma_5$ can be treated as an adjustable parameter independent of the lepton variables, the corresponding pseudoscalar contribution $\int\beta\gamma_5$ cannot. By application of the Foldy-Wouthuysen[25] transformation, $\int\beta\gamma_5$ can be transformed into $i(2M)^{-1}\int\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}$, where M= nucleon mass and the gradient operator acts on the lepton wave functions only. Thus, in the absence of Coulomb corrections, the contribution of the pseudoscalar to first forbidden transitions is of the order of magnitude of third forbidden transitions. Coulomb effects change this result and one finds that in order to obtain from P contributions of the same order as those from T or A, it is necessary that $(C_P/C_X)\simeq(M/\xi)$ where $X=A$ or T. In the present work we assume instead that $C_P\simeq C_T\simeq C_A$ and consequently the pseudoscalar's contribution is in general ignorable. In those few cases where the pseudoscalar's contribution is of the same order as other terms considered, it was evaluated separately and not by use of the substitution law.

APPENDIX C

Since in β decay the assumption $pR\ll1$, $qR\ll1$ (R= nuclear radius) is well justified, we shall calculate the radial functions in that approximation. For the antineutrino we have (omitting the subscript ν on κ and l)

$$g_\kappa=(1/\sqrt{2})j_l(qR)\rightarrow(1/\sqrt{2})(qR)^l/(2l+1)!!, \tag{C-1}$$

$$f_{-\kappa}=(1/\sqrt{2})S(-\kappa)j_l(qR)\rightarrow (1/\sqrt{2})S(-\kappa)(qR)^l/(2l+1)!!, \tag{C-2}$$

where $(2l+1)!!=(2l+1)(2l-1)\cdots3\cdot1$ and $S(\kappa)$= sign of κ.

For the electron we take the radial functions as given by Rose,[20] except that they are multiplied by $S(\kappa)$ and normalized in a box of unit volume. If one ignores finite-nuclear-size effects (and in the approximation $pR\ll1$), the combination of radial functions and Coulomb phase shifts of interest is given by

$$\begin{pmatrix} f_\kappa \\ g_\kappa \end{pmatrix} e^{i\Delta(\kappa)}\rightarrow(2pR)^{\gamma_\kappa-1}\left(\frac{1\mp E}{2E}\right)^{\frac{1}{2}}\frac{\Gamma(\gamma_\kappa-i\Lambda E)}{\Gamma(2\gamma_\kappa+2)} \times\exp\{(\pi/2)[\Lambda E+i(l-1-\gamma_\kappa)]\} \times\{(1+2\gamma_\kappa+ipR)(-\kappa+i\Lambda) \mp(1+2\gamma_\kappa-ipR)(\gamma_\kappa-i\Lambda E)\}, \tag{C-3}$$

where the notation is the same as explained following Eq. (A-5).

From Eq. (C-3) the functions $K_{\kappa\kappa'}$, $L_{\kappa\kappa'}$, $M_{\kappa\kappa'}$, $N_{\kappa\kappa'}$ defined by Eq. (6) may be obtained for any values of the subscripts that may be needed. In many cases it is sufficient to have these functions in the approximation $(\alpha Z)^2\ll1$ [but $\xi=(\alpha Z)/(2R)\gtrsim1$] and we therefore list them explicitly in this approximation. Three cases have to be distinguished: (1) both κ and κ' are negative integers, (2) κ (or κ') is a positive integer and κ' (or κ) is a negative integer, and (3) both κ and κ' are positive integers. In the first case we have only terms proportional to the zeroth power of ξ, in the second case the first power of ξ appears, and in the third case the second power of ξ also appears.

Let m and n be positive integers. Then

$$K_{-m-n}\rightarrow S_{mn}(2pR)^{l(-m)+l(-n)}[4mn+2i\Lambda(m-n)p^2/E], \tag{C-4}$$

$$L_{-m-n}\rightarrow S_{mn}(2pR)^{l(-m)+l(-n)}4mn/E, \tag{C-5}$$

$$M_{-m-n}\rightarrow S_{mn}(2pR)^{l(-m)+l(-n)}[4mnp/E+2i\Lambda(m-n)p], \tag{C-6}$$

$$N_{-m-n}\rightarrow S_{mn}(2pR)^{l(-m)+l(-n)}2i\Lambda(m+n)p/E, \tag{C-7}$$

$$K_{m-n}\rightarrow S_{mn}(2pR)^{l(m)+l(-n)}\left[2n\frac{\xi}{E}+\frac{2mn}{2m+1}+i\Lambda\left(\frac{m-n}{m}\xi+\frac{m-n}{2m+1}\frac{p^2}{E}-\frac{2n}{2m+1}\frac{1}{E}\right)\right], \tag{C-8}$$

$$L_{m-n}\rightarrow S_{mn}(2pR)^{l(m)+l(-n)}\left[\frac{2mn}{2m+1}\frac{1}{E}-i\Lambda\left(\frac{m+n}{m}\frac{\xi}{E}+\frac{2n}{2m+1}\right)\right], \tag{C-9}$$

$$M_{m-n}\rightarrow S_{mn}(2pR)^{l(m)+l(-n)}\left[2n\frac{\xi}{p}+\frac{2mn}{2m+1}\frac{p}{E}+i\Lambda(m-n)\left(\frac{1}{m}\frac{\xi p}{E}+\frac{1}{2m+1}p\right)\right], \tag{C-10}$$

[24] M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954).
[25] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

$$N_{m-n}\to S_{mn}(2pR)^{l(m)+l(-n)}\left[-2n\frac{\xi}{pE}+i\Lambda\frac{m-n}{2m+1}\frac{p}{E}\right], \tag{C-11}$$

$$K_{mn}\to S_{mn}(2pR)^{l(m)+l(n)}\left\{\frac{\xi^2}{p^2}+\left(\frac{m}{2m+1}+\frac{n}{2n+1}\right)\frac{\xi}{E}+\frac{mn}{(2m+1)(2n+1)}\right.$$
$$\left.+\tfrac{1}{2}i\Lambda(m-n)\left[\frac{1}{mn}\xi^2+\left(\frac{1}{(2m+1)n}+\frac{1}{(2n+1)m}\right)\xi E+\frac{1}{(2m+1)(2n+1)}(E^2+1)\right]\frac{1}{E}\right\}, \tag{C-12}$$

$$L_{mn}\to S_{mn}(2pR)^{l(m)+l(n)}\left\{-\frac{\xi^2}{p^2}+\frac{mn}{(2m+1)(2n+1)}+\tfrac{1}{2}i\Lambda(m-n)\right.$$
$$\left.\times\left[\left(\frac{1}{(2m+1)n}+\frac{1}{(2n+1)m}\right)\xi+\frac{2}{(2m+1)(2n+1)}E\right]\right\}\frac{1}{E}, \tag{C-13}$$

$$M_{mn}\to S_{mn}(2pR)^{l(m)+l(n)}\left\{\xi^2+\left(\frac{m}{2m+1}+\frac{n}{2n+1}\right)\xi E+\frac{mn}{(2m+1)(2n+1)}p^2+\tfrac{1}{2}i\Lambda(m-n)\right.$$
$$\left.\times\left[\frac{1}{mn}\xi^2E+\frac{1}{(2m+1)(2n+1)}(4E^2-p^2)\xi+\frac{1}{(2m+1)(2n+1)}p^2E\right]\right\}\frac{1}{pE}, \tag{C-14}$$

$$N_{mn}\to S_{mn}(2pR)^{l(m)+l(n)}\left\{\frac{m-n}{(2m+1)(2n+1)}\xi-\tfrac{1}{2}i\Lambda\right.$$
$$\left.\times\left[\left(\frac{1}{m}+\frac{1}{n}\right)\xi^2+4\frac{m+n+1}{(2m+1)(2n+1)}\xi E+\frac{m+n}{(2m+1)(2n+1)}p^2\right]\right\}\frac{1}{pE}, \tag{C-15}$$

where

$$S_{mn}=\frac{(m-1)!(n-1)!}{(2m)!(2n)!}\{1-i\Lambda E[\psi_1(m)-\psi_1(n)]\}, \tag{C-16}$$

$$\psi_1(m)\equiv\Gamma'(m)/\Gamma(m),\quad \psi_1(m+1)=\psi_1(m)+1/m. \tag{C-17}$$

Certain of the functions discussed in this appendix can be found in the literature in various notations. The relation between our notation and, for example, that of Rose[24] is

$$\begin{aligned} &L_{k-1}=R^{2-2k}K_{-k-k}, && P_{k-1}=R^{2-2k}L_{-k-k},\\ &M_{k-1}=R^{-2k}K_{kk}, && Q_{k-1}=R^{-2k}L_{kk},\\ &N_{k-1}=-R^{1-2k}M_{-kk}, && R_{k-1}=-R^{1-2k}N_{-kk},\end{aligned} \tag{C-18}$$

and for the additional functions of, for example, Curtis and Lewis,[1]

$$L_{k-1}'=R^{2-2k}M_{-k-k},\qquad M_{k-1}'=R^{-2k}M_{kk},$$
$$N_{k-1}'=-R^{1-2k}\,\mathrm{Re}\,K_{-kk},\quad R_{k-1}'=R^{1-2k}\,\mathrm{Im}\,K_{-kk}.$$

APPENDIX D

Consider the β interaction in the approximation $Z=0$. The density matrix ρ summed over the lepton spins is readily obtained by the standard trace methods. For example, for a pure A interaction law one has

$$\begin{aligned}\rho=\;&i\beta_{AA}\mathbf{M}(\boldsymbol{\sigma})\times\mathbf{M}(\boldsymbol{\sigma})\cdot(\mathbf{q}-\mathbf{p}p/E)\\ &+\alpha_{AA}[\mathbf{M}(\boldsymbol{\sigma})\cdot\mathbf{M}(\boldsymbol{\sigma})(\mathbf{p}\cdot\mathbf{q}p/E-1)\\ &-2\mathbf{M}(\boldsymbol{\sigma})\cdot\mathbf{q}\;\mathbf{M}(\boldsymbol{\sigma})\cdot\mathbf{p}p/E\\ &-2M(\gamma_5)\mathbf{M}(\boldsymbol{\sigma})\cdot(\mathbf{q}+\mathbf{p}p/E)\\ &-M(\gamma_5)M(\gamma_5)(\mathbf{p}\cdot\mathbf{q}p/E+1)].\end{aligned} \tag{D-1}$$

Equation (D-1) contributes to all orders of forbiddenness, no retardation expansion having been carried out. The nuclear matrix elements $\mathbf{M}(\boldsymbol{\sigma})$, $M(\gamma_5)$ are assumed real[6]; they are defined by

$$\begin{aligned}\mathbf{M}(\mathbf{O})&\equiv\langle\psi_p{}^*|\,\mathbf{O}e^{i\mathbf{P}\cdot\mathbf{r}}|\psi_n\rangle,\\ M(O)&\equiv\langle\psi_p{}^*|\,Oe^{i\mathbf{P}\cdot\mathbf{r}}|\psi_n\rangle,\end{aligned} \tag{D-2}$$

where $\mathbf{P}$ is the recoil momentum.

Only the β_{AA} term of Eq. (D-1) contributes to a resonance fluorescence experiment of the type performed by Goldhaber *et al.*[9] More generally, if the interaction law is pure VA or pure ST the following terms can contribute:

$$i[\beta_{VV}\mathbf{M}(\boldsymbol{\alpha})\times\mathbf{M}(\boldsymbol{\alpha})+\beta_{AA}\mathbf{M}(\boldsymbol{\sigma}) \times\mathbf{M}(\boldsymbol{\sigma})]\cdot(\mathbf{q}-\mathbf{p}p/E)-(\beta_{VA}+\beta_{AV}) \times[M(\gamma_5)\mathbf{M}(\boldsymbol{\alpha})+M(1)\mathbf{M}(\boldsymbol{\sigma})]\cdot(\mathbf{q}+\mathbf{p}p/E), \quad \text{(D-3)}$$

or

$$-i\beta_{TT}[\mathbf{M}(\beta\boldsymbol{\alpha})\times\mathbf{M}(\beta\boldsymbol{\alpha})+\mathbf{M}(\beta\boldsymbol{\sigma}) \times\mathbf{M}(\beta\boldsymbol{\sigma})]\cdot(\mathbf{q}+\mathbf{p}p/E)+(\beta_{ST}+\beta_{TS}) \times M(\beta)\mathbf{M}(\beta\boldsymbol{\sigma})\cdot(\mathbf{q}-\mathbf{p}p/E). \quad \text{(D-4)}$$

For application to a β emitter, Eqs. (D-3) and (D-4) must be integrated over the lepton momenta for a fixed direction and magnitude of the recoil $\mathbf{P}$, followed by an integration over $\mathbf{P}$ subject to the resonance condition. For these purposes it is convenient to write the phase space factor as

$$(2\pi)^{-6}P^2dPd\Omega_P d\omega,$$
$$d\omega\equiv q^2(dq/dW)d\Omega_q=q^2E(W-E_P+P\cos\theta_q)^{-1}d\Omega_q, \quad \text{(D-5)}$$

where W is the total energy available, E_P is the total energy of the recoil, and θ_q is the antineutrino angle measured relative to the recoil. Expressing all quantities in terms of θ_q and P, we find[26]

$$\int\mathbf{q}d\omega=\int\mathbf{p}(p/E)d\omega$$
$$=\frac{\pi}{3}\mathbf{P}(W-E_P)\left[1+\frac{2m^2}{(W-E_P)^2-P^2}\right]\times\left[1-\frac{m^2}{(W-E_P)^2-P^2}\right]^2. \quad \text{(D-6)}$$

Hence, for example, unique transitions will give vanishing circular polarization for the resonant fluorescent gamma if the interaction law is VA, large polarization if the law is ST. On the other hand, allowed transitions in which the Gamow-Teller-Fermi interference dominates the Gamow-Teller part would give the opposite result.

In practice Coulomb effects should be taken into account and the considerations of this appendix appropriately modified, as discussed at the end of Sec. IV.

[26] In this appendix only, to avoid confusion, we do not set the electron mass m equal to unity.