Resonance Theory of Neutron Cross Sections of Fissionable Nuclei

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The general Wigner-Eisenbud resonance theory is used to develop a method of analysis for the neutron cross sections of fissionable nuclei. The method is employed in giving a reasonable description of the low-energy cross sections in U^{235}. The single-level fit for U^{235} is known to be unreasonable. Many-level expressions for the cross sections are derived—the only approximation to the general theory being the neglect of all but a small group of resonances. No explicit reference to fission channels is needed and the many-level expressions require few level parameters: the E_λ, $\Gamma_{\lambda n}$, $\Gamma_{\lambda\gamma}$, and $\Gamma_{\lambda f}$ of the single-level theory for each resonance and a few additional parameters pertinent to the interference between levels. The interference terms are described and shown to be important. Their average value yields information about the number of channels involved in fission. The shape and size of the U^{235} cross sections below 2 ev are fitted to within one or two percent using: (a) only one negative energy resonance of smaller size than in the single-level fit; (b) no additional levels to fit the shape other than the observed levels at positive energies; (c) three interference parameters whose size suggests that there are several fission channels in U^{235}.

1. INTRODUCTION

THE low-energy neutron cross sections of U^{235}, measured by Sailor and Shore,[1] are among the best-known nuclear cross-section data. In spite of this fact many difficulties have been encountered in the description of these cross sections with the resonance theory—difficulties which also seem to occur in the other common fissionable isotopes, U^{233} and Pu^{239}.[2] The troublesome features of the cross sections are: (1) at thermal energies both the total cross section, σ_{nT}, and the fission cross section, $\sigma_{n,f}$, are uncommonly large; (2) the shape of the resonances, particularly of the first two at positive energies in U^{235}, is quite asymmetric about the resonance energies; and (3) the value of the cross sections between resonances is anomalously large. The data of Sailor and Shore,[1] for U^{235}, clearly show all of these features.

If only the first difficulty, the large thermal cross section, had occurred the problem could be easily solved by assuming the existence of a single resonance, of normal size, at negative energies. By such a device the U^{235} cross sections could be fitted, to within experimental error, from zero energy up to the first maximum at 0.29 ev. However the Breit-Wigner single-level theory does not lend itself very readily to providing solutions for the difficulties (2) and (3) above. To fill in the cross sections in between the 0.29-ev and 1.13-ev resonances of U^{235}, the negative-energy resonance can be moved further from zero neutron energy and increased in size. In effect the two peaks (at 0.29 and 1.13 ev) are made to lie on the wings of a huge negative-energy resonance. Such a fit requires that the reduced neutron width of the negative-energy resonance be unreasonably large (roughly 25 times the average size of the reduced widths of the levels observed at positive energies) and even then it does not give a solution to the difficulty (3) above. The single-level fit[3] required additional resonances at -0.02, 0.4, and 0.9 ev to fill in the unusual shape of the cross sections of U^{235}. These latter three levels do not all correspond to actual peaks in the cross section. The difficulties with the shapes of the cross section are such that these levels had to have both an unusually small value of the reduced neutron width and a large fission width. Such levels have a low probability of occurring.

Since the single-level Breit-Wigner formula does not give a reasonable description of the U^{235} cross sections, the explanation of those cross sections lies in either the more general resonance theory which includes the effects of the interference between levels or in a breakdown of the resonance theory. Interference between the levels of U^{235} was first considered by Sailor.[4] There are many reasons to suspect that the interference between levels should be important for the fissionable isotopes.

The single-level Breit-Wigner formula is strictly valid only when the widths, Γ_λ, of the neighboring resonances are much smaller than their spacing, D, from the resonance being considered. In the fissionable nuclei Γ_λ/D is frequently larger than 0.1. To be more precise, the capture width should be subtracted from the total width in this criterion since the capture width is really a sum of a very large number of partial widths each pertaining to the decay of the given level into one of the lower lying states. The reduced width amplitudes for capture presumably have sign fluctuations so that

[1] F. J. Shore and V. L. Sailor, preceding paper [Phys. Rev. **112**, 191 (1958)].
[2] See Fig. 3. A summary of the U^{235} data as well as that for the other fissionable isotopes is contained in *Neutron Cross Sections*, compiled by D. J. Hughes and J. A. Harvey, Brookhaven National Laboratory Report BNL-325 (U. S. Government Printing Office, Washington, D. C., 1955), and *Neutron Cross Sections*, compiled by D. J. Hughes and R. Schwartz, Brookhaven National Laboratory Report BNL-325, Supplement No. 1 (U. S. Government Printing Office, Washington, D. C., 1957).
[3] J. A. Harvey and J. E. Sanders, *Progress in Nuclear Energy* (McGraw-Hill Book Company, Inc., New York, 1957), Series I; see also, H. Bethe, *Progress in Nuclear Energy* (McGraw-Hill Book Company, Inc., New York, 1957), Series I.
[4] V. L. Sailor, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (United Nations, New York, 1956), Paper P/832.

it is very unlikely that the capture process can ever cause interference between the levels of heavy nuclei (see Sec. 2).

In the nonfissionable nuclei the average level spacing is usually several orders of magnitude larger than the neutron width of any level. Where this is not so, as, for example, in Mn^{55}, it is necessary to consider interference between the levels in order to describe the cross section accurately. This has been done for Mn^{55} by Krotkov.[5] In the nuclei U^{233}, U^{235}, and Pu^{239} the fission width is usually only about an order of magnitude smaller than the spacing between adjacent levels (the neutron width is much smaller). If the principal contribution to the fission width, $\Gamma_{\lambda f}$, comes from only a few fission channels then interference between the levels can occur.

At first sight it would appear that each set of fission products in each state of excitation is a separate fission channel. However, Bohr[6] pointed out that in the fission process the nucleus passes through an intermediate state which is a shape deformation. Furthermore, if the energy is low enough then the number of these states is small. Bohr suggested then that the term channel should be applied to the intermediate states of deformation. In his picture many different sets of fission products arise in the same channel by small interactions between the particles in the channel during the existence of the intermediate state. If Bohr's suggestion is correct then it should be possible to analyze resonances involving fission using only a few channels to describe fission. The distribution in size of the fission widths, as analyzed by Porter and Thomas,[7] is in agreement with the idea that there are only a few fission channels. If there were many independent channels for fission the fission widths would be expected to remain constant from level to level as the radiation widths apparently do.[8] From the distribution in size of the fission widths Porter and Thomas inferred that there were two or three important fission channels. In this case the large fission width combined with the small level spacing of the fissionable nuclei can cause serious error in the application of the single-level formulas to the cross sections of these isotopes.

The analysis of Sailor[4] showed that the shape of the cross sections could be changed considerably by the interference of resonances. The formulas he used were based on an analysis for which Γ_λ/D is small, where D is the average spacing between levels and, as is shown in the next section, under the assumption that there is only one fission channel. Neither of these assumptions are likely to be good for U^{235}.

The present study was undertaken to show that the low-energy U^{235} cross sections could be explained in a reasonable and simple way with the general Wigner-Eisenbud[9,10] theory. Toward this end the next section provides a method which is derived from the completely general theory by only one weak assumption—that the analysis of the cross sections at a given energy can be made using only a small *group* of resonances. The method can be used for any of the fissionable isotopes. It involves only a few parameters: apart from the usual widths and energies of the Breit-Wigner formula, only one extra parameter for each pair of levels of the same spin and parity. The application of this method to U^{235} is given in Sec. 3.

2. MANY-LEVEL CROSS SECTIONS

The resonance theory developed by Wigner and Eisenbud[9,10] contains almost no specific assumptions. On the other hand, the theory is, in general, very complicated and contains a discrete infinity of free parameters. Simple expressions for the cross sections can be obtained for two types of approximations in which the cross sections are assumed to involve: (i) a few channels and an arbitrary number of resonance levels; (ii) a few levels but an arbitrary number of channels. Which approximation is the best depends on the physical situation. The general formula for cross sections near a single resonance line is a special case of (ii).

Of the two types of approximations to the general resonance theory the one which applies most naturally to the low-energy neutron cross sections of the fissionable nuclei is (ii). The reason for this fact is that the number and definition of the fission channels is ambiguous and that, strictly speaking, the number of channels for resonance capture is very large. On the other hand, it will be shown below, and in Sec. 4, that even for resonances involving fission one needs to consider only a small number of resonance levels at a time. Thus approximation (ii), above, can be justified for the neutron cross sections of fissionable nuclei. In addition, the cross-section expressions derived from this approximation require no explicit reference to the number or definition of the fission channels: indeed, the number of fission channels is deduced from the results of this method rather than being inserted into the method initially.

The nature of the many-level approximation (ii), above, can be pointed out by examining the cross sections of the fissionable isotopes. As was pointed out in the preceding section the single-level formulas break down for these isotopes because Γ_λ/D for these resonances is only slightly smaller than unity. However,

[5] R. Krotkov, Can. J. Phys. **33**, 622 (1955).
[6] A. Bohr, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (United Nations, New York, 1956), Vol. 2, p. 151.
[7] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).
[8] J. S. Levin and D. J. Hughes, Phys. Rev. **101**, 1328 (1956).
[9] E. P. Wigner, Phys. Rev. **70**, 15 (1946); **70**, 606 (1946); and E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[10] For a summary of resonance theory the reader is referred to A. M. Lane and R. G. Thomas, Revs. Modern Phys. (to be published) or to E. Vogt, *Nuclear Reactions* [North-Holland Publishing Company, Amsterdam (to be published)].

if we consider in full detail all of the states lying in some small energy interval around the energy of interest we are ignoring only the overlap effects due to the many smaller amplitudes from the states outside this interval. The average magnitude of these smaller amplitudes falls off as $(E_\lambda - E)^{-1}$ where $E_\lambda - E$ is the energy separation between the distant states and the energy E, of interest. Even though the magnitude does not decrease very quickly, the sign of the small overlapping amplitudes presumably fluctuates from one distant level to another (see Sec. 4) so that their actual contribution to the cross section near E should be small. A completely similar assumption about the distant resonances must be made for the single-level theory even when Γ_λ/D is very small. In the latter case the single-level formula has always worked very well. In considering all of the levels in an interval about E, the many-level theory [approximation (ii), above] should be valid whenever $\Gamma_\lambda/(E_\lambda - E)$ is small for the *ignored* levels. As long as the interval contains at least several levels, $\Gamma_\lambda/(E_\lambda - E)$ for the ignored levels will be much smaller than Γ_λ/D, where D is the average spacing between levels. Thus, by taking enough levels at a time, the many-level theory [approximation (ii), above] for closely spaced levels ($\Gamma_\lambda \sim D$) can presumably be made as good as the single-level theory for widely spaced levels ($\Gamma_\lambda \ll D$). In the remainder of this section the many-level expressions for the cross sections of fissionable nuclei will be developed using the approximation (ii) above.

The commonly measured cross sections of the fissionable isotopes are the fission cross section, $\sigma_{n,f}$, the scattering cross section, $\sigma_{n,n}$, the capture cross section, $\sigma_{n,\gamma}$, the total cross section $\sigma_{nT}(=\sigma_{n,n}+\sigma_{n,f}+\sigma_{n,\gamma})$, and the absorption cross section $\sigma_{nX}(=\sigma_{n,\gamma}+\sigma_{n,f})$. Each cross section is integrated over the angles of the emitted particles. In the Wigner-Eisenbud theory[9,10] the general expression for a cross section, $\sigma_{cc'}$, proceeding from an incident channel c to an outgoing channel c', integrated over the possible angles of c' is

$$\sigma_{cc'} = \frac{\pi}{k_c^2} \sum_J \frac{(2J+1)}{(2I+1)(2i+1)} |\delta_{cc'} - U_{cc'}{}^J|^2, \qquad (1)$$

where I is the spin of the target nucleus, i the spin of the incident nucleon, J the total angular momentum of the system, k_c the relative momentum of the incident particle and the target nucleus, and $U_{cc'}{}^J$ the collision matrix component of a given J referring to the channels c and c'. In addition to being integrated over all angles, (1) has been averaged over the possible polarizations of the incident nucleon and the target nucleus. For the low-energy neutrons which we are considering, only s waves are important so that $J = I \pm \frac{1}{2}$ and the sum over J, in (1), contains only two terms corresponding to the two spins of the compound nucleus that can be reached by s-wave neutrons. In all of the cross sections which we are considering, there are these two species of levels: as far as the cross-section calculations are concerned, the two species are treated completely independently.

If we label the s-wave neutron channel by n then, in (1), $c=n$ for each of our cross sections. The scattering cross section is $\sigma_{n,n}$; the fission cross section is a sum of $\sigma_{nc'}$ over all the c' which correspond to fission. Similarly, the capture cross section is a sum of $\sigma_{nc'}$ over all the channels corresponding to capture. The total cross section is a sum of $\sigma_{nc'}$ over all c' but, because of conservation laws, the collision matrix is always chosen to be unitary and symmetric, a fact which leads to the following simple expression for the total neutron cross section:

$$\sigma_{nT} = \sum_{c'} \sigma_{nc'} = \frac{\pi}{k_n^2} \sum_J 2g \,\mathrm{Re}(1 - U_{nn}{}^J), \qquad (2)$$

where Re stands for the real part of the expression in parentheses, and g is the statistical spin factor:

$$g = \frac{2J+1}{(2I+1)(2i+1)}. \qquad (3)$$

(2) follows at once from (1) and the symmetry and unitarity of U^J, that is, from

$$\sum_{c'} U_{c'c} U_{c'c}{}^* = 1, \qquad (4)$$

where the asterisk indicates the complex conjugate. Equation (2) states that σ_{nT} depends only on the diagonal component of U^J pertaining to the incident s-wave neutron channel, and only linearly on this component. This fact makes σ_{nT} particularly simple to calculate.

In order to calculate the cross sections we must evaluate the components of the collision matrix, $U_{cc'}{}^J$, to be used in (1). It is at this stage that we write down the many-level form of the collision matrix.[11] As is shown in Appendix A, $U_{cc'}{}^J$ may be written

$$U_{cc'}{}^J = e^{i(\varphi_c + \varphi_{c'})} [\delta_{cc'} + i \sum_{\lambda\lambda'} (\Gamma_{\lambda c})^{\frac{1}{2}} (\Gamma_{\lambda' c'})^{\frac{1}{2}} A_{\lambda\lambda'}], \qquad (5)$$

where $(\Gamma_{\lambda c})^{\frac{1}{2}}$ is the square root of the observed partial width $\Gamma_{\lambda c}$ for the decay of the level λ into the channel c. The square root is to be taken with the sign appropriate to the reduced width amplitude $\gamma_{\lambda c}$, that is

$$(\Gamma_{\lambda c})^{\frac{1}{2}} \equiv (2P_c)^{\frac{1}{2}} \gamma_{\lambda c}, \qquad (6)$$

where P_c is the penetration factor of the channel c. The $(\Gamma_{\lambda c})^{\frac{1}{2}}$ can, in general, have either sign. We use this notation to avoid referring explicitly to the penetration factors of fission and capture channels, which we shall neither use nor define. The φ_c of (5) are potential scattering phase shifts, that is, the sum of a Coulomb phase shift and a hard-sphere potential scattering phase shift. The only φ_c we shall need are those for s-wave neutrons. In that case $\varphi_n = -ka$ where

[11] E. P. Wigner, Revs. Modern Phys. 70, 606 (1946).

a is the radius of the nucleus. The $A_{\lambda\lambda'}$ of (5) was derived in Appendix A. The rows and columns refer to levels not to channels, and we must remember that the levels are of the same species since U^J refers only to levels of a given spin and parity. The reciprocal of A has the components

$$(A^{-1})_{\lambda\lambda'}=(E_\lambda-E)\delta_{\lambda\lambda'}-\tfrac{1}{2}i\textstyle\sum_c(\Gamma_{\lambda c})^{\frac{1}{2}}(\Gamma_{\lambda' c})^{\frac{1}{2}}, \tag{7}$$

where c runs over all channels and E_λ is the resonance energy of the level λ.

The cross-section expressions (1) with the collision matrix components (5) and, in turn, the level matrix A determined from (7), are still those of the completely general Wigner-Eisenbud theory. However, in general, a physical scattering process involves an infinite number of levels so that the matrix inversion of (7) cannot be accomplished. We shall ignore all but a small number of levels so that the inversion (7) can be accomplished. With this one approximation we can use (1), (5), and (7) not only to write down but also to calculate the cross sections which are of interest to us.

To simplify the expressions for the cross sections we define a reduced neutron width, $\Gamma_{\lambda n}{}^0$, as

$$\Gamma_{\lambda n}{}^0=2g\Gamma_{\lambda n}E^{-\frac{1}{2}}, \tag{8}$$

where E is the neutron energy (in ev). We have absorbed the statistical spin factor into the definition of the reduced width. Then using the fact that $\pi/k_n{}^2=6.52\times10^5/E$ barns, where E is in ev, we obtain

$$\begin{aligned}\sigma_{nT}E^{\frac{1}{2}}=6.52\times10^5\,\mathrm{Re}\{&(1-e^{-2ika})2gE^{-\frac{1}{2}}\\&+ie^{-2ika}\textstyle\sum_{\lambda,\lambda'}(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}(\Gamma_{\lambda' n}{}^0)^{\frac{1}{2}}A_{\lambda\lambda'}\}\\&+\text{similar term for the other spin,}\end{aligned} \tag{9}$$

and

$$\begin{aligned}\begin{pmatrix}\sigma_{n,f}\\ \sigma_{n,\gamma}\\ \sigma_{nX}\end{pmatrix}E^{\frac{1}{2}}=\left(\frac{6.52\times10^5}{2}\right)\textstyle\sum_c\left|\sum_{\lambda\lambda'}(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}(\Gamma_{\lambda' c})^{\frac{1}{2}}A_{\lambda\lambda'}\right|^2\\+\text{similar term for the other spin,}\end{aligned} \tag{10}$$

where c refers to fission channels for $\sigma_{n,f}$, to capture channels for $\sigma_{n,\gamma}$, or to both for σ_{nX}. The scattering cross section, $\sigma_{n,n}$, is

$$\begin{aligned}\sigma_{n,n}E^{\frac{1}{2}}=(6.52\times10^5)gE^{-\frac{1}{2}}|1-e^{-2ika}\\\times\{1+i\textstyle\sum_{\lambda\lambda'}(\Gamma_{\lambda n})^{\frac{1}{2}}(\Gamma_{\lambda' n})^{\frac{1}{2}}A_{\lambda\lambda'}\}|^2\\+\text{similar term for the other spin.}\end{aligned} \tag{11}$$

Since there are assumed to be very many capture channels and the $(\Gamma_{\lambda c})^{\frac{1}{2}}$ for these channels are assumed to fluctuate in sign, we can simplify $\sigma_{n,\gamma}E^{\frac{1}{2}}$ at once to

$$\begin{aligned}\sigma_{n,\gamma}E^{\frac{1}{2}}=(6.52\times10^5/2)\\\times\textstyle\sum_{\lambda,\lambda',\lambda''}\Gamma_{\lambda\gamma}(\Gamma_{\lambda' n}{}^0)^{\frac{1}{2}}(\Gamma_{\lambda'' n}{}^0)^{\frac{1}{2}}A_{\lambda\lambda'}A_{\lambda\lambda''}{}^*\\+\text{similar term for the other spin.}\end{aligned} \tag{12}$$

In (12), $\Gamma_{\lambda\gamma}$ is the total radiation width for the level λ.

If A^{-1} (and therefore A) is assumed to be a diagonal matrix, then the cross-section expressions (9) to (12) reduce to the sum of single-level contributions. It is this case which we have referred to above as the single-level formula. A^{-1} will be diagonal if the sum $\sum_c(\Gamma_{\lambda c})^{\frac{1}{2}}(\Gamma_{\lambda' c})^{\frac{1}{2}}$ of (7) vanishes for all $\lambda\neq\lambda'$. The fact that the cross terms between resonances in the cross sections vanish in this case is obvious for σ_{nT} from (9) and for $\sigma_{n,\gamma}$ from (12). The single-level expressions for the cross sections involve four independent parameters for each level: E_λ and any three of Γ_λ, $\Gamma_{\lambda n}$, $\Gamma_{\lambda f}$, or $\Gamma_{\lambda\gamma}$.

When the nondiagonal elements of A^{-1} are included in the cross-section calculations, terms arise which correspond to interference between levels. These interference terms can be important in the low-energy neutron cross sections of fissionable nuclei. Their use will be described below and their importance in the cross sections of U^{235} is described in the next section. At first sight it would seem that the formulas for N levels ($N>1$) and nondiagonal A are difficult to handle because they involve inverting the complex matrix A^{-1}, and that this general case involves a large number of additional parameters—such as the signs of the $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$ and the partial widths for individual fission channels—which did not occur in the single-level formulas. The first of these impressions is correct but the second is not. As is shown below, the cross sections are independent of the signs of the $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$ and the *only* new parameters which occur in the cross sections are the $N(N-1)/2$ off-diagonal components of A^{-1}, where N is the number of resonance levels which are being considered. That the latter fact is so is evident at once from a look at $\sigma_{n,n}$, σ_{nT}, and $\sigma_{n,\gamma}$, (9) to (12) above, which do not involve the partial fission widths. Since $\sigma_{n,f}=\sigma_{nT}-\sigma_{n,\gamma}-\sigma_{n,n}$, this fact must also hold for $\sigma_{n,f}$.

To make the new independent parameters more perspicuous we shall look at the fission cross section itself. We define a vector $\mathbf{g}_{\lambda f}$ whose components refer to fission channels and are equal to $(\Gamma_{\lambda c})^{\frac{1}{2}}$, where c refers to a fission channel. With this definition the nondiagonal terms of A^{-1} are

$$(A^{-1})_{\lambda\lambda'}=-(i/2)\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}\quad(\lambda\neq\lambda'). \tag{13}$$

We have neglected the contributions of the capture channels to $(A^{-1})_{\lambda\lambda'}$ since they are negligible because of sign fluctuations. The neutron channel is also neglected because the value of the neutron widths are so small: more precisely, for the fissionable isotopes, $(\Gamma_{\lambda n})^{\frac{1}{2}}(\Gamma_{\lambda' n})^{\frac{1}{2}}$ is always small compared to any Γ_λ so that the contribution of the neutron channel to the off-diagonal elements of A^{-1} is several orders of magnitude smaller than any of the diagonal elements of A^{-1}. Because of this fact we can neglect the neutron channel in the off-diagonal elements. Using the vectors $\mathbf{g}_{\lambda f}$ to write the fission cross section, (10), we obtain

$$\begin{aligned}\sigma_{n,f}E^{\frac{1}{2}}=(6.52\times10^5/2)\textstyle\sum_{\lambda,\lambda',\lambda'',\lambda'''}(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}(\Gamma_{\lambda' n}{}^0)^{\frac{1}{2}}\mathbf{g}_{\lambda'' f}\\\cdot\mathbf{g}_{\lambda''' f}A_{\lambda\lambda''}A_{\lambda'\lambda'''}{}^*.\end{aligned} \tag{14}$$

Thus precisely the $N(N-1)/2$ parameters $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$

which occur in the off-diagonal elements of A^{-1} also suffice for the description of the fission cross sections. The fission channels enter into the cross sections only in this way.

To show that the cross sections discussed above do not depend on the signs of the $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$ [or, equivalently, on the signs of the $(\Gamma_{\lambda n})^{\frac{1}{2}}$], we look, first of all, at the signs of the $A_{\lambda\lambda'}$. From (8) and (13) it is apparent at once that changing the sign of the vector $\mathbf{g}_{\lambda f}$ simultaneously changes the sign of the components of A^{-1} in the λth row and the λth column [$(A^{-1})_{\lambda\lambda}$ does not change sign]. Therefore the determinant of A^{-1} does not change sign along with $\mathbf{g}_{\lambda f}$ but the co-factor of the element $(A^{-1})_{\mu\nu}$, $\mu\neq\nu$, does change sign. As a result the sign of $A_{\mu\nu}$ is proportional to $\mathbf{g}_{\mu f}\cdot\mathbf{g}_{\nu f}$. Using this result an inspection of the cross sections, (9) to (12), shows that as far as signs are concerned the amplitudes $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$—or $(\Gamma_{\lambda n})^{\frac{1}{2}}$—always occur as a product with $\mathbf{g}_{\lambda f}$, and since the latter are assumed to have arbitrary signs, all the $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$ or $(\Gamma_{\lambda n})^{\frac{1}{2}}$ can be chosen to be positive without imposing any limitation on the cross-section description. No measurement of σ_{nT}, $\sigma_{n,n}$, $\sigma_{n,f}$, σ_{nX}, or $\sigma_{n,\gamma}$ will tell us anything about the sign of the neutron reduced width amplitude $(\Gamma_{\lambda n}{}^0)^{\frac{1}{2}}$.

The $N(N-1)/2$ new parameters in the many-level cross sections are the scalar products, $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$, between the vectors $\mathbf{g}_{\lambda f}$. Since the length of these vectors is known, $|\mathbf{g}_{\lambda f}|^2=\Gamma_{\lambda f}$, each of the scalar products $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ can be characterized by the angle, $\vartheta_{\lambda\lambda'}$, between $\mathbf{g}_{\lambda f}$ and $\mathbf{g}_{\lambda' f}$, that is,

$$\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}=|\mathbf{g}_{\lambda f}|\,|\mathbf{g}_{\lambda' f}|\,\cos\vartheta_{\lambda\lambda'}. \tag{15}$$

For a set of N levels, and therefore N vectors $\mathbf{g}_{\lambda f}$, we can always choose the unit vector $\mathbf{g}_{\lambda f}/|\mathbf{g}_{\lambda f}|$ for the first level to be the polar axis. Then the same unit vector for the second level defines a polar angle (from its scalar product with the $\mathbf{g}_{\lambda f}$ of the first level) and, together with the vector of the first level, a plane from which the azimuthal angle of the next vector can be measured. Continuing in this way the N vectors $\mathbf{g}_{\lambda f}$ map out a space of N dimensions. This means that the values of the $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ determined by fitting an N-level formula to the cross sections of fissionable isotopes can always be described in terms of only N fission channels. If it turns out that according to some natural definition of the concept fission channel there are a great number of such channels, for the purpose of our N-level fit we could have chosen N independent linear combinations of the original channels. The new linear combinations would suffice completely for describing the cross sections. Thus with the N-level fit we can never tell whether or not there are more than N channels for fission. If there are less than N channels some of the $\mathbf{g}_{\lambda f}$ will be linear combinations of the other $\mathbf{g}_{\lambda f}$.

It is important to remember that the $N(N-1)/2$ parameters $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ are not completely independent of each other. For example, if $\mathbf{g}_{\lambda f}$ is approximately parallel to $\mathbf{g}_{\lambda' f}$ (that is, $\vartheta_{\lambda\lambda'}\sim 0$), then for any third vector $\mathbf{g}_{\lambda'' f}$ we must have $\vartheta_{\lambda\lambda''}\approx\vartheta_{\lambda'\lambda''}$. To make sure that the $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ actually correspond to the scalar products of a set of N vectors, it is simplest to obtain the scalar products by first constructing the vectors as outlined in the preceding paragraph. An example of the construction is given in Table I of the next section.

Although the values of the scalar products $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ are not directly related to the number of fission channels, the average size of these scalar products is. For example, if there is only one fission channel all the $\mathbf{g}_{\lambda f}$ are "parallel" or "antiparallel," that is $\cos\vartheta_{\lambda\lambda'}=\pm 1$. For more than a few levels, it would be very unlikely to find the $\mathbf{g}_{\lambda f}$ closely parallel if there were more than one channel of importance in the fission process. Similarly if the fission process can proceed at an appreciable rate through a large number of channels we would expect all of the scalar products, $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ to vanish, that is $\vartheta_{\lambda\lambda'}\approx\pm\pi/2$ so that the vectors $\mathbf{g}_{\lambda f}$ are mutually perpendicular. This fact would be true if the components of the $\mathbf{g}_{\lambda f}$ have random sign fluctuations from level to level. If the fission process is assumed to occur with equal probability in exactly m channels, then the average value of $\cos\vartheta_{\lambda\lambda'}$ for random sign fluctuations of the components and a Gaussian distribution[7] of their sizes is the average value of $|\cos\vartheta|$ over a sphere in m dimensions, ϑ being the polar angle. The magnitude of the average scalar product calculated in this way is

$$\begin{aligned}\langle|\cos\vartheta_{\lambda\lambda'}|\rangle &= \frac{(m-1)!}{2^{m-1}\{[\frac{1}{2}(m-1)]!\}^2}, && m \text{ odd}\\ &= \frac{2(m-1)!}{\pi}\,\frac{1}{[1\cdot3\cdot5\cdots(m-1)]^2}, && m \text{ even}\\ &\approx [2/\pi(m-1)]^{\frac{1}{2}}, && m \text{ large.}\end{aligned} \tag{16}$$

The variation of the magnitude of the average scalar product, (16), with the number of fission channels m is shown on Fig. 1. Thus when the interference parameters $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ have been computed by fitting the cross section, Fig. 1 can be used, with the average of the interference parameters taken as in (16), to give a rough indication of the number of channels which are important in fission.

To illustrate the effect of the interference terms on the cross section, the contribution of the $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ term to the cross section has been calculated, where λ is the 0.29-ev resonance of U^{235} and λ' the 1.13-ev resonance of the same element. The calculation was made by taking the difference between the successful four-level fit of Fig. 3 to the U^{235} data for the total cross section and the values of the total cross section obtained by setting $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}=0$ for the two resonances, leaving all the other parameters the same. Since the value of $\cos\vartheta_{\lambda\lambda'}$ was approximately -1 for the fit of Fig. 3 (see Table I) the result obtained in this calculation shows the maximum contribution of the interference from two

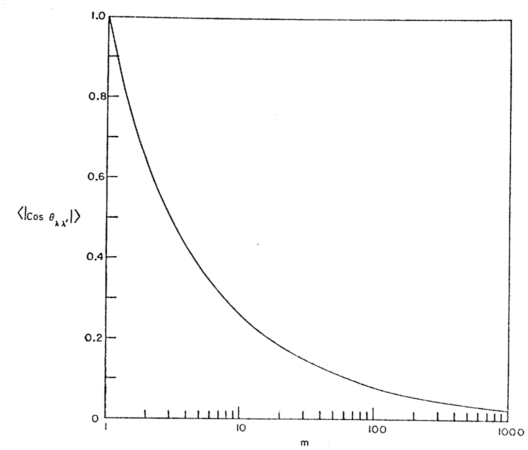

FIG. 1. The average value, $\langle|\cos\vartheta_{\lambda\lambda'}|\rangle$, Eq. (16), of the interference parameters (15) as a function of the number of fission channels, m.

particular resonances in the presence of other interfering resonances. The result is shown by the dashed curve on Fig. 2. As is evident from Fig. 2, the interference term between these two resonances makes an appreciable contribution to the total cross section of U^{235} even though these resonances are not larger than average and are not very close together.

The important property of the cross-section contribution from the interference of a given pair of levels is that it changes sign very near the resonance energy of each of the two levels. Furthermore, the absolute value of the contribution has four maxima of roughly equal size, one on each side of both resonances. The maxima occur at energies differing from the energy of the nearest resonance by the half width, $\Gamma_\lambda/2$, of the resonance. The minimum in the absolute value of the contribution midway between the two resonances is deeper when the separation of the resonances is greater. The sign of the interference contribution is determined by the sign of $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$. A rough approximation to the interference contribution, $\sigma_{nT}^{(\lambda\lambda')}$, of the resonances λ and λ' to the total cross section is given by

$$\sigma_{nT}^{(\lambda\lambda')}E^{\frac{1}{2}}\approx 6.52\times 10^5 \cos\vartheta_{\lambda\lambda'} \times\frac{(E_\lambda-E)(E_{\lambda'}-E)(\Gamma_{\lambda n}{}^0\Gamma_{\lambda' n}{}^0\Gamma_{\lambda f}\Gamma_{\lambda' f})^{\frac{1}{2}}}{[(E_\lambda-E)^2+(\Gamma_\lambda/2)^2][(E_\lambda-E)^2+(\Gamma_{\lambda'}/2)^2]}, \quad (17)$$

which is the approximate formula used by Sailor[4] if we set $\cos\vartheta_{\lambda\lambda'}=1$ and $\Gamma_{\lambda f}=\Gamma_\lambda$, $\Gamma_{\lambda' f}=\Gamma_{\lambda'}$. This approximation is not necessarily very accurate, as is shown on Fig. 2 by a comparison of (17) with the actual $\sigma_{nT}^{(\lambda\lambda')}$ as computed above. It can be shown to be a good approximation for the interference between a pair of levels belonging to a set of well-separated resonances, and it can be used to give rough quantitative estimates of the importance of the interference terms.

The interference described on Fig. 2 shows that we must always pay a price for the contribution to the cross section from these terms. If we choose them to contribute constructively at one energy, they will always contribute destructively—in roughly equal measure—at some other energy. Because of their change in sign at resonance energy, the interference terms can cause a shift in the position of the peaks from the energies E_λ.

The fact that the approximate treatment of interference by means of (17) is only very rough means that we must use the complete expressions (9) to (12) in the cross-section computations. Although the actual number of parameters is not large the computation becomes laborious for more than two levels.

3. ANALYSIS OF U^{235} CROSS SECTIONS

Of the various low-energy neutron cross sections that can be measured, detailed data are available, in U^{235}, only for σ_{nT} and $\sigma_{n,f}$. Because of the complication of Doppler broadening in the higher energy resonances, the analysis of this section will limit itself to σ_{nT} and $\sigma_{n,f}$ below 2 ev in U^{235}. The fact that $\sigma_{n,n}$ or $\sigma_{n,\gamma}$ have not been measured in detail for U^{235} is not disturbing. A crude analysis of the two peaks below 2 ev in U^{235} shows that for these peaks the neutron width is about 10^4 times smaller than the total width so that the resonance scattering is at most a small fraction of a barn, compared to the potential scattering of about 12 barns. Thus $\sigma_{n,n}$ is smooth and fairly constant at about 12 barns (see Fig. 4) and therefore $\sigma_{n,\gamma}$ and σ_{nX} can both be deduced from σ_{nT} and $\sigma_{n,f}$. We shall apply the method of the preceding section to the analysis of σ_{nT} and $\sigma_{n,f}$. Figure 3 gives the data of Sailor and Shore[1] for $\sigma_{n,f}$ and σ_{nT} below 2 ev, and, in addition, the world value[12] of these cross sections at thermal energies.

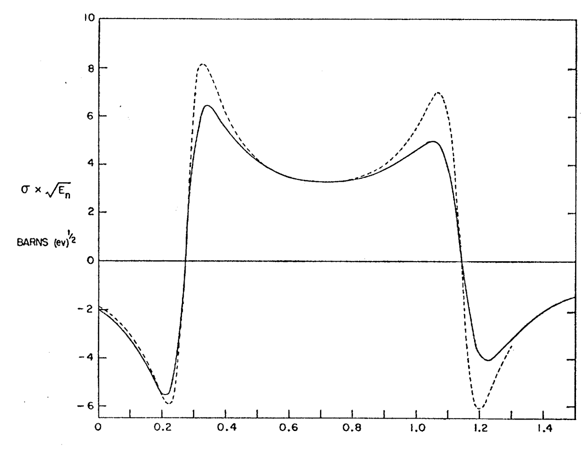

FIG. 2. The contribution of the interference between the 0.29-ev resonance with the 1.13-ev resonance to the total neutron cross section of U^{235} as a function of the neutron energy, E_n. The solid curve corresponds to the approximation (17), the dotted curve to a calculation of the contribution as made with the many-level theory in Sec. 2.

[12] At the 1957 meeting of the American Physical Society, L. M. Bollinger, reported a thermal fission cross section five percent

In describing σ_{nT} and $\sigma_{n,f}$ we shall from the outset use only one negative-energy resonance. We also note that the sharp resonance at 2.04 ev has a small total width and from $\sigma_{nT}/\sigma_{n,f}$ at resonance, a very small fission width so that it doesn't interfere appreciably with any other resonances. It can therefore be treated by the single-level formulas, regardless of its spin. The contribution of the 2.04-ev resonance to the cross section is negligible below 1.7 ev. The analysis was begun by assuming that the negative-energy resonance, the 0.29-ev resonance, the 1.13-ev resonance, and one of the resonances (see reference 1) between three and four ev had the same spin. If the former three resonances do not all have the same spin, this fact will emerge from the analysis inasmuch as it should then be possible to choose the $\mathbf{g}_{\lambda f}$ for one of the resonances to be perpendicular to the $\mathbf{g}_{\lambda f}$'s of the other two. If this turns out to be not possible, this fact is strong evidence that these levels actually have the same spin. The details of the one high-energy resonance chosen are unimportant —we could as well have chosen one level at, say, 8 ev. The high-energy resonance was included to illustrate the effect it has on the cross sections above the energy of the 1.13-ev level. Below this energy it makes no appreciable contribution. In the language used at the beginning of the preceding section, the contribution of the high-energy resonance will tell us how important the "external" resonances are at the edge of the energy interval whose resonances are being considered.

First of all, an attempt was made to fit the cross sections of U^{235} with a negative-energy resonance near zero energy and therefore of reasonable size. By use of the optimum interference between such a level and the two positive-energy resonances (below 2 ev) the cross sections can be increased sufficiently at 0.70 ev to bring them into agreement with the observed cross sections. The amount of interference required to do this is, roughly, the maximum possible; the $\mathbf{g}_{\lambda f}$ of the negative-energy resonance is parallel to that of the 0.29-ev level and antiparallel to that of the 1.13-ev level, as in the case of a single fission channel. This amount of constructive interference at 0.70 ev is, however, balanced by a large amount of destructive interference at 0.15 ev and at energies above 1.13 ev. The two resonance peaks are then very asymmetrical. This fact precludes the possibility that the negative-energy resonance is near zero energy and of average size.

We then put the negative-energy resonance increasingly further from zero energy and make it correspondingly larger so as to fit the thermal cross sections. We do not have to go as far as is necessary in the single-level analysis,[3,4] that is up to -1.4 ev, because we can now receive assistance from interference. The size of the interference terms necessary to fit the cross sections at 0.70 ev becomes smaller as the negative-energy resonance moves from the origin—because the negative energy resonance makes an increasingly larger direct contribution. For the same reason the Γ_λ, $\Gamma_{\lambda n}$, and $\Gamma_{\lambda f}$ of the 0.29-ev resonance become increasingly smaller.

It was pointed out above that the strong interference which is present if there is only one fission channel makes the observed peaks too asymmetrical when the negative-energy resonance is near zero energy. This effect becomes even stronger when the negative-energy resonance is placed further from zero energy, because the amplitude of the negative-energy resonance becomes larger at the positions of the positive-energy resonance. It seems almost certain from this fact that U^{235} cannot be analyzed in terms of a single channel for fission, as long as one invokes the use of only a single negative-energy resonance.

By the time that the negative-energy resonance has gone as far as -0.8 ev from zero neutron energy (roughly half the way toward the energy required by the single-level picture)[3,4] the amount of constructive interference required to fit the 0.7-ev cross sections is only about half of the maximum possible. As pointed out above, the maximum interference was needed when the level was near zero energy. For this value of $E_\lambda(=-0.8$ ev) the cross sections σ_{nT} and $\sigma_{n,f}$ can be fitted everywhere up to 1.2 ev to within the one or two percent error in the data. However, the interference parameters $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$ required to to produce such a fit still provide too much destructive interference in the vicinity of 1.4 ev. Roughly speaking, it would require the optimum interference from *several* of the stronger levels above 2 ev to fit the cross section at 1.4 ev as accurately as at lower energies. This is invoking too much help from external levels.

One needs to proceed only slightly further from zero energy. Figure 3 shows the fit that is obtained at $E_\lambda=-0.95$ ev for the one negative-energy resonance. The remaining parameters are given in Table I. The calculation was carried out using (9) and (14), programmed for the Chalk River Datatron computer. No approximation was made in these formulas. One of the resonances from above 2 ev was used, interfering moderately with the -0.95-, the 0.29-, and the 1.13-ev resonances. Thus four levels were employed—the 2.04-ev resonance was ignored since no attempt was made to fit the cross sections near 2 ev.

Of the six possible interference parameters, $\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$, in the four-level formulas only three independent ones were required for the fit of Fig. 3. This means that our fit allows the possibility that there are only two independent fission channels, a fact which is brought out clearly by the vectors $\mathbf{g}_{\lambda f}$ of Table I. The fact that such a two-channel fit was possible does not imply, even mildly, that there are only two fission channels. Its meaning lies in the fact that the other three interference parameters, even when they are of moderate

higher than the world value used on Fig. 3. If his thermal cross section is correct then all the fission cross section data of Fig. 3 should be increased by this amount since they were normalized with the world value of the thermal fission cross section. In turn, the fission widths in our analysis of U^{235} should then be increased by the same five percent, all other parameters being left the same.

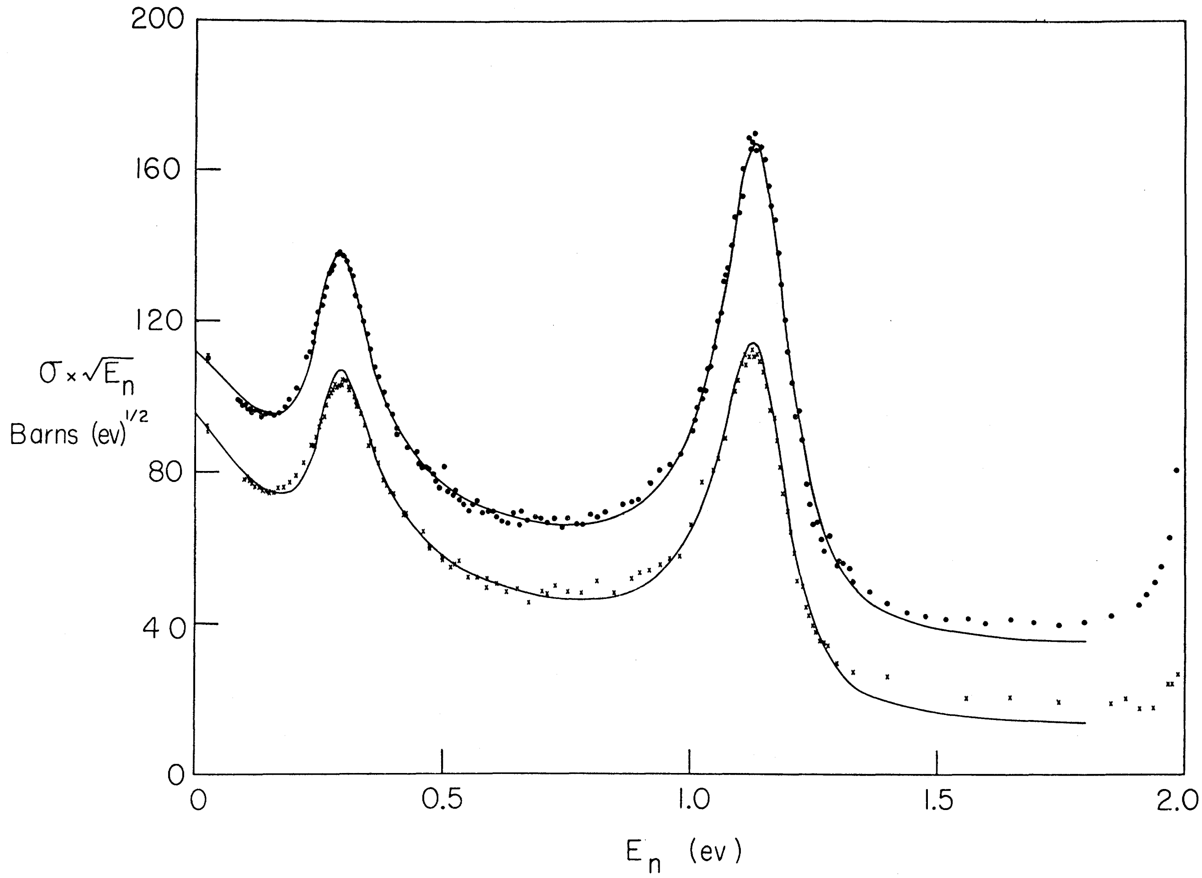

FIG. 3. The data for the total neutron cross section, σ_{nT}, and the fission cross section, $\sigma_{n,f}$, of U^{235} as a function of the neutron energy, E_n. The solid points (upper set of data) are the data of reference 1 for σ_{nT} and the crosses (lower set of data) are the data for $\sigma_{n,f}$ from the same reference. The world values of the thermal cross sections (i.e., at 0.025 ev) are also shown. The solid curves are the many-level fit to the data, as discussed in Sec. 3. The parameters of the fit are given in Table I.

size, do not contribute much to the cross section. The important interference parameters are those involving the scalar product of $\mathbf{g}_{\lambda f}$ for the strong negative-energy resonance with the vectors of the other three resonances. The value of the average $\langle\cos\vartheta_{\lambda\lambda'}\rangle$ is 0.530 for the six interference parameters, which lies close to the average expected for three fission channels (see Fig. 1).

The fit of Fig. 3 is about as good as the data everywhere up to about 1.3 ev. In the vicinity of 1.5 ev both of the computed cross sections are too low. The cross section in that region depends on the resonances above 2 ev. The contribution from a higher energy level can be chosen to interfere so that it makes an appreciable contribution at 1.5 ev without affecting the cross section elsewhere. The 3.14-ev resonance used in Table I, i.e., on Fig. 3, was used in this way. Its contribution at 1.5 ev is twice the difference between the computed and observed cross sections at that energy. Thus only a minor contribution from the higher energy resonances is required to fit the cross sections up to 2 ev.

The fit of Fig. 3 is fairly unique. The position of the negative-energy resonance cannot be placed much further from zero energy than -0.95 ev. In the single-level fit[3,4] the slope of the computed cross sections near zero energy was very small, a fact which was corrected by assuming the existence of a second negative-energy resonance near zero energy. On Fig. 3 the slope of the computed cross sections near thermal energies is much greater than in the single-level fit without the additional negative-energy resonance. The difference in slopes is due to two facts of almost equal importance. First, from the fact that the negative-energy resonance of Table I is closer to thermal energies than the similar resonance in the single-level fit; second, from the interference between the negative-energy resonance and the 0.29-ev resonance. The interference contributes a dip near 0.20 ev. Because of the two facts we get away without the additional resonance near zero energy that was necessary for the single-level fit. However, if we move the negative-energy resonance of Table I further out—even by 0.15 ev to -1.10 ev—the decrease in slope at thermal energies must be compensated for by more interference between the negative-energy resonance and the 0.29-ev resonance. This, in turn, makes

TABLE I. The many-level fit to U^{235} of Fig. 3. The table contains the parameters employed in the many-level fit of Fig. 3. The second column gives the level energy, the third column gives the total level width, the fourth column is the reduced neutron width (8), and the fifth column gives the fission width. The vectors $\mathbf{g}_{\lambda f}$ are defined in Sec. 2. In the table the vector $\mathbf{g}_{1f}$ was chosen as the polar axis so that it has a component only for "channel 1" (see Sec. 2). Channel 2 was chosen so that $\mathbf{g}_{2f}$ had nonzero components only for it and channel 1. The components of $\mathbf{g}_{3f}$ in channel 3 and $\mathbf{g}_{4f}$ in channels 3 and 4 were chosen to be zero since these do not affect the cross sections appreciably. (Such a choice makes the fit compatible with a two-channel picture for fission.) Since the length of each $\mathbf{g}_{\lambda f}$ is connected to $\Gamma_{\lambda f}$, $|\mathbf{g}_{\lambda f}|^2=\Gamma_{\lambda f}$, only three of the seven nonzero components are not derivable from $\Gamma_{\lambda f}$. The interference parameters $\cos\vartheta_{\lambda\lambda'}$, listed below, were defined by (15). $\Gamma_{\lambda\gamma}$, which is also given below, is $\Gamma_{\lambda\gamma}=\Gamma_\lambda-\Gamma_{\lambda f}-\Gamma_{\lambda n}\approx\Gamma_\lambda-\Gamma_{\lambda f}$. Quantities derived from the table are $\cos\theta_{12}=0.148$, $\cos\theta_{13}=-0.326$, $\cos\theta_{14}=-0.707$, $\cos\theta_{23}=-0.906$, $\cos\theta_{24}=0.630$, $\cos\theta_{34}=-0.452$, and thus $\langle|\cos\theta_{ij}|\rangle=0.53$. In addition, $\Gamma_{1\gamma}=0.028$ ev, $\Gamma_{2\gamma}=0.029$ ev, $\Gamma_{3\gamma}=0.044$ ev, and $\Gamma_{4\gamma}=0.031$ ev.

Level number	E_λ ev	Γ_λ ev	$\Gamma_{\lambda n}^0$ $10^{-3}(\text{ev})^{\frac{1}{2}}$	$\Gamma_{\lambda f}$ ev	$\mathbf{g}_{\lambda f}(\text{ev})^{\frac{1}{2}}$ "channel 1"	"channel 2"	"channel 3"	"channel 4"
1	−0.095	0.197	1.49	0.169	0.411	0	0	0
2	0.275	0.128	0.00570	0.099	0.047	0.311	0	0
3	1.144	0.169	0.0157	0.125	−0.113	−0.334	0	0
4	3.16	0.186	0.0182	0.155	−0.278	0.278	0	0

the 0.29-ev resonance too asymmetrical so that we cannot make such a change in the energy of the negative-energy level without invoking the use of an additional state. The size of the important interference terms cannot be changed much without disturbing the shape.

The size of the huge negative-energy resonance has been decreased by a factor of two from the single-level analysis.[3,4] This leaves it still a very large resonance but quite comparable now to the state at 35 ev in U^{235}. The present fit makes it appear almost certain that a negative-energy resonance as strong as this is a fact of nature for U^{235}. The one strong resonance could be replaced by two weaker resonances, each one-third as strong if their mutual interference is optimum. This possibility, or the consideration of any larger number of negative-energy resonances seems less attractive (because it involves more free parameters) and is certainly no more probable.

It seems highly likely that the 1.13-ev resonance has the same spin as the negative-energy resonance because we cannot move the latter (without invoking the use of additional resonances) and at its present position it must interfere with the 1.13-ev state to describe the cross section. The evidence for the fact that the 0.29-ev resonance also has the same spin is weaker since this resonance is weaker and its interference terms are therefore less important. In the present analysis it seemed difficult to avoid using the 0.29-ev resonance as interfering with its two neighbors.

Some independent evidence toward the confirmation of the above analysis of U^{235} is given by the few data for the scattering cross section, $\sigma_{n,n}$. It was pointed out in the beginning of this section that the resonance scattering from the 0.29- and 1.13-ev resonances was negligible compared to the potential scattering cross section. However a negative-energy resonance of the size needed for the fit of Fig. 3 can contribute appreciably to $\sigma_{n,n}$ particularly through the interference of resonance and potential scattering. The value of $\sigma_{n,n}$, computed from (11) with the parameters of Table I, is shown by the solid line of Fig. 4. The most recent data[13] for $\sigma_{n,n}$ are also shown on the figure. The broken line of Fig. 4 gives the potential scattering, $4\pi a^2$ computed with a radius $[a=1.32\times(A^{\frac{1}{3}}+1)\times10^{-13}\text{ cm})]$ chosen to make $\sigma_{n,n}$ at 0.27 ev coincide with the experimental value at that energy. The large deviation of $\sigma_{n,n}$ from $4\pi a^2$ is caused by the negative-energy resonance. The effect of the 0.29- and 1.13-ev resonances on $\sigma_{n,n}$ is clearly evident on Fig. 4. The few experimental points on Fig. 4 agree with the computed curve and therefore with a negative-energy resonance of the size given by Table I. The present data for $\sigma_{n,n}$ are not very compatible with an increase in the size of the negative-energy resonance by a factor of two as required by the single-level fit.[3,4] The agreement of Fig. 4 would also be made worse by decreasing the size of the negative-energy resonance by a factor of two.

When the single-level fits[3,4] to the U^{235} data were made, a few years ago, the old data[4] for $\sigma_{n,n}$ implied the existence of a larger negative-energy resonance

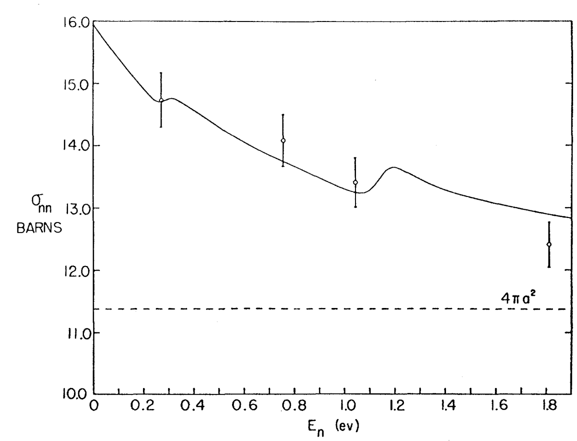

FIG. 4. The scattering cross section $\sigma_{n,n}$ of U^{235} as a function of the neutron energy. The experimental data of reference 13 are shown as open circles with bars indicating the probable error due to counting statistics. The solid line is the value of $\sigma_{n,n}$ computed from (11) with the parameters of Table I. The broken line, $4\pi a^2$, is the computed nonresonant part of $\sigma_{n,n}$.

[13] H. L. Foote, Jr., Phys. Rev. **109**, 1641 (1958).

than that of Table I: the value of $\sigma_{n,n}$ at 0.27 ev was then 18.3 barns instead of 14.7 barns. At that time the data for $\sigma_{n,n}$ were actually strong supporting evidence[4] for the large negative-energy resonance of the single-level fit. We are fortunate that the new data for $\sigma_{n,n}$ prefer our picture to that of the single-level fit.

4. CONCLUSIONS

The analysis of the cross sections of fissionable nuclei was begun to determine whether or not a reasonable and simple description of the cross sections could be made with the Wigner-Eisenbud resonance theory. The preceding sections of this paper have shown that the general resonance theory, with one approximation (consideration of only a finite number of levels), could be used in a straightforward manner for the description of the cross sections of these nuclei. The derived method does not make explicit reference to the number or definition of the fission channels but can yield information on the number. Again without referring to the fission channels, we can say something about the justification for neglecting the distant levels. It was shown in Sec. 2 that the sign of the interference term between any pair of levels, λ and λ', was given by $(\Gamma_{\lambda n}^0)^{\frac{1}{2}}(\Gamma_{\lambda' n}^0)^{\frac{1}{2}}\mathbf{g}_{\lambda f}\cdot\mathbf{g}_{\lambda' f}$. From considerable experience with the cross sections of nonfissionable nuclei, we suspect that the reduced neutron widths, $(\Gamma_{\lambda n}^0)^{\frac{1}{2}}$, have random sign fluctuations. Therefore, even if the vectors $\mathbf{g}_{\lambda f}$ for a large number of levels are strongly correlated the signs of the interference terms will still have random sign fluctuations. The interference from the many distant states will therefore cancel regardless of the fission process. Consequently our principal assumption should be a good one.

The application of our method of analysis to the U^{235} cross sections has given a good fit to the data below 2 ev. It has given evidence for the fact that there are only several channels important in the fission process and that the 0.29-ev and the 1.13-ev resonance have the same spin as the large negative-energy resonance near -1.0 ev. Many artificial features of previous analyses have been removed: only *one* negative-energy resonance was required and no additional resonances at positive energies. An equally good or better fit could have been obtained by using two negative-energy resonances, say one of each spin type. However, such a fit would be less desirable (because it involves more arbitrary parameters) than the present one and unnecessary (because the present fit is about as good as the data).

We are still left with an uncomfortably large negative-energy resonance. The $\Gamma_{\lambda n}^0$ of this resonance is smaller than in previous fits by about a factor of two—which is important in the determination of the probability of finding a resonance as large as this. According to the Porter-Thomas distribution[7] the probability that a given level has a $\Gamma_{\lambda n}^0$ of at least this size lies roughly between 0.1 and 0.01%. A level with twice the $\Gamma_{\lambda n}^0$ is less probable by several orders of magnitude.

It has been suspected for some time that the need for a huge negative-energy resonance is common to the common fissionable isotopes, U^{233}, Pu^{239}, and U^{235}. Such a fact would be very disturbing. The proof of such a fact must await the analyses of the cross sections of the former two isotopes by the methods of the present paper. A preliminary fit[14] of the data[2] for Pu^{239} up to the second resonance at 7.8 ev did not meet such a difficulty. Without interference a huge negative-energy resonance was required to describe the cross section in the vicinity of 3 ev. With interference one negative-energy resonance with a $\Gamma_{\lambda n}^0$ only three times average (therefore a 25% probability, see above) was needed, in addition to the first few levels at positive energy, to fit the data everywhere up to 7 ev.

ACKNOWLEDGMENTS

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APPENDIX A. MANY-LEVEL FORM OF THE COLLISION MATRIX

In the usual form of the Wigner-Eisenbud resonance theory,[9,10] the collision matrix is written in matrix notation (with all rows and columns referring to channels) as

$$U=k^{\frac{1}{2}}(O-RO'-RbO)^{-1}(I-RI'-RbI)k^{-\frac{1}{2}}, \quad \text{(A.1)}$$

where the matrices R and b refer to the stationary states of the compound nucleus which are characteristic of the Wigner-Eisenbud theory, and the matrices k, O, O', I and I' refer to the wave functions of the various reaction alternatives or channels. More specifically, b is a diagonal matrix whose diagonal components, b_c, are the arbitrary set of boundary-condition numbers which, together with the Hamiltonian of the compound nucleus define the stationary states X_λ of the resonance theory: b_c is the negative of the logarithmic derivative of X_λ at the nuclear surface. R is the matrix whose components $R_{cc'}$ are

$$R_{cc'}=\sum\nolimits_\lambda \gamma_{\lambda c}\gamma_{\lambda c'}/(E_\lambda-E), \quad \text{(A.2)}$$

where $\gamma_{\lambda c}$ is the reduced width amplitude for the decay

[14] E. Vogt (unpublished).

of the state X_λ into the channel c and E_λ is the characteristic energy of the stationary state X_λ. The sum in (A.2) runs over all the states.

The matrices $(k)^{\frac{1}{2}}$ and $(k)^{-\frac{1}{2}}$ are diagonal matrices whose diagonal components are, respectively, $(k_c)^{\frac{1}{2}}$ and $(k_c)^{-\frac{1}{2}}$, k_c being the relative momentum of the channel c. I, I' and O, O' are also diagonal matrices. The diagonal components, I_c and O_c, of I and O are, respectively, the radial wave functions of incoming and outgoing waves evaluated at the nuclear surface, $r_c=a_c$. The diagonal components, I_c' and O_c', of I' and O' are the derivatives with regard to r_c of I_c and O_c, also evaluated at $r_c=a_c$. The radial wave functions are normalized so that

$$O_c'I_c-I_c'O_c=2ik_c. \tag{A.3}$$

For the derivation of (A.1) and a more complete discussion of the quantities involved in it the reader is referred to the original papers by Wigner and Eisenbud[9] or the review by Lane and Thomas.[10] The expression (A.1) is the familiar form of the collision matrix in the resonance theory. The purpose of the present Appendix is to derive the many-level form of U from (A.1). The derivation of this less familiar form of U was first given by Wigner.[11]

To simplify the derivation of the many-level form of U we define a diagonal matrix, L:

$$L\equiv O'O^{-1}+b \tag{A.4}$$

in terms of which (A.1) may be written

$$U=k_c^{\frac{1}{2}}O(1-RL)^{-1}(1-RL^*)Ik_c^{-\frac{1}{2}}, \tag{A.5}$$

where the star implies the complex conjugate. (A.5) follows from the fact that $I^*=0$ and $I'^*=O'$. For a large number of channels the matrix inversion of $(1-RL)^{-1}$ may be very difficult. The problem of the inversion of a matrix whose rows and columns refer to channels can be converted into a problem involving the inversion of a matrix whose rows and columns refer to levels, not channels. To accomplish the conversion we define a level matrix $A_{\lambda\lambda'}$ by

$$[(1-RL)^{-1}]_{c''c'}=\delta_{c''c'}+\textstyle\sum_{\lambda\lambda'}\gamma_{\lambda c''}\gamma_{\lambda'c'}L_{c'}A_{\lambda\lambda'}. \tag{A.6}$$

To determine what $A_{\lambda\lambda'}$ is, we multiply both sides of (A.6) by $(1-RL)_{cc''}$ and sum over c'', obtaining

$$\textstyle\sum_{c''}\{\delta_{cc''}-\sum_{\lambda''}\gamma_{\lambda''c}\gamma_{\lambda''c''}L_{c''}/(E_{\lambda''}-E)\}$$
$$\times\{\delta_{cc''}+\textstyle\sum_{\lambda\lambda'}\gamma_{\lambda c''}\gamma_{\lambda'c'}L_{c'}A_{\lambda\lambda'}\}=\delta_{cc'}, \tag{A.7}$$

which is

$$\textstyle\sum_{\lambda\lambda'}(\gamma_{\lambda c}\gamma_{\lambda'c'}L_{c'})[-\delta_{\lambda\lambda'}(E_\lambda-E)+A_{\lambda\lambda'}$$
$$-\textstyle\sum_{\lambda''}\xi_{\lambda\lambda''}A_{\lambda''\lambda'}/(E_\lambda-E)]=0, \tag{A.8}$$

where

$$\xi_{\lambda\lambda''}=\textstyle\sum_{c''}\gamma_{\lambda c''}\gamma_{\lambda''c''}L_{c''}. \tag{A.9}$$

The components of ξ also refer to levels not channels. ξ is a complex matrix whose real part, $-\Delta$, leads to level shifts and whose imaginary part $\frac{1}{2}i\Gamma$ leads to level widths.

Since (A.8) must hold for arbitrary values of $\gamma_{\lambda c}$ and $\gamma_{\lambda'c}$ we can set the square bracket of (A.8) equal to zero for each λ and λ' separately. From (A.8) we have, therefore,

$$(A^{-1})_{\lambda\lambda'}=(E_\lambda-E)\delta_{\lambda\lambda'}-\xi_{\lambda\lambda'}. \tag{A.10}$$

The components of A itself can be obtained from (A.10) by simply inverting the level matrix A^{-1}.

We can use (A.8) and (A.6) to obtain at once

$$[(1-RL)^{-1}(1-RL^*)]_{cc'}$$
$$=\delta_{cc'}+\textstyle\sum_{\lambda\lambda'}\gamma_{\lambda c}\gamma_{\lambda'c'}A_{\lambda\lambda'}(L_{c'}-L_{c'}^*). \tag{A.11}$$

In order to arrive at the desired result for $U_{cc'}{}^J$ we define the level width

$$\Gamma_{\lambda c}=-i(L_c-L_c^*)\gamma_{\lambda c}^2, \tag{A.12}$$

and note that the radial wave functions normalized according to (A.3) yield immediately that

$$k_c^{\frac{1}{2}}O_c^{-1}I_{c'}k_{c'}^{-\frac{1}{2}}$$
$$=e^{i(\varphi_c+\varphi_{c'})}[(L_c-L_c^*)/(L_{c'}-L_{c'}^*)]^{\frac{1}{2}}, \tag{A.13}$$

where φ_c is the sum of a Coulomb phase shift and a hard-sphere potential scattering phase shift. The product of (A.13) and (A.11) yields

$$U_{cc'}=e^{i(\varphi_c+\varphi_{c'})}[\delta_{cc'}+i\textstyle\sum_{\lambda\lambda'}(\Gamma_{\lambda c})^{\frac{1}{2}}(\Gamma_{\lambda'c})^{\frac{1}{2}}A_{\lambda\lambda'}], \tag{A.14}$$

where

$$(\Gamma_{\lambda c})^{\frac{1}{2}}\equiv[-i(L_c-L_c^*)]^{\frac{1}{2}}\gamma_{\lambda c}. \tag{A.15}$$

The result (A.14) is the one required in Sec. 2.

The matrix A to be used in (A.14) is computed from (A.10). For the low-energy cross sections of the fissionable nuclei we can ignore the real part, Δ, of ξ which, in general leads to shifts in the energies of cross-section maxima from the level energies E_λ. For s-wave neutrons O_c'/O_c is imaginary and the proper choice of b_c for this channel is $b_c=0$. Thus the s-wave neutrons cannot contribute to Δ. For the fission channels (as well as for the radiation channels) the definition of O_c'/O_c is not clear. However it should be true that the value of O_c'/O_c does vary much in an energy interval of a few electron volts near the neutron binding energy so that we can choose the constant b_c for these channels to make Δ vanish. The use of (A.9) and (A.15) in (A.10) gives directly Eq. (7) of Sec. 2.

The expression (A.14) for $U_{cc'}{}^J$ is useful whenever the matrix A^{-1} can be truncated so that only a few rows and columns of A and A^{-1} are involved in the inversion (A.10). This truncation is equivalent to the neglect of all but a few of the resonances. It is possible to include some of the effects due to the bulk of the very distant resonances in an approximate way. For example, giant resonances in the neutron strength function several Mev away from the energy of interest could make an appreciable contribution $R_{nn}{}^\infty$ to the diagonal element of R which refers to neutron channels. Such an

R_{nn}^{∞} would change the "radius" of the potential scattering in our cross sections. However, for the fissionable nuclei, R_{nn}^{∞} is expected to be small compared to the radius, a, because these nuclei lie almost halfway in between the 4s and 5s peaks in the s-wave neutron strength function. Consequently we have not included R_{nn}^{∞} in the discussion above or in Sec. 2.

The truncation of the matrix A^{-1} refers actually only to the off-diagonal components. We can include, in addition to the small set of levels whose interference is considered exactly, any number of levels for which only the diagonal component of A^{-1} is retained. These additional levels then appear in the analysis exactly as they do in the single-level formula. As pointed out in Sec. 4, the contributions to the cross section from interference between a level close to E and the many levels far away will roughly cancel because of sign fluctuations. Therefore the contribution to the cross section of the neglected levels—the nearest neglected levels as well as those very far away—can be treated with the single-level formula. These contributions can be important. The very distant levels, as discussed above, will make a contribution R_{nn}^{∞} to the scattering length, which has been neglected. The effect of the nearby neglected levels, estimated in the above way, is unimportant for U^{235} but is significant in Pu^{239}.

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Scattering of Low-Energy Neutrons by Deuterons*†

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Neutron-deuteron scattering lengths are calculated by a new variational method which makes provision for polarization (space distortion) of the deuteron. Central Gaussian potentials are employed to represent the two-body interaction, and the trial function is also Gaussian in nature. The results indicate that polarization is not important in either spin state, except for that which is automatically provided by the exclusion principle. The calculated scattering lengths are not in good agreement with either of the two experimentally allowed sets.

I. INTRODUCTION

THE nuclear three-body problem provides a logical first test of any conclusions drawn from the two-body data concerning the nature of nuclear forces. For the description of low-energy phenomena, these conclusions are adequately expressed in terms of a static potential which is charge-independent, contains a tensor component, and has a definite spin dependence. The low-energy two-body data do not specify the exact shape of the potential and yield no information concerning the interaction in states of odd parity. They are not incompatible with the existence of a repulsive core at small separation, which is indicated by experiments at higher energies.

Calculations on the three-nucleon bound state are in general agreement with these conclusions. It is known, for example, that purely central forces, chosen to fit the deuteron's binding energy, lead to too much binding for the triton, whereas the inclusion of a tensor force brings the calculated binding energy into reasonable agreement with experiment.[1] The addition of repulsive cores to the central potentials does not appear to alter this result,[2] although the effect of including tensor forces and hard cores together has not been calculated. Again, little information is obtained from the triton concerning the odd-parity two-body interaction, because the ground state is almost completely space symmetric.[1] In the scattering problem, however, the spatial wave function must necessarily have an antisymmetric component, so the results may be sensitive to the odd-parity two-body potentials even at very low energies.

In the limit of zero energy, the n-d scattering is characterized by two parameters, the quartet and doublet scattering lengths a_4 and a_2. Unfortunately, these parameters have not been uniquely determined experimentally. There are two possible sets of scattering

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