High-Energy Bremsstrahlung in Electron-Proton Collisions

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(Received August 29, 1958)

The cross section for the bremsstrahlung resulting from high-energy collisions of electrons with protons is calculated, considering the electrons as extreme relativistic and taking into consideration the proton's recoil, anomalous magnetic moment, bremsstrahlung, and form factors. The final cross section is integrated over the recoil proton and final photon to obtain a formula of interest for the Stanford accelerator experiments. The computation of the integrated cross section, made without regard for radiative corrections and mesonic contributions, is estimated to be accurate to 2% in the energy ranges of interest. This cross section is corrected for radiative effects, which produce an additional uncertainty of less than 5% and possibly as little as 1%. Resonance effects due to meson interactions in the "virtual Compton effect" diagrams, however, become significant in the range of interest. Their contribution to the cross section is discussed.

I. INTRODUCTION

The calculations described here have been motivated by the experiments now being carried out at Stanford[1] on the production of pions by electrons. Bremsstrahlung (above the pion threshold) will be a competing background effect for the primary process of pion production. A theoretical knowledge of its size in the range of interest is useful as a supplement to experimental procedures of subtracting the effect.

For the inelastically scattered electron of laboratory momentum p_2^l, observed at a fixed laboratory angle Ω_2^l, the cross section may be considered as a function of two invariants, which are chosen for convenience to be W, the center-of-mass energy of the photon and final nucleon, and ϵ_1^l, the incident energy of the electron in the laboratory, so

$$\frac{d^2\sigma}{d\Omega_2^l dp_2^l}=\frac{d^2\sigma}{d\Omega_2^l dp_2^l}(W,\epsilon_1^l).$$

The present range of interest extends for values of W from the pion threshold, $W\approx 1079$ Mev, to about the resonance energy, $W\approx 1235$ Mev, while values of ϵ_1^l up to 700 Mev (present limit of the Stanford accelerator) have been considered.

The cross section $d^2\sigma/d\Omega_2^l dp_2^l$, hereafter referred to as the integrated cross section, is an average over the recoil proton and final photon. We have explicitly evaluated it as a function of ϵ_1^l, for fixed values of W, corresponding to the kinematics[2] of pion experiments completed or in progress. An expression for the unintegrated cross section, also given here, is applicable to coincidence experiments being planned on the production of pions by electrons,[1] though the bremsstrahlung background is less important in this case.

Our work is an extension of previous calculations of the bremsstrahlung matrix element and integrated cross sections.[3] We take into account the effects of proton recoil, bremsstrahlung by both the charge and anomalous magnetic moment distributions of the proton (using the Hofstadter proton form factors[4]), and radiative corrections.

The diagrams considered are pictured in Figs. 1 and 2. Since the processes considered are extreme relativistic for the electrons, terms involving the electron mass may usually be neglected. The interference of I and II was considered up to terms of order q^3/M^3, where q^μ is the four-momentum transferred to the proton. The maximum value of q^2 in the range described above is about (500 Mev)^2. Since the interference terms (when integrated over final photon and proton) only contribute at most 3% to the final integrated cross section, this will result in an inaccuracy of less than 2% in this region.

The radiative corrections were estimated for the integrated cross section by taking into consideration the fact that the major contribution to the integrated cross section comes when the bremsstrahlung photon is parallel to either the incident or final electron. The major uncertainty in our estimate of the radiative correction comes from the non-infrared-divergent part of the real radiative corrections to the electrons in Fig. 1. We have adopted directly the expressions given by Bjorken, Drell, and Frautschi[5] for the closely related

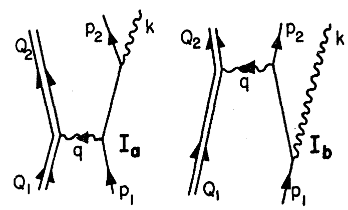

FIG. 1. Electron bremsstrahlung diagrams.

* Supported in part by the U. S. Air Force through the Air Force Office of Scientific Research.

[1] W. K. H. Panofsky and E. A. Allton, Phys. Rev. **110**, 1155 (1958). Also W. K. H. Panofsky (private communication).

[2] The superscript l refers always to a quantity in the laboratory frame. Kinematics are discussed in Appendix A. The electron momentum transfer r^2 is also relevant for comparison with pion kinematics.

[3] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934); S. D. Drell, Phys. Rev. **87**, 753 (1952); L. I. Schiff, Phys. Rev. **87**, 750 (1952); P. V. C. Hough, Phys. Rev. **74**, 80 (1948).

[4] R. Hofstadter, in *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1957), Vol. 7, p. 231.

[5] Bjorken, Drell, and Frautschi, Phys. Rev. **112**, 1412 (1958).

process of pair production, and refer to their paper for details.

The "virtual Compton effect" of the diagrams in Fig. 2 is treated here as a purely electrodynamic phenomenon. However, higher-order corrections to Fig. 2 involving virtual meson processes may become important in the region we consider. These effects are discussed in Appendix D.

The charge and magnetic form factors of the proton used here are those determined by the Stanford electron-scattering experiments. These experiments determine the form factors to be used in the electron bremsstrahlung (Fig. 1) for the real proton vertex, but the form factors to be used in the proton bremsstrahlung (Fig. 2) are unknown. Since more than one invariant describes the proton vertex with both photon and proton virtual in Fig. 2, the two Hofstadter form factors may not be adequate for this process. However, for simplicity, we have assumed that the form factors are the same for both cases. We also treat the magnetic and charge form factors as identical, in agreement with the scattering results.[4]

II. CALCULATIONS

A. Unintegrated Cross Section

The notation used is the same as that of Schweber *et al.*,[6] with boldface italics replacing letters with slashes through them.[6] Symbols and kinematics are defined in Appendix A. The matrix elements used are (aside from a common factor)

$$\mathbf{M}_{\mathrm{I}}=\frac{1}{q^2}\left\langle p_2\left|\boldsymbol{e}\frac{1}{\boldsymbol{p}_2+\boldsymbol{k}-m}\gamma^\nu+\gamma^\nu\frac{1}{\boldsymbol{p}_1-\boldsymbol{k}-m}\boldsymbol{e}\right|p_1\right\rangle \times\langle Q_2|\Gamma_\nu(q)|Q_1\rangle=\frac{1}{q^2}J_{e\nu}{}^{\mathrm{I}}J_p{}^{\mathrm{I}\nu},$$

$$\mathbf{M}_{\mathrm{II}}=\frac{-1}{r^2}\langle p_2|\gamma^\mu|p_1\rangle \times\left\langle Q_2\left|\boldsymbol{E}\frac{1}{\boldsymbol{Q}_2+\boldsymbol{k}-M}\Gamma_\mu'(r)+\Gamma_\mu'(r)\frac{1}{\boldsymbol{Q}_1-\boldsymbol{k}-M}\boldsymbol{E}\right|Q_1\right\rangle =\frac{-1}{r^2}J_{e\mu}{}^{\mathrm{II}}J_p{}^{\mathrm{II}\mu},$$

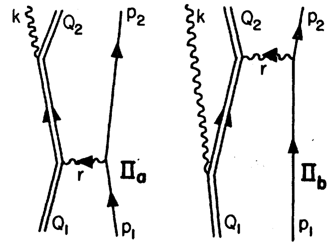

FIG. 2. Proton bremsstrahlung diagrams. (Virtual Compton effect.)

where

$$\Gamma_\gamma(q)=F_1(q^2)\gamma_\nu+(\mu/4M)F_2(q^2)(\boldsymbol{q}\gamma_\nu-\gamma_\nu\boldsymbol{q}),$$
$$\Gamma_\mu'(r)=G_1(r^2)\gamma_\mu+(\mu/4M)G_2(r^2)(\boldsymbol{r}\gamma_\mu-\gamma_\mu\boldsymbol{r}),$$
$$\boldsymbol{E}=\Gamma_\lambda(k^2=0)e^\lambda.$$

We assume that the "virtual Compton effect" form factors $G_{1,2}$ (from II) are identical with those from I, and that the charge and moment form factors are identical:

$$F_1=F_2=G_1=G_2=F.$$

The usual techniques[6] are used to compute the cross section, averaged and summed over initial and final spin and polarization states, neglecting terms of order q^3/M^3 in the interference of $\mathbf{M}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}}$. In addition, $|\mathbf{M}_{\mathrm{II}}|^2$ has been neglected.

The differential cross section is then

$$d^3\sigma=\frac{\alpha^3}{4\pi^2M^2\mu_1}\frac{d^3k}{k}\frac{d^3p_2}{\epsilon_2}\frac{d^3Q_2}{E_2}\delta^4(p_1+Q_1-p_2-Q_2-k)\mathbf{Y},$$

where

$$\mathbf{Y}=4m^2M^4S_f\bar{S}_i|\mathbf{M}_{\mathrm{I}}+\mathbf{M}_{\mathrm{II}}|^2=F^2(q^2)\mathbf{Y}_{el}+F(q^2)F(r^2)\mathbf{Y}_{in}.$$

The dimensionless quantities $\mathbf{Y}_{el}$ and $\mathbf{Y}_{in}$ will be written here in a form which is convenient for integration over the recoil proton and the final photon. (A more concise form for $\mathbf{Y}_{el}$ is given in Appendix B. This form agrees with the analogous result for pair production.[5]) Here terms involving the electron mass m were neglected, with the exception of terms involving λ_1^{-2} and λ_2^{-2}, since these give nonzero results when integrated:

$$\begin{aligned}\mathbf{Y}_{el}=&\frac{M^6}{\lambda_1q^4}\left(a_1+a_3\frac{q^2}{M^2}+a_5\frac{q^4}{M^4}\right)+\frac{M^6}{\lambda_2q^4}\left(a_2+a_4\frac{q^2}{M^2}+a_6\frac{q^4}{M^4}\right)+\frac{M^4}{q^4}\left(a_7+a_8\frac{q^2}{M^2}\right)\\&+\frac{M^4}{2\lambda_1\lambda_2}a_9+\frac{m^2M^6}{\lambda_1^2q^4}\left(a_{11}+a_{13}\frac{q^2}{M^2}+a_{15}\frac{q^4}{M^4}\right)\\&+\frac{m^2M^6}{\lambda_2^2q^4}\left(a_{10}+a_{12}\frac{q^2}{M^2}+a_{14}\frac{q^4}{M^4}\right),\end{aligned}$$

$$\begin{aligned}-\mathbf{Y}_{in}=&\frac{M^4}{\lambda_1q^2}\left(b_1+b_3\frac{q^2}{M^2}\right)+\frac{M^4}{\lambda_2q^2}\left(b_2+b_4\frac{q^2}{M^2}\right)\\&+\frac{M^4}{2\lambda_1\Lambda_1}b_5+\frac{M^4}{2\lambda_2\Lambda_1}b_6+\frac{M^2}{q^2}b_7+\frac{M^2}{2\Lambda_1}b_8\\&+\left(\frac{1}{q^2}-\frac{1}{2\Lambda_1}\right)(\lambda_1+\lambda_2)b_9.\end{aligned}$$

[6] Schweber, Bethe, and de Hoffmann, *Mesons and Fields* (Row-Peterson and Company, Evanston, 1955), Vol. I.

$$a_1=-a_2=r^2/M^2,$$

$$a_3=a_3(\mu_1,\mu_3)=1+\frac{2r^2}{M^2}x+\frac{4(\mu_1{}^2+\mu_3{}^2)}{M^2r^2}+\frac{\mu_1}{M^2},$$

$$a_4=-a_3(-\mu_3,\,-\mu_1),$$

$$a_5=a_5(\mu_3)=-2x-\mu^2\mu_3/2M^2,\qquad a_6=-a_5(-\mu_1),$$

$$a_7=-4,$$

$$a_8=-8x,$$

$$a_9=-2-\frac{4r^2x}{M^2}+\left(\mu^2-\frac{4M^2}{r^2}\right)\left(\frac{r^2\mu_{13}+2(\mu_1{}^2+\mu_3{}^2)}{2M^4}\right),$$

$$a_{10}=a_{10}(\mu_1)=-4\mu_1{}^2/M^4,\qquad a_{11}=a_{10}(-\mu_3),$$

$$a_{12}=a_{12}(\mu_1)=-1-2\mu_1/M^2+\mu^2\mu_1/M^4,\quad a_{13}=a_{12}(-\mu_3),$$

$$a_{14}=a_{14}(\mu_1)=-2x+\mu^2\mu_1/2M^2,\qquad a_{15}=a_{14}(-\mu_3),$$

$$b_1=b_1(\mu_1,\mu_3)=\frac{4\mu_3{}^3}{\Lambda_2\mu_{13}M^2}+\frac{2\mu_1\mu_3}{M^2}\left(\frac{1}{\mu_{13}}+\frac{1}{\Lambda_2}\right)+\frac{\mu_3r^2}{M^2\Lambda_2}\left(1+\frac{M^2}{2\mu_{13}}\right),$$

$$b_3=b_3(\mu_1,\mu_3)=-1-\frac{M^2}{2\Lambda_2}+\frac{2\mu_1\mu_3}{\Lambda_2r^2}-\frac{2\mu_1}{r^2}-\frac{M^2\mu_1}{\Lambda_2r^2}+\frac{\mu_3}{\Lambda_2},$$

$$b_5=b_5(\mu_1,\mu_3)=\frac{2\mu_1}{r^2}-\frac{\mu_1}{\mu_{13}}-\frac{8\mu_1{}^2\mu_3}{M^2\mu_{13}r^2}-\frac{4\Lambda_2\mu_1{}^2}{M^2\mu_{13}r^2},$$

$$b_7=\frac{2(\mu_1+\mu_3)}{r^2}\left(\frac{M^2r^2}{\Lambda_2\mu_{13}}+2+\frac{2\mu_{31}}{\Lambda_2}\right),\qquad b_2=b_1(-\mu_3,\,-\mu_1),$$

$$b_8=\frac{2(\mu_1+\mu_3)}{r^2}\left(\frac{2M^2r^2}{\Lambda_2\mu_{31}}+4-\frac{4M^2}{\Lambda_2}\right),\qquad b_4=b_3(-\mu_3,\,-\mu_1),$$

$$b_9=4M^4/\mu_{31}r^2,\qquad b_6=b_5(-\mu_3,\,-\mu_1).$$

B. Integrated Cross Section

The cross section of particular interest is then given by integrating over the recoil proton and the final photon. For convenience, all integrations will be carried out in the photon-proton center-of-mass system (i.e., where $\mathbf{k}+\mathbf{Q}_2=0$). Unless otherwise specified, all energies and momenta will be in this system. The integrated cross section then becomes

$$\frac{d^2\sigma}{dp_2{}^ld\Omega_2{}^l}=\frac{\alpha r_0{}^2m^2}{4\pi^2M^3}\left(\frac{\epsilon_2{}^l}{\epsilon_1{}^l}\right)\frac{k}{k+E_2}\int d\Omega_k\mathbf{Y}.$$

The coefficients a_i and b_i are not functions of the photon direction and can therefore be taken outside the integrals. The form factors are expanded in a power series in q^2/M^2 for the integration. The integrals which occur are given in Appendix C. The following expansion is used for the form factor:

$$F^2(q^2)=1+4.23\left(\frac{q^2}{M^2}\right)+7.40\left(\frac{q^4}{M^4}\right)+4.47\left(\frac{q^6}{M^6}\right).$$

This expansion deviates from the exponential form used by Hofstadter to describe the experimental results, by at most $\pm4\%$ for the largest values of q^2 considered. The net error introduced by integrating over these fluctuations is expected to be much smaller.

The calculation of the integrated cross section was programmed for an IBM 650 computer and the results are shown in Fig. 3 for different center-of-mass energies. We find the results to be substantially in agreement with previous calculations (e.g., Panofsky[1]) using the Schiff-Rosenbluth formula.[3,7] Deviations at the lower energies can be ascribed to the more precise evaluation of the integrals used here.

C. Radiative Corrections

We have directly adopted the results of Bjorken, Drell, and Frautschi,[5] who calculated the radiative corrections to the closely related process of pair production. For the integrated cross section, the corrections to the diagrams of Fig. 2 are neglected. Keeping only terms of order $\alpha\ln(E/m)$, the radiative correction is then given by

$$\sigma(\text{radiative})\simeq\frac{\alpha}{2\pi}\left[\ln\left(-\frac{r^2}{m^2}\right)\left\{\frac{13}{3}-2\ln\frac{\epsilon_1\epsilon_2}{(\Delta E)^2}\right\}\right]\times\sigma(\text{brems.})+\sigma_1,$$

where σ_1 is due to the non-infrared-divergent part of

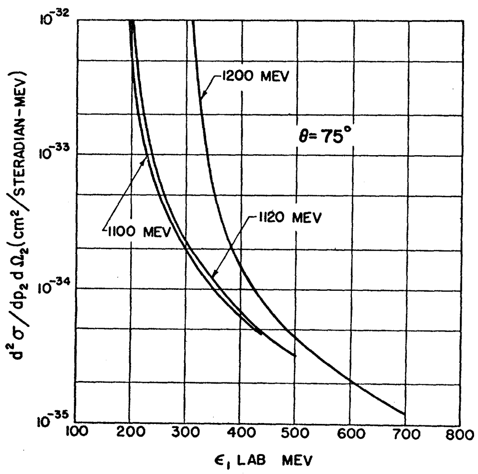

FIG. 3. Integrated cross section as a function of the incident electron energy in the laboratory for different values of the center-of-mass energy, W. The inelastically scattered electron is observed at $\theta_l=75°$. The explicit contribution of the anomalous moment included here varies from 1% at $\epsilon_1{}^l=350$ Mev to 60% at $\epsilon_1{}^l=700$ Mev.

[7] M. Rosenbluth, Phys. Rev. 79, 615 (1950).

the two-real-photon emission. The energy resolution for the integrated case we take as $\Delta E \approx k$. We have not made an estimate of σ_1. The radiative correction (not including σ_1) is about 5% for the regions of interest. In the closely related process of pair production, there are strong cancellations between σ_1 and the remainder of the radiative correction (for low energy resolution) lowering the 5% estimate for pair production to about 1%.[5] It is possible that a similar type of cancellation may occur in the radiative corrections (with low resolution) considered here, and result in a correction much less than 5%. We do not consider the unintegrated cross section here. In the latter case the radiative corrections to the proton diagrams may also be significant.

ACKNOWLEDGMENTS

We thank Professor S. D. Drell, Dr. S. C. Frautschi, and Dr. J. D. Bjorken for communicating the results of their paper to us before publication, and in particular we are grateful for valuable comments from Dr. Frautschi with regard to the radiative corrections. We thank Professor D. R. Yennie for suggesting the problem. We are grateful also to Professof L. I. Schiff for guiding our research and to Professor Drell, Professor Schiff, and Dr. Frautschi for reading our manuscript.

APPENDIX A

Definitions of Symbols

p_1, Q_1 = energy-momentum 4-vector of initial electron and proton;

e.g.: $p_1=(\mathbf{p}_1,\epsilon_1)$ $Q_1=(\mathbf{Q}_1,E_1)$

p_2, Q_2, k = energy-momentum 4-vector of final electron, proton, and photon, respectively.

$$(ab)=a_\mu b^\mu = a^0b^0-\mathbf{a}\cdot\mathbf{b},$$

$$q=Q_2-Q_1=Q_{21},\qquad x=\tfrac{1}{4}(1+\mu)^2-\tfrac{1}{8}\mu^2,$$

$$-r=p_2-p_1=p_{21},$$

$$P=Q_2+Q_1,$$

$$\lambda_j=(kp_j),\qquad \Lambda_j=(kQ_j),\quad \mu_1=(p_1Q_1),\ \mu_2=(p_2Q_2),$$

$$K_1=(p_1p_2),\qquad K_2=(Q_1Q_2),\ \mu_3=(p_2Q_1),\ \mu_4=(p_1Q_2),$$

$$r^2=2m^2-2(p_1p_2),\quad q^2=2M^2-2(Q_1Q_2),\quad \mu_{13}=\mu_1-\mu_3.$$

Kinematics

$$\mu_1=Mp_1^l,\quad \mu_3=Mp_2^l,\quad r^2/2=-p_1^lp_2^l(1-\cos\theta^l),$$

$$\Lambda_2=\mu_{13}+r^2/2.$$

For the electron energies considered we assume that $\epsilon_j=p_j$. The energies in the photon-proton center-of-mass system are

$$\epsilon_1=W^{-1}(\Lambda_2+\mu_3),\quad E_1=W^{-1}(M^2+\mu_{13}),\quad k=W^{-1}\Lambda_2,$$

$$\epsilon_2=W^{-1}(\mu_1-\Lambda_2),\quad E_2=W^{-1}(M^2+\Lambda_2).$$

$W=E_2+k$ = center-of-mass energy of photon plus proton $=(M^2+2\Lambda_2)^{\frac{1}{2}}$.

$$g_1=k(\epsilon_1-\epsilon_2\cos\theta),$$

$$g_2=k(\epsilon_1\cos\theta-\epsilon_2),$$

$$f=M^2-E_1E_2.$$

Here g_1, g_2, f are defined in terms of center-of-mass system energies, and θ is the angle between p_1 and p_2 in the center-of-mass system.

APPENDIX B

$$\mathbf{Y}_{el}=\mathbf{Y}_{el}^{(1)}+\mathbf{Y}_{el}^{(2)},$$

$$\mathbf{Y}_{el}^{(1)}=\frac{M^4}{2q^4}\left\{\left[(1+\mu)^2\frac{q^2}{M^2}+\frac{P^2}{2M^2}\left(1-\frac{\mu^2q^2}{4M^2}\right)\right]\left[-\frac{2\lambda_1}{\lambda_2}-\frac{2\lambda_2}{\lambda_1}-\frac{m^2q^2}{\lambda_1^2}-\frac{m^2q^2}{\lambda_2^2}-\frac{r^2q^2}{\lambda_1\lambda_2}\right]\right.$$
$$\left.+\left[1-\frac{\mu^2q^2}{4M^2}\right]\left[-\frac{2m^2(p_1P)^2}{M^2\lambda_2^2}-\frac{2m^2(p_2P)^2}{M^2\lambda_1^2}-\frac{q^2(p_1P)^2+q^2(p_2P)^2}{M^2\lambda_1\lambda_2}\right]\right\},$$

$$\mathbf{Y}_{el}^{(2)}=\frac{M^4}{q^4}\left\{(1+\mu)^2\frac{q^2}{M^2}\left[\frac{2m^2}{\lambda_1}-\frac{2m^2}{\lambda_2}-\left(\frac{m^2}{\lambda_2}-\frac{m^2}{\lambda_1}\right)^2\right]+\left(1-\frac{\mu^2q^2}{4M^2}\right)\left[\frac{m^2P^2q^2-2m^2(\mathbf{k}P)^2+2m^2(p_1P)^2+2m^2(p_2P)^2}{2M^2\lambda_1\lambda_2}\right]\right\}.$$

The $\mathbf{Y}_{el}^{(2)}$ term is neglected in the limit $m^2\to 0$.

We also note that the $|\mathbf{M}_{\text{II}}|^2$ contribution can be obtained from the above by a permutation (if $\mu=0$).

$$\mathbf{Y}_{\text{proton}}=M^2\times[\text{permutation of } (1/M^2)\,\mathbf{Y}_{el}]$$

$$\text{Permutation: }\left.\begin{array}{ll} M\rightleftharpoons m, & P\to p_1+p_2 \\ \lambda_i\rightleftharpoons\Lambda_i & \\ q\rightleftharpoons r & \end{array}\right\}\text{i.e.,}\left\{\begin{array}{l} P_i\to Q_i, \\ \\ m\rightleftharpoons M. \end{array}\right.$$

APPENDIX C

The integrals which occur in the integrated cross section are: (See Appendix A for definitions of symbols—all energies, angles, and momenta are in the center-of-mass system.)

$$\int d\Omega\,\frac{M^6}{\lambda_1q^4}=\frac{\pi M^6}{2k\epsilon_1(f+g_1)}\ln\left[\frac{4\epsilon_1^2(f+g_1)^2}{m^2(f^2-k^2Q_1^2)}\right]+\frac{\pi^2M^6(fg_1+k^2Q_1^2)}{k\epsilon_1(f+g_1)^2(f^2-k^2Q_1^2)},$$

$$\int d\Omega\,\frac{M^4}{\lambda_1q^2}=\frac{\pi M^4}{k\epsilon_1(f+g_1)}\ln\left[\frac{4\epsilon_1^2(f+g_1)^2}{m^2(f^2-k^2Q_1^2)}\right],$$

$$\int d\Omega \frac{M^2}{\lambda_1} = \frac{2\pi M^2}{k\epsilon_1} \ln\left[\frac{4\epsilon_1^2}{m^2}\right],$$

$$\int d\Omega \frac{q^2}{\lambda_1} = \frac{4\pi(f+g_1)}{k\epsilon_1} \ln\left[\frac{4\epsilon_1^2}{m^2}\right] - \frac{8\pi g_1}{k\epsilon_1},$$

$$\int d\Omega \frac{q^4}{M^2\lambda_1} = \frac{8\pi(f+g_1)^2}{k\epsilon_1 M^2} \ln\left[\frac{4\epsilon_1^2}{m^2}\right] + \frac{8\pi(k^2Q_1^2-3g_1^2-4fg_1)}{M^2k\epsilon_1},$$

$$\int d\Omega \frac{m^2q^2}{\lambda_1^2} = \frac{8\pi(f+g_1)}{k^2},$$

$$\int d\Omega \frac{m^2M^2}{\lambda_1^2} = \left[\frac{M^2}{2(f+g_1)}\right] \int d\Omega \frac{m^2q^2}{\lambda_1^2},$$

$$\int d\Omega \frac{m^2M^4}{\lambda_1^2q^2} = \left[\frac{M^2}{2(f+g_1)}\right]^2 \int d\Omega \frac{m^2q^2}{\lambda_1^2},$$

$$\int d\Omega \frac{m^2M^6}{\lambda_1^2q^4} = \left[\frac{M^2}{2(f+g_1)}\right]^3 \int d\Omega \frac{m^2q^2}{\lambda_1^2},$$

$$\int d\Omega \frac{M^4}{q^4} = \frac{\pi M^4}{f^2-k^2Q_1^2},$$

$$\int d\Omega \frac{M^2}{q^2} = \frac{\pi M^2}{kQ_1} \ln\left[\frac{(f+kQ_1)}{(f-kQ_1)}\right],$$

$$\int d\Omega \frac{q^2}{M^2} = \frac{8\pi f}{M^2},$$

$$\int d\Omega \frac{M^4}{2\lambda_1\lambda_2} = -\frac{4\pi M^2}{k^2r^2} \ln\left(\frac{-r^2}{m^2}\right),$$

$$\int d\Omega \frac{\lambda_1+\lambda_2}{q^2} = \left(\frac{k(\epsilon_1+\epsilon_2)}{M^2} + \frac{f(g_1\epsilon_1+g_2\epsilon_2)}{kQ_1^2M^2}\right) \times \int d\Omega \frac{M^2}{q^2} - \frac{2\pi(g_1\epsilon_1+g_2\epsilon_2)}{kQ_1^2}.$$

The corresponding integrals where λ_1 has been replaced by λ_2 can be obtained from the above by the substitution $g_1 \to g_2$, $\epsilon_1 \to \epsilon_2$.

Integrals of the same form as above, but where q^2 has been replaced by $2\Lambda_1$ can be obtained from the above set of integrals by substitution; $f \to f+\mu_1-\mu_3$.

Note: $Q_1 = |\mathbf{Q}_1|$ (in center-of-mass system).

APPENDIX D

Our calculation has taken into consideration only the electrodynamic interaction. We examine here the mesonic contribution of the virtual Compton process to the diagrams II of Fig. 2. Karzas, Watson, *et al.*[8] have computed the matrix element for the real Compton effect in the static limit. This includes essentially the transverse part of the mesonic addition to $J_p^{\mathrm{II}\nu}$. We denote the sum of these as $J_p^{\mathrm{II}\nu*}$. The $|J_p^{\mathrm{II}}|^2$ contribution of the bremsstrahlung can be related to the real proton Compton effect by a standard Weizsäcker-Williams treatment.[9] In the limit of $r^2 \to 0$, the relation between these real and virtual processes becomes a purely kinematic one,

$$\lim_{r^2\to 0} \frac{d^2\sigma(\text{brems.})}{dp_2{}^l d\Omega_2{}^l} = \frac{\alpha\sigma(\text{Compton})}{4\pi^2 p_1(1-\cos\theta^l)}.$$

Therefore an estimate of this contribution for this nonphysical limit can be made directly from the work of Karzas *et al.*[8] At resonance, where we expect the effect to be most significant, we find

$$\lim_{r^2\to 0} \frac{d^2\sigma(\text{brems.})}{dp_2{}^l d\Omega_2{}^l} \simeq 1.6\times 10^{-36} \frac{\text{cm}^2}{\text{sterad-Mev}}.$$

For small values of r^2 the result can be expected to be insignificant.

We have not carried out the calculation for $r^2>0$. If we assume small longitudinal and scalar contributions to the matrix element and a monotonic decrease behaving roughly as $|F|^2$, as in the corresponding meson process (which is more slowly varying than the electrodynamic process) then we estimate from Fig. 3 that $|J_p^{\mathrm{II}*}|^2$ may contribute roughly 4% correction to the pure electrodynamic result for an incident electron energy of 700 Mev at resonance. This estimate, however, does not include the mesonic part of the $|J_p^{\mathrm{II}\nu*}J_{e\nu}|$ interference. We note that integration considerably reduced the pure electrodynamic part of the interference term, but we have not made any attempt here to ascertain whether the mesonic interference is likewise affected. It is, however, probably the dominant correction to our result at resonance, at least for still larger values of ϵ_1 than those considered here, although bremsstrahlung is less significant as the background to pion production by electrons in this case.

Note added in proof.—After this manuscript was submitted for publication we received a communication from M. Dresden, T. Fulton, and D. S. Moroi informing us of similar work[10] which confirms the expression for $\mathbf{Y}_{el}$. We wish to thank them for pointing out some errors in our preprint.

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