Electron Number Distribution in Electron-Photon Showers

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Numerical results for the number distribution of electrons above a given energy due to primary electrons and photons are given. The average numbers derived from the distributions are also given. Values are presented both for the case of approximation A and for more accurate cross sections at low energies in air. Collision losses and Compton effect are taken into account but scattering at low energies is neglected. The calculations were carried out using Monte Carlo methods on the electronic digital computer SILLIAC. The results of the calculations are discussed and compared with previous work on the subject.

1. INTRODUCTION

MUCH theoretical work has been done on cascades since Bhabha and Heitler[1] first suggested that the formation of an electron-photon cascade is an important factor in the penetration of electrons through matter. This work has been roughly divided into two branches, one dealing with the one-dimensional longitudinal development which considers the number development of shower particles, the other with the three-dimensional angular and radial spread of a cascade shower as it penetrates matter. Only the number fluctuation problem is considered in the present paper.

The analytical solution of the fluctuation problem in cosmic-ray shower theory has been presented in a series of publications.[2–6] The formal solution is effected when we have obtained an expression, however complicated, for the distribution function Φ which satisfies the diffusion equations describing the multiplication of particles in the cascade. Physically Φ represents the probability of finding a given number of particles of a particular kind, above a given energy, at a certain depth below the cascade origin, the cascade having been originated by a given type of particle with a particular primary energy.

Two approximations, usually referred to as approximations A and B, are commonly used in dealing with an electromagnetic cascade. In approximation A, full screening cross sections are used for bremsstrahlung and pair production; other processes are completely neglected. Ionization loss, assumed to be at a constant rate, is introduced for approximation B but no other change is made. Even in the case of these two approximations the diffusion equations for Φ are of considerable complexity and their solution[2–6] of such a complex nature that it is well-nigh impossible to evaluate the results numerically.

The best that one has been able to do to date numerically is to evaluate the first two moments in approximation A,[7] and the first moment in the case of approximation B.[8] The first few moments do give some useful information about the probability distribution, such as average numbers, spread, symmetry, and so forth. However, the lack of any realistic numerical tables for Φ has made the qualitative interpretation of many cosmic-ray and machine experiments next to impossible, and in many cases has undoubtedly led to results of dubious value.

Attempts have also been made to reconstruct the distribution function Φ from its first two moments. Arley[9] in his book on the electron-photon distribution *assumed* that Φ was a Polya distribution, used the calculated first moment of the electron-photon distribution and a *guessed* value of the second moment to fit the two Polya parameters. Messel[10] likewise *assumed* that the distribution was Polya, but rather than using a guessed value for the second moment, used the calculated values obtained by Janossy and Messel.[7] Perhaps the only comment one can make about these attempts is that they were the best one could do in the circumstances. One could not expect, even in the range where calculation of the first and higher moments are reliable, to get sufficiently precise information about Φ.

Aside from these comments, it is also clear that the meager results on either the moments or the distribution itself, based on approximation A and B, are likely to be in great error at low energies, where the pair-production cross section and the rate of bremsstrahlung radiation become smaller and where the Compton effect plays an important role.

These considerations have led us to a full-scale attack on obtaining numerical results for Φ using the automatic

* Also supported by the Nuclear Research Foundation within the University of Sydney.

[1] H. S. Bhabha and W. Heitler, Proc. Roy. Soc. (London) **A159**, 423 (1937).
[2] H. Messel, Proc. Phys. Soc. (London) **A65**, 465 (1952).
[3] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) **A65**, 473 (1952).
[4] H. Messel and R. B. Potts, Phys. Rev. **86**, 847 (1952).
[5] H. Messel and R. B. Potts, Phys. Rev. **87**, 759 (1952).
[6] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) **A65**, 854 (1952).
[7] L. Janossy and H. Messel, Proc. Phys. Soc. (London) **A63**, 1101 (1950).
[8] H. S. Bhabha and S. K. Chakrabarty, Phys. Rev. **74**, 1352 (1948).
[9] N. Arley, *On the Theory of Stochastic Processes* (Gads Forlag, Copenhagen, 1943).
[10] H. Messel, Proc. Phys. Soc. (London) **A64**, 807 (1951).

digital computer SILLIAC. The present paper presents the results of our calculation so far.

We have been successful to date in evaluating Φ in the cases of electrons being produced in air by primary electrons and photons, both in approximation A and in the case where accurate cross sections for bremsstrahlung and pair production are used. In addition both collision losses and Compton effect are taken into account; however, scattering at low energies is neglected.

In Secs. 2 and 3 we give the relevant details of the method used; in Sec. 4, results and their discussion are presented. The SILLIAC is now being used six hours daily in the evaluation of Φ for a wide variety of cases and in different absorbers. During the next twelve months most of the cases of interest to experimentalists will have been completed.

2. MONTE CARLO METHOD IN CASCADE SHOWER CALCULATIONS

The Monte Carlo or simulation method is the device of constructing an artificial statistical model whose statistics are known to bear a relation to the problem under consideration and by sampling from the model to study the properties of the relevant statistics and thus of the original problem. In our case, as the problem is itself a stochastic one, it is convenient to preserve a close analogy between the model and the physical problem.

Thus our model would be of "electrons" and "photons" that have numbers associated with them representing the position, orientation, and energy of the "particles." Each "particle" would have a certain probability of suffering a given type of "collision" while traversing a given "distance."

All the "particles" that are produced are considered in turn, and the whole process continues until all "particles" have moved outside the region of "energies" and "positions" that interest us. At the end of each "shower" the number of "electrons," say, that had "energies" above a given value at a given "depth" is assessed and added into a histogram.

A new "shower" with a specified "primary particle" is started and the whole process is repeated sufficiently often for a given accuracy to be obtained.

The difficulty that most obviously arises in doing as we have described is in keeping track of all the "particles" that are produced. While one "particle" is under consideration the appropriate information about the other ones must be stored in the limited space of the computer. However, the method we use overcomes in a convenient way all the obvious difficulties that might arise.

A directory is kept of all particles that are to be considered. When, because of a collision, two particles replace one, the particle of higher energy has its information stored at the end of the directory, and the other is the next considered. When a particle is to be taken from the directory the last one in the directory is always chosen. It is easily seen that doing as we have described puts a limit on the total number of sets of entries that will ever be needed in the directory, and in fact the number cannot exceed the integral part of $\log_2(E_0/E)$, where E_0 is the energy of the primary particle and E is the energy below which particles are no longer considered.

3. SIMULATING THE PHYSICAL PROCESSES

The cross sections for bremsstrahlung and pair production are taken to be as follows (see Rossi[11]):

$$\Phi_1(E,\epsilon)=\frac{4\alpha NZ^2r_0^2\rho}{A}\frac{1}{\epsilon}\{[1+(1-\epsilon)^2][\tfrac{1}{4}f_1(\gamma)-\tfrac{1}{3}\ln Z]-\tfrac{2}{3}(1-\epsilon)[\tfrac{1}{4}f_2(\gamma)-\tfrac{1}{3}\ln Z]\}, \quad (1)$$

$$\Phi_2(E,\epsilon)=\frac{4\alpha NZ^2r_0^2\rho}{A}\{[\epsilon^2+(1-\epsilon)^2][\tfrac{1}{4}f_1(\gamma)-\tfrac{1}{3}\ln Z]+\tfrac{2}{3}\epsilon(1-\epsilon)[\tfrac{1}{4}f_2(\gamma)-\tfrac{1}{3}\ln Z]\}, \quad (2)$$

where Φ_1 is the differential cross section per unit distance for bremsstrahlung, that is for the production of a photon of energy ϵE by an electron of primary energy E. Φ_2 is the differential cross section per unit distance for a photon of energy E producing a pair of electrons one of which has energy ϵE. α, N, and r_0 are, respectively, the fine structure constant, Avogadro's number, and the classical electron radius e^2/m (m is expressed in energy units throughout). A and Z are the atomic weight and number of the absorbing medium. ρ is the density of the absorber. γ is a variable that depends on the influence of the screening effect of the outer electrons on the electromagnetic interaction, being given in the bremsstrahlung case by

$$\gamma=100\frac{m}{E}\frac{\epsilon}{1-\epsilon}Z^{-\frac{1}{3}}, \quad (3)$$

and in the pair production case by

$$\gamma=100\frac{m}{E}\frac{1}{\epsilon(1-\epsilon)}Z^{-\frac{1}{3}}. \quad (4)$$

f_1 and f_2 are numerically tabulated functions which become, in the full screening case ($\gamma=0$),

$$f_1(0)=4\ln 183,$$
$$f_2(0)=f_1(0)-\tfrac{2}{3}.$$

For convenience we use the variable $\delta=(136\gamma/100)$ instead of γ. We have found that f_1 and f_2 can be closely approximated as follows:

For $\delta\leqslant 1$,

$$f_1=20.838-3.600\gamma-2.767\gamma^2+6.081\gamma^3-2.943\gamma^4$$
$$=20.838-2.647\delta-1.496\delta^2+2.417\delta^3-0.860\delta^4, \quad (5)$$

$$f_2=20.170-1.599\gamma-4.405\gamma^2+5.275\gamma^3-1.661\gamma^4$$
$$=20.170-1.176\delta-2.382\delta^2+2.097\delta^3-0.486\delta^4; \quad (6)$$

[11] B. Rossi, *High-Energy Particles* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1952).

for $\delta>1$

$$f_1=f_2=19.80-4.184\ln(\gamma+0.695)$$
$$=21.12-4.184\ln(\delta+0.952). \quad (7)$$

If we measure distances in terms of radiation lengths, where a radiation length R is defined by

$$1/R=4\alpha NZ^2r_0^2\rho\ln(183Z^{-\frac{1}{3}})/A,$$

then the cross sections become

$$\phi_1(E,\epsilon)=\Phi_1(E,\epsilon)R$$
$$=\left[\ln 2\left(\tfrac{4}{3}+\frac{1}{9\ln(183Z^{-\frac{1}{3}})}\right)\right]\left[\frac{1}{\ln 2}\left(\frac{1-\epsilon}{\epsilon}\right)\right]$$
$$\times[A(\delta)]+[\tfrac{1}{2}][2\epsilon][B(\delta)], \quad (8)$$

$$\phi_2(E,\epsilon)=\Phi_2(E,\epsilon)R$$
$$=\left[\tfrac{2}{3}-\frac{1}{36\ln(183Z^{-\frac{1}{3}})}\right][1][C(\delta)]$$
$$+\left[\tfrac{1}{12}\left(\tfrac{4}{3}+\frac{1}{9\ln(183Z^{-\frac{1}{3}})}\right)\right]$$
$$\times[12(\epsilon-\tfrac{1}{2})^2][A(\delta)], \quad (9)$$

where

$$A(\delta)=[9f_1(\delta)-3f_2(\delta)-8\ln Z]/[9f_1(0)-3f_2(0)-8\ln Z], \quad (10)$$

$$B(\delta)=[3f_1(\delta)-4\ln Z]/[3f_1(0)-4\ln Z], \quad (11)$$

$$C(\delta)=[9f_1(\delta)+3f_2(\delta)-16\ln Z]/[9f_1(0)+f_2(0)-16\ln Z]. \quad (12)$$

The reason for the factorization used in Eqs. (8), (9) will be apparent later, but we may observe here that the first factor in each case is a constant, and the last is a factor not greater than unity which alone depends on the absolute energy and becomes unity in the full screening approximation.

In each case but one, the middle factor is normalized in the sense that its integral from 0 to 1 is unity. The "infrared catastrophe" term in the bremsstrahlung cross section is "normalized" in a sense that will be explained later.

For the time being we may ignore the existence of processes other than radiation and materialization and consider how we might simulate the physical events that may take place.

To begin with, we clearly need a method of generating some random event in a computer. In the computer SILLIAC, as in most computers, every operation the machine can perform is completely deterministic. Hence it is necessary to use some sequence of operations which produces numbers which, though precisely determined from some initial set of conditions, have many of the appearances of randomness. For convenience both in generating these numbers and using them it is convenient to seek numbers that appear to be uniformly distributed.

The method used to generate such numbers in the SILLIAC was to take a certain linear combination of the preceding four random numbers and a number formed by permuting the digits of the fifth from last number as the next member of the sequence at each step. Various tests applied to these numbers give results completely acceptable under the hypothesis that these numbers are uniformly distributed and independent.

In what follows, it is assumed that a supply of random numbers uniformly distributed between 0 and 1 is always available. We shall denote such numbers by the symbols ξ, ξ_1, ξ_2, $\cdots$ as required.

A theorem that makes it possible in principle to write down a rule for computing variates from any distribution is that, if $F(x)$ be the cumulative distribution function of a variate x, then $F(x)$ is itself distributed uniformly. $F(x)$ is defined by the property that if x is sampled from the corresponding distribution, the probability that the value of x will be less than x' is $F(x')$. $F(x)$ is a monotonically increasing function of x and hence

$$\text{Prob}[F(x)<F(x')]=\text{Prob}(x<x')$$
$$=F(x'),$$

$F(x)$ is therefore uniformly distributed between 0 and 1. Thus we may draw variates from this distribution by drawing a uniform random number ξ, and solving the equation

$$F(x)=\xi.$$

In the important case of the exponential distribution,

$$F(x)=1-\exp(-x/\bar{x}),$$

the general method yields

$$x=\bar{x}\ln[1/(1-\xi)],$$

or, more simply, replacing $1-\xi$ by ξ,

$$x=\bar{x}\ln(1/\xi).$$

This method is the one used to compute exponential variates in our program. However, as a rule, it is advantageous to proceed less directly. Two general methods sometimes called "the composition method" and "the rejection method"[12] may be combined in the applications we have made of them to the following (see Appendix).

If the frequency function of a distribution is $f(x)$ and if $f(x)$ can be written

$$f(x)=\sum_{i=1}^{n}\alpha_i f_i(x)g_i(x), \quad (13)$$

[12] J. W. Butler, *Symposium on Monte Carlo Methods* (John Wiley and Sons, Inc., New York, 1956), p. 249.

where

$$0\leqslant g_i(x)\leqslant 1,$$
$$\alpha_i>0,$$
$$f_i(x)\geqslant 0,$$
$$\int f_i(x)dx=1,$$

then samples may be drawn from the required distribution by first choosing an integer from 1 to n with probability proportional to α_i of being i. This may be done by drawing a uniform variate ξ and forming in turn

$$x_1=\xi-\frac{\alpha_1}{\alpha_1+\alpha_2+\cdots\alpha_n},$$
$$x_2=x_1-\frac{\alpha_2}{\alpha_1+\alpha_2+\cdots\alpha_n},$$
$$x_3=x_2-\frac{\alpha_3}{\alpha_1+\alpha_2+\cdots\alpha_n},$$

and terminating the process for the first x_i that becomes negative and choosing the corresponding i. Having selected the value of i, a sample is drawn from the distribution function with frequency function $f_i(x)$, and the value is accepted or rejected by computing the value of $g_i(x)$ and comparing with a random variate ξ. If

$$\xi>g_i(x), \text{ reject } x;$$
$$\xi\leqslant g_i(x), \text{ accept } x;$$

so that the probability of accepting x is $g_i(x)$.

In the case of rejection the process is repeated again and again until a value is finally accepted.

The value of the method lies in the fact that a decomposition of the frequency function as in (13) can usually be found, without much difficulty, such that n is a low number, $f_i(x)$ corresponds to a distribution easily sampled, and $g_i(x)$ is never much less than 1 (so that rejection of a sample is a rare event).

Examples of distributions easily sampled are those with frequency functions of the form

$$f(x)=\frac{(m+n+1)!}{m!n!}x^m(1-x)^n, \quad 0<x<1,$$

where m and n are positive integers, for such a distribution may be realized by drawing $m+n+1$ uniform random numbers, ordering them in increasing order, say,

$$\xi_1\leqslant\xi_2\leqslant\xi_3\leqslant\cdots\leqslant\xi_{m+n+1},$$

and selecting as the value of x the number ξ_{m+1}. For the joint frequency function of $\xi_1, \xi_2, \cdots\xi_{m+n+1}$ is $(m+n+1)!$ and hence the frequency function of $x=\xi_{m+1}$ is

$$f(x)=(m+n+1)!\int_0^x d\xi_1\int_{\xi_1}^x d\xi_2\cdots\int_{\xi_{m-1}}^x d\xi_m \times\int_x^1 d\xi_{m+2}\int_{\xi_{m+2}}^1 d\xi_{m+3}\cdots\int_{\xi_{m+n}}^1 d\xi_{m+n+1}$$
$$=\frac{(m+n+1)!}{m!n!}x^m(1-x)^n.$$

The reason for the factorization of the pair production cross section (9) is now obvious, for we may set

$$\alpha_1=\tfrac{2}{3}-1/36\ln(183Z^{-\frac{1}{2}}),$$
$$\alpha_2=\tfrac{1}{12}[\tfrac{4}{3}+1/9\ln(183Z^{-\frac{1}{2}})],$$
$$f_1=1,$$
$$f_2=12(\epsilon-\tfrac{1}{2})^2,$$
$$g_1=C(\delta),$$
$$g_2=A(\delta).$$

The distribution corresponding to f_1 is simply the uniform distribution, and that corresponding to f_2 may be sampled by drawing three uniform random numbers and selecting the one that differs most from one half.

For bremsstrahlung, the cross section is of course not so simple. An artificial cutoff must be introduced to avoid the infrared catastrophe, so for convenience we replace the term $(1/\ln 2)(1-\epsilon)/\epsilon$ by a function $f_0(\epsilon)$, say, which has this value for

$$2^{-r}\leqslant\epsilon\leqslant 1,$$

and has the constant value $(1/\ln 2)(2^r-1)$ for $\epsilon\leqslant 2^{-r}$. If we now define

$$f_i(\epsilon)=0, \quad \epsilon\geqslant 2^{1-i}$$
$$=\frac{1}{\ln 2}\frac{(1-\epsilon 2^{i-1})}{\epsilon}, \quad 2^{-i}\leqslant\epsilon\leqslant 2^{1-i}$$
$$=\frac{1}{\ln 2}2^{i-1}, \quad \epsilon\leqslant 2^{-i}$$

for $i=1, 2, \cdots r$, then we have

$$f_0(\epsilon)=\sum_{i=1}^{r} f_i(\epsilon).$$

Furthermore all the $f_i(\epsilon)$ have unit integral, and a variate from the f_i distribution may be drawn by multiplying a variate from the f_1 distribution by 2^{1-i}.

Thus we have, as a suitable decomposition of the bremsstrahlung cross section,

$$\sum_{i=1}^{r+1}\alpha_i f_i(\epsilon)g_i(\epsilon),$$

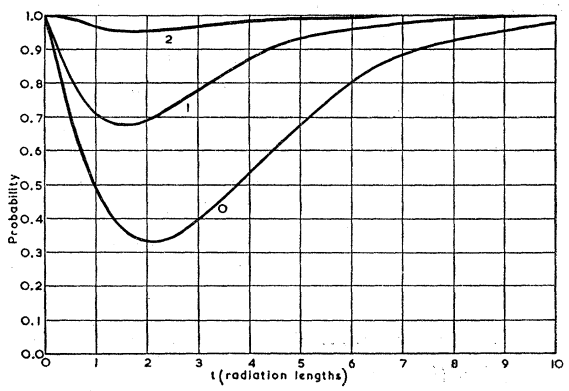

FIG. 1. Approximation A, $E_0/E=10$, primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

where

$$\alpha_i=\ln 2\left[\tfrac{4}{3}+\frac{1}{9\ln(183Z^{-\frac{1}{3}})}\right],\quad i=1, 2, \cdots r,$$

$$\alpha_{r+1}=\tfrac{1}{2}.$$

The f_i are as defined for $i=1, 2, \cdots, r$, and

$$f_{r+1}(\epsilon)=2\epsilon.$$

Also

$$g_i(\epsilon)=A(\delta),\quad i=1, 2, \cdots r$$
$$g_{r+1}(\epsilon)=B(\delta).$$

Variates from the distribution with frequency function 2ϵ may be drawn by taking the higher of two random numbers, or more simply on a computer by drawing two random numbers, ξ_1 and ξ_2, and computing ϵ thus:

$$\epsilon=1-|\xi_1-\xi_2|.$$

We need now consider only how to draw variates from the distribution with frequency function

$$f(\epsilon)=(1/\ln 2)(1-\epsilon)/\epsilon,\quad \tfrac{1}{2}\leqslant\epsilon\leqslant 1$$
$$=1/\ln 2,\qquad\qquad 0\leqslant\epsilon\leqslant\tfrac{1}{2}.$$

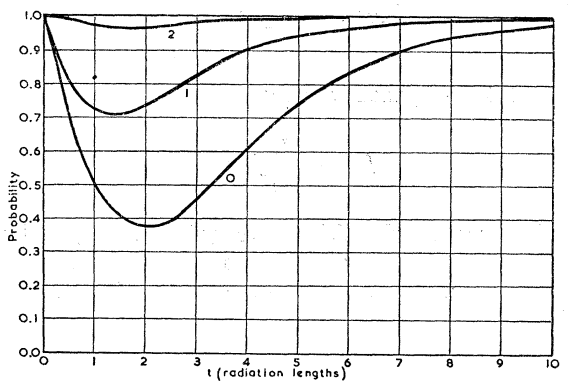

FIG. 2. $E_0=500$ Mev, $E=50$ Mev; absorber: air; primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

As

$$\text{Prob}(\epsilon\leqslant\tfrac{1}{2})=1/2\ln 2\approx 0.72,$$

we may first compare a random number ξ_1 with $1/2\ln 2$, and if $1/2\ln 2\geqslant\xi_1$, we may take as our value of ϵ

$$\epsilon=\tfrac{1}{2}\xi_2.$$

When $1/2\ln 2<\xi_1$, we must draw from the distribution with frequency function

$$f(\epsilon)=[1/(\ln 2-\tfrac{1}{2})](1-\epsilon)/\epsilon,\quad \tfrac{1}{2}\leqslant\epsilon\leqslant 1.$$
$$=0,\qquad\qquad \epsilon<\tfrac{1}{2}.$$

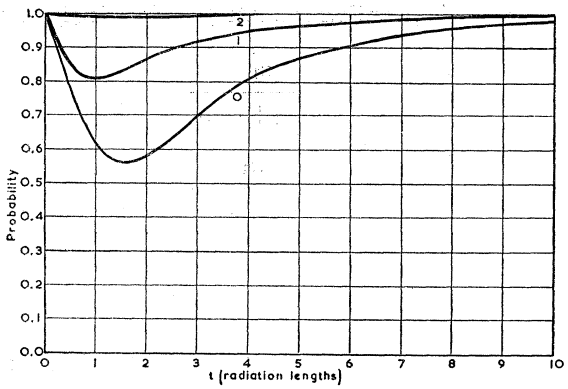

FIG. 3. $E_0=50$ Mev, $E=5$ Mev; absorber: air; primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

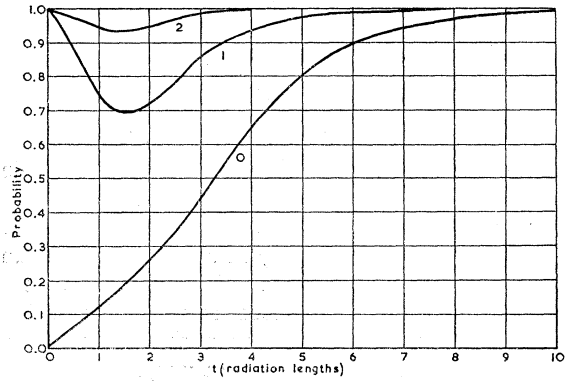

FIG. 4. Approximation A, $E_0/E=10$, primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

If we substitute $\epsilon=1-\frac{1}{2}x$ so that x will range from zero to 1, we have as the frequency function of x, $h(x)$ say,

$$h(x)=\frac{1}{(4\ln 2)-2}\left[\frac{x}{1-\frac{1}{2}x}\right]$$
$$=\frac{1}{(4\ln 2)-2}[x+\tfrac{1}{2}x^2+\tfrac{1}{4}x^3+\cdots]$$
$$=C_2(2x)+C_3(3x^2)+C_4(4x^3)+\cdots,\text{ say.}$$

The series is terminated at a point such that the combined weight of the terms neglected is negligible compared with the accuracy to which we are working and the coefficients $C_2, C_3 \cdots$ are tabulated.

As a distribution with frequency function rx^{r-1} may be sampled by choosing the highest of r uniformly distributed random numbers, we have a suitable decomposition of $h(x)$ as in Eq. (13) with all the $g(x)$ actually unity. From the value of x sampled from $h(x)$, ϵ is found from the relation $\epsilon=1-\frac{1}{2}x$.

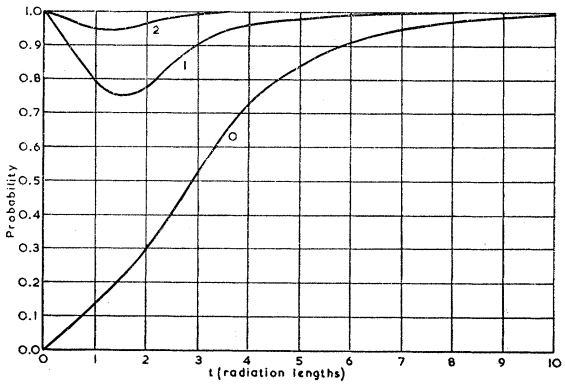

FIG. 5. $E_0=500$ Mev, $E=50$ Mev; absorber: air; primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

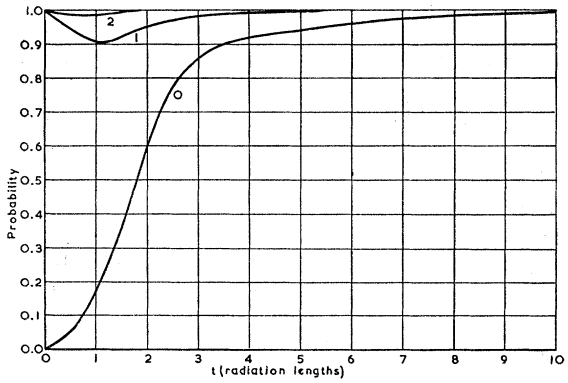

FIG. 6. $E_0=50$ Mev, $E=5$ Mev; absorber: air; primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

We have seen how to sample from the distribution of secondary energies as defined by the cross sections for the various events. However, the task of computing the total cross section for a given primary energy still appears to remain. This problem may be completely avoided by observing that if we assume the cross sections to have the form

$$f(\epsilon)=\sum_{i=1}^{n}\alpha_i f_i(\epsilon)g_i(\epsilon),$$

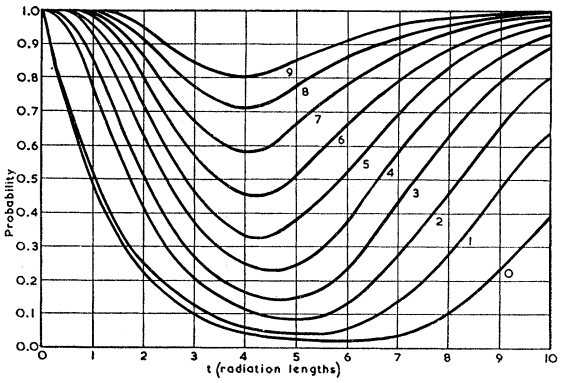

FIG. 7. Approximation A, $E_0/E=100$, primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

where the restrictions applying to the functions as in Eq. (13) are satisfied, then taking the total cross section as $\sum_{i=1}^{n}\alpha_i$ introduces no error if we proceed as follows.

Draw an exponential variate with total cross section $\sum_{i=1}^{n}\alpha_i$, i.e., with mean free path $1/\sum_{i=1}^{n}\alpha_i$. Simulate the movement of the particle being considered through this distance. Draw a sample of ϵ as previously described, either accepting or rejecting the value of ϵ by comparing a random number with the appropriate $g_i(\epsilon)$. However, in the event of rejection, instead of assuming that the distance that the particle has moved is to be rejected too, suppose that the particle has moved but is to be moved again and again until a value of ϵ is accepted. If the true cross section is σ and the value of $\sum_{i=1}^{n}\alpha_i$ is σ', then clearly the probability of acceptance of the ϵ at a given step will be just σ/σ'. Thus, if we require n attempts before acceptance, the distance traveled t will be distributed as $(\sigma')^n\xi^{n-1}e^{-\sigma' t}/(n-1)!$. But the probability of having exactly n attempts is $(\sigma/\sigma')(1-\sigma/\sigma')^{n-1}$. Hence the distance

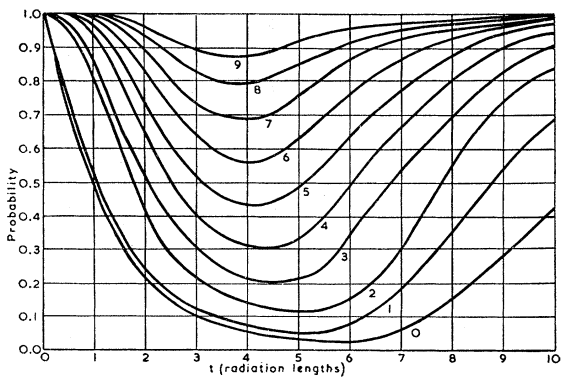

FIG. 8. $E_0=5000$ Mev, $E=50$ Mev; absorber: air; primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

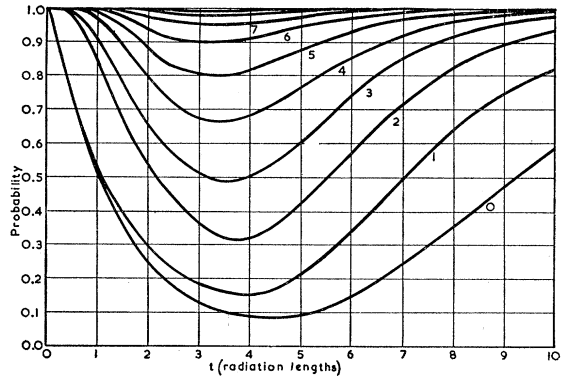

FIG. 9. $E_0=500$ Mev, $E=5$ Mev; absorber: air; primary photon. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

traveled will be distributed as

$$\sum_{n=1}^{\infty}\frac{(\sigma')^n t^{n-1}e^{-\sigma' t}}{(n-1)!}\frac{\sigma}{\sigma'}\left(1-\frac{\sigma}{\sigma'}\right)^{n-1}=\sigma e^{-\sigma t}.$$

Thus the effective cross section is correct.

The cross section for Compton effect is taken to be given by the Klein-Nishina formula. That is, the cross section for a photon of energy E giving a scattered photon of energy ϵE per radiation length is given by

$$\phi_3(E,\epsilon)=\frac{\pi m}{4\alpha Z\ln(183Z^{-\frac{1}{3}})}\frac{1}{E\epsilon}[1+\epsilon^2-\epsilon\sin^2\theta],$$

where θ, the scattering angle, satisfies the relation

$$m(1-\epsilon)=\epsilon E(1-\cos\theta).$$

The computation of variates from this cross section is straightforward.

Ionization loss is allowed for by subtracting from the electron energy, after each flight through a distance t radiation lengths, the quantity

$$E_0-E=\frac{\pi m}{2\alpha Z\ln(183Z^{-\frac{1}{3}})}\left[20.2+3\ln\frac{E_0}{m}-2\ln Z\right]t,$$

where E_0 was the energy at the beginning of the flight and E the corrected energy. This formula is based on Eq. (1.12) of reference 11 but differs by a negligible amount in that E_0 in the logarithmic term replaces p, the momentum of the electron, and that it is assumed that the rate of energy loss is constant as the electron moves through the small distance t.

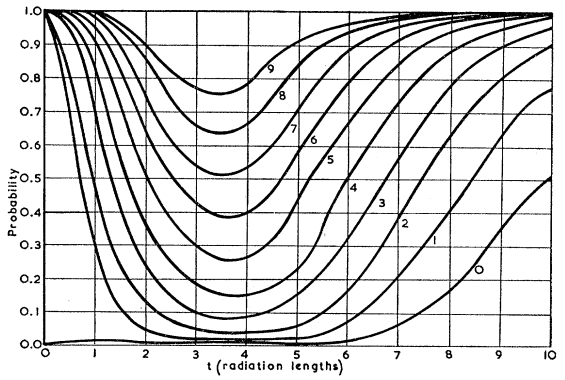

FIG. 10. Approximation A, $E_0/E=100$, primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

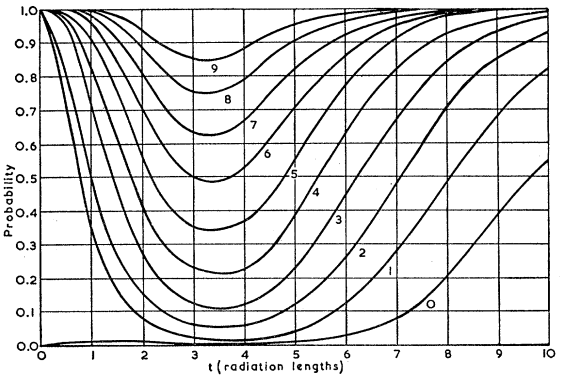

FIG. 11. $E_0=5000$ Mev, $E=50$ Mev; absorber: air; primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

4. DISCUSSION OF RESULTS

Results are here presented for 12 different cases which are summarized in Figs. 1–12. In each graph we have plotted the probability of having no more than n electrons with energies greater than E in a shower of the specified type, as a function of depth measured in radiation lengths. The number n is attached below the corresponding curve. Thus to read from the graph the probability of exactly n electrons, one need simply read the distance between the two adjacent curves between which the number n is written.

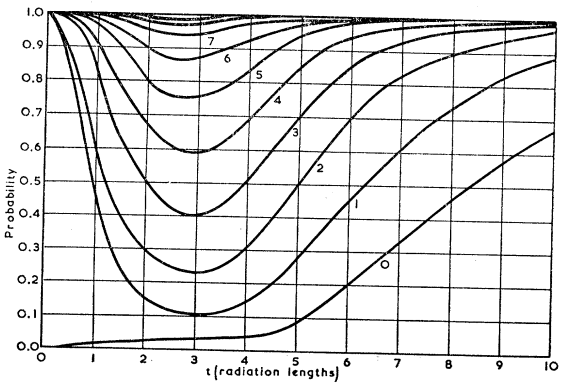

FIG. 12. $E_0=500$ Mev, $E=5$ Mev; absorber: air; primary electron. Probability of no more than n electrons with energies greater than E; n is attached to the curves.

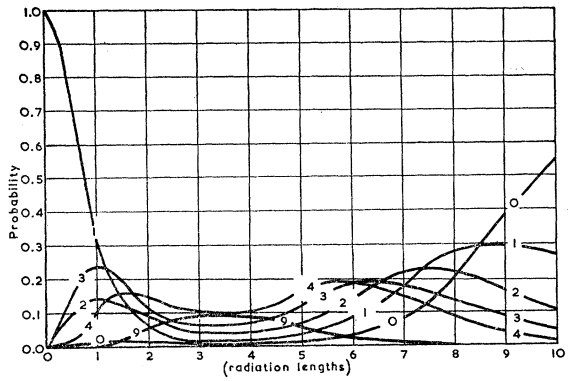

FIG. 13. $E_0=5000$ Mev, $E=50$ Mev; absorber: air; primary electron. Probability of exactly n electrons with energies greater than E; n is attached to the curves.

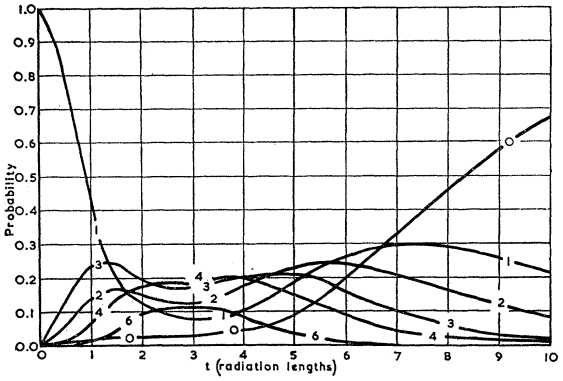

FIG. 15. $E_0=500$ Mev, $E=5$ Mev; absorber: air; primary electron. Probability of exactly n electrons with energies greater than E; n is attached to the curves.

The first of each trio of figures (1–3, 4–6, etc.) corresponds to approximation A (Figs. 1, 4, 7, 10) while the second and third are for the more accurate approximations for showers in air with different values of E_0 and E but with the same value of $E_0/E(=10$ or $100)$. For Figs. 1–6 the ratio E_0/E is taken as 10 and for Figs. 7–12 it is taken as 100. Figures 1–3 and 7–9 correspond to showers initiated by a primary photon while Figs. 4–6 and 10–12 are for showers initiated by a primary electron.

To obtain a more ready comparison of the striking way in which the properties of showers depend on energy and depth, we have plotted in Figs. 13–15 the probability of finding exactly n electrons with energies greater than E, with n attached to the curves for the cases $n=0$, 1, 2, 3, 4, etc.

Generally speaking, the quantitative results appear at first sight to bring out pretty well what one would expect from arguing on a purely qualitative basis. However, we shall see that the detailed behavior is most complex and defies purely qualitative reasoning. Furthermore, the complexity of the curves bears out the contention that the task of trying to find a simple analytical solution of the problem, amenable to easy numerical evaluation, was a hopeless one.

Many interesting features are highlighted by the curves. For instance, in Figs. 13, 14, and 15 the odd-even effect in a shower is clearly shown. In the case of a primary electron, the probability of an odd number of electrons before the cascade maximum—which corresponds roughly to the minimum of the curves—is greater than the probability for an even number. Thus at a depth of one radiation unit the probability of one or three electrons is considerably greater than the probability for zero, two, or four electrons. A similar plot in the case of a photon primary using Figs. 7–9 would show the reverse with the highest probabilities showing up for even numbers of electrons. The two-maxima behavior for the curves $n=2$, 3, 4, and higher numbers is easily understood. The number development of the shower builds up through the small values of n, eventually reaching the maximum value for a certain depth t, and then the numbers decrease as the particles die off.

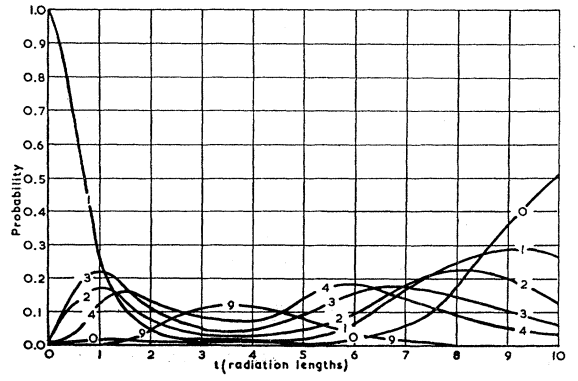

FIG. 14. Approximation A, $E_0/E=100$, primary electron. Probability of exactly n electrons with energies greater than E; n is attached to the curves.

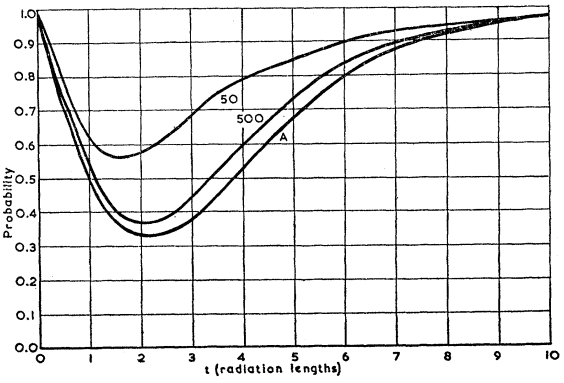

FIG. 16. Probability of finding exactly 0 electrons with energy greater than E (primary photon) in the case of approximation A with $E_0/E=10$, and in the case of accurate cross sections, Compton effect and ionization losses being taken into account, $E_0=500$ Mev, $E=50$ Mev; $E_0=50$ Mev, $E=5$ Mev. The energy E_0 or the letter A in the case of approximation A is attached to the curves. The absorber is air or, in the case of approximation A, a general absorber.

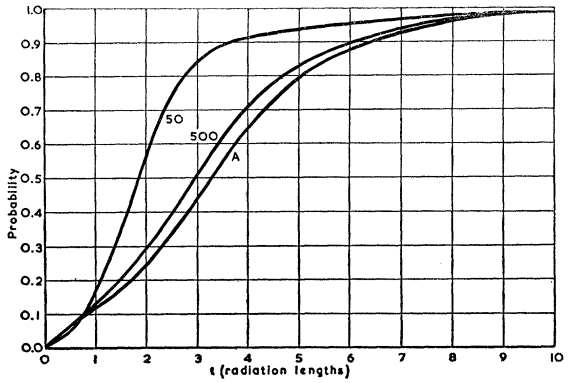

FIG. 17. Probability of finding exactly 0 electrons with energy greater than E (primary electron) in the case of approximation A with $E_0/E=10$, and in the case of accurate cross sections, Compton effect and ionization losses being taken into account, $E_0=500$ Mev, $E=50$ Mev; $E_0=50$ Mev, $E=5$ Mev. The energy E_0 or the letter A in the case of approximation A is attached to the curves. The absorber is air or, in the case of approximation A, a general absorber.

Hence before and after the cascade maximum the probability of finding n electrons, where n is less than the average number at the maximum, must be greater than at the cascade maximum. The value of the depth t for which each curve reaches its maximum value must therefore also approach more and more closely to the value of the depth at the cascade maximum as n approaches the value at the cascade maximum. Near and beyond the average number at the shower maximum there is only one peak for the curve—namely near the cascade maximum. It is interesting to note that immediately past the cascade maximum the value of the probability of finding $n=0$, 1, 2, 3, 4 electrons (n less than the average value of the maximum) increases with increasing n. This behavior is understandable since the odd-even effect brought about by pair production becomes masked as the other shower factors come into play and overshadow pair production effects. On the other hand, at sufficiently great depths the order mentioned above is completely reversed.

In each of Figs. 13, 14, and 15 the ratio of primary to secondary energy is 100. However, the effect of decreasing primary and secondary energies brings into play the accurate cross sections for pair production and bremsstrahlung as well as Compton effect and ionization loss. The resultant complex interplay of all these factors with the odd-even effect thrown in gives rise to the complicated detailed behavior depicted in the figures. For instance, at a depth of one radiation length the probability of finding exactly four particles in the case of approximation A is about 0.125; for $E_0=5000$ Mev, $E=50$ Mev the value is 0.110, and for $E_0=500$ Mev, $E=5$ Mev the value is about 0.075. As would be expected, this shows that at higher energies the probability of finding a high number of particles is greater than at low energies. However, at a depth of only two cascade units the corresponding figures are 0.135, 0.150, and 0.175, respectively, showing that the effect has completely reversed. This single example chosen from dozens of similar ones shows the danger in the analysis of shower results without the aid of quantitative numerical results.

Figures 16 and 17 give the probability of zero electrons with energies greater than E in the cases for which $E_0/E=10$, plotted as a function of depth. Again for convenience of comparison we have shown each set of three comparable cases, namely: approximation A with $E_0/E=10$, and in the low-energy case in air with $E_0=500$ Mev, $E=50$ Mev; $E_0=50$ Mev, $E=5$ Mev. Figure 16 is for the case of a photon-initiated shower and Fig. 17 for an electron primary. Both figures again demonstrate the strong energy dependence of the probability distribution and the important roles which Compton effect, ionization loss, and accurate cross sections play. Figure 17 for a primary electron exhibits an interesting feature, namely the behavior of the curves between the depths of zero and three quarters of

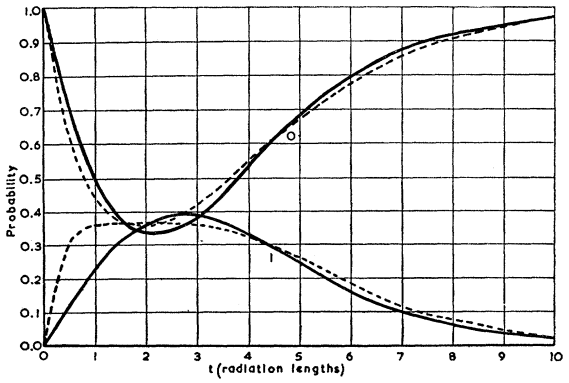

FIG. 18. Probability of finding exactly n electrons with energy greater than E for $n=0$, 1 (attached to the curves) in the case of a primary photon and using approximation A with $E_0/E=10$. The broken lines give probabilities of n electrons when one assumes that the distribution was Poissonian with the correct mean value.

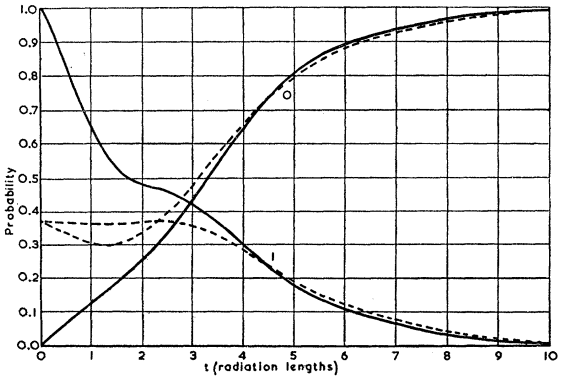

FIG. 19. Probability of finding exactly n electrons with energy greater than E for $n=0$, 1 (attached to the curves) in the case of a primary electron and using approximation A with $E_0/E=10$. The broken lines give probabilities of n electrons when one assumes that the distribution was Poissonian with the correct mean value.

TABLE I. Approximation A, $E_0/E=100$, primary electron. The probability of exactly n electrons with energies greater than E at a depth t. The numbers in parentheses are those of Messel[a]; the others are the values of the actual distribution found in the present paper.

t \ n	0	1	2	3	4	5	6	7	8	9
2	0.008 (0.003)	0.045 (0.018)	0.078 (0.049)	0.112 (0.096)	0.132 (0.140)	0.138 (0.164)	0.140 (0.160)	0.104 (0.134)	0.097 (0.098)	0.058 (0.064)
4	0.008 (0.001)	0.011 (0.004)	0.023 (0.016)	0.049 (0.041)	0.069 (0.075)	0.116 (0.112)	0.125 (0.139)	0.142 (0.147)	0.125 (0.137)	0.110 (0.113)
6	0.013 (0.008)	0.061 (0.040)	0.089 (0.095)	0.151 (0.152)	0.183 (0.182)	0.162 (0.175)	0.123 (0.140)	0.091 (0.096)	0.055 (0.057)	0.030 (0.031)
8	0.167 (0.129)	0.244 (0.240)	0.225 (0.246)	0.148 (0.184)	0.091 (0.111)	0.056 (0.058)	0.036 (0.027)	0.015 (0.012)	0.008 (0.005)	0.002 (0.002)
10	0.507 (0.452)	0.261 (0.310)	0.128 (0.147)	0.057 (0.060)	0.031 (0.022)					

[a] See reference 2.

a radiation length. In this interval the probability of there being exactly zero electrons is smaller for the lower-energy case than for the high-energy one. The opposite behavior might normally be expected. Apparently in this interval the radiation losses are so small that the electron has a considerable chance of travelling a long distance.

It is of some interest to compare our results with those obtained by assuming that the distribution is of a simple form with one or two parameters, and computing these parameters from the accurate values of the mean and, if necessary, the second moment. The two forms that are most naturally chosen are the Poisson, with probability generating function $e^{m(u-1)}$, and its generalization, the binomial, with probability generating function $[(\lambda+um)/(\lambda+m)]^\lambda$. [The probability generating function $g(u)$ of a distribution with probability $f(n)$ of having the value n is defined as $g(u)=\sum_{n=0}^{\infty} u^n f(n)$.] In the case where the parameter λ is negative, the binomial distribution is usually called the negative binomial or the Polya distribution.

In Figs. 18 and 19 we have plotted the probability of finding exactly zero and one electron for the case of a primary photon and electron, respectively, in approximation A with $E_0/E=10$. Also plotted in broken lines are the values that these functions would have if the distribution was Poissonian with the same value of the mean. It is seen that only after the region of the shower maximum is agreement close in this case.

For the case of an electron primary and $E_0/E=100$, we may compare our results in the approximation-A case with those of Messel[2] who computed the distribution function using calculated values of the first two moments and assuming that the distribution is of the negative binomial type. This comparison is shown in Table I.

As would be expected, the agreement shown in Table I is better at large depths than at small ones. However, again as in the case of the Poisson distribution, we see that the reliability of using reconstructed distribution functions is very poor indeed. Using our results we have tried to find some simple empirical formula which would yield these; however, to date we have been unsuccessful. By the time one takes the odd-even and various other factors into account, one seems to end up with an empirical relation too complicated to be of much value.

It should be noted that throughout this paper for probabilities whose values are of the order of 0.1 the standard deviation of our results is about 0.005, and for values of the order of 0.02 it is about 0.002.

Our results yield in a direct and accurate manner the average-number curves for the cases considered. In Figs. 20 and 21 the logarithm of the average number of electrons with energies greater than E at a depth of t radiation lengths, due to a primary photon, and primary

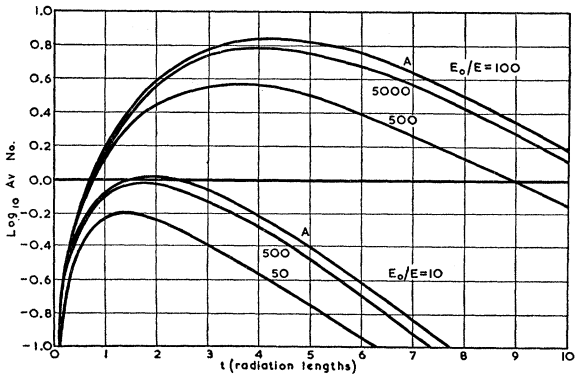

FIG. 20. The logarithm of the average number of electrons with energies greater than E at a depth of t radiation lengths due to a primary photon. The value of E_0/E is shown beside the two sets of curves. The energy E_0 in Mev of the primary photon, or the letter A in the case of approximation A, is attached to the curves. The absorber is air or, in the case of approximation A, a general absorber.

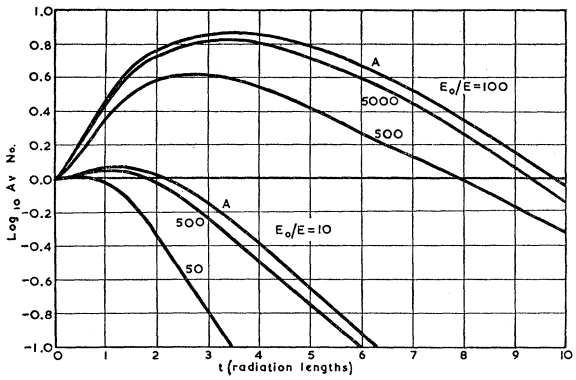

FIG. 21. The logarithm of the average number of electrons with energies greater than E at a depth of t radiation lengths due to a primary electron. The value of E_0/E is shown beside the two sets of curves. The energy E_0 in Mev of the primary electron, or the letter A in the case of approximation A, is attached to the curves. The absorber is air or, in the case of approximation A, a general absorber.

electron, are given. The values of E_0/E, 10 and 100, are shown beside the respective curves. The energy E_0 in Mev of the primary or the letter A in the case of approximation A is attached to the curves.

It is of great interest to observe the behavior with energy of the position of the shower maximum. It will be noted that the depth of the shower maximum decreases with decreasing energy of the primary and secondary particles, whereas it may have been expected that the opposite would be the case. Clearly the decreasing rate of bremsstrahlung and materialization at low energies would be expected to extend the region in which the shower builds up to a maximum, whereas the higher rates of Compton effect and ionization loss should give the opposite effect. The results show clearly that the Compton effect and ionization loss outweigh considerably the effect of decreasing cross sections for bremsstrahlung and pair production.

A further surprising feature is the large variation of the average numbers with primary and secondary energy, even though the ratio of E_0/E is constant. For instance, in the case of $E_0=5000$ Mev, $E=50$ Mev, it is reasonable to expect that the results for this would be yielded accurately by using approximation A. However, this is simply not the case, the error being in most cases as high as 15%. In the case of $E_0=500$ Mev and $E=5$ Mev the differences are so large that approximation-A results would be of little or no value.

APPENDIX

Consider the distribution with frequency function $f(x)$ given by Eq. (13) with the same restrictions on the symbols. We shall show that using the composition and rejection methods as we have described, we do in fact arrive at this distribution.

First we choose an integer from 1 to n with probability proportional to α_i of being i. By following the procedure we outlined, i is to be chosen when $x_i<0$, $x_{i-1}\geqslant 0$, which is equivalent to

$$\sum_{j=1}^{i-1}\alpha_j/\sum_{i=1}^{n}\alpha_i\leqslant\xi<\sum_{j=1}^{i}\alpha_j/\sum_{i=1}^{n}\alpha_i.$$

Since ξ is uniformly distributed, the probability of this occurring is simply $\alpha_i/\sum_{i=1}^{n}\alpha_i$.

The joint frequency function of (i,x) before we allow for rejection is

$$\alpha_i f_i(x)/\sum_{i=1}^{n}\alpha_i.$$

However, there is a probability of $1-g_i(x)$ that we shall reject this value once we have chosen it. Letting $j=0$ denote acceptance and $j=1$ rejection, we have as the joint frequency function of (i,j,x)

$$[\alpha_i f_i(x)/\sum_{i=1}^{n}\alpha_i]\,|\,j-g_i(x)\,|.$$

Hence, as we repeat the process until we reach a zero value of j, the joint frequency function of the first accepted values of (i,x) will be $c\alpha_i f_i(x)g_i(x)$, where c is a constant required to normalize the function. To find the frequency function of x, we must sum over i, obtaining

$$c\sum_{i=1}^{n}\alpha_i f_i(x)g_i(x)=cf(x).$$

Thus, as it was supposed that $f(x)$ is normalized, c is in fact unity and we have established the required result.

It is of interest to note that the probability of rejection $(j=1)$ is

$$\sum_{i=1}^{n}\int[\alpha_i f_i(x)/\sum_{i=1}^{n}\alpha_i][1-g_i(x)]dx=1-(1/\sum_{i=1}^{n}\alpha_i).$$

Thus $\sum_{i=1}^{n}\alpha_i$ is the average number of attempts that must be made until a value of x is accepted.