Inelastic Scattering from Light Nuclei—the Alpha-Particle Model for Be^9†*

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The inelastic scattering of nucleons, deuterons, and alpha particles from light nuclei is discussed in terms of direct interactions. The validity of this description, and of the approximations made here and by other authors in calculating cross sections, are analyzed in detail. Physical arguments are given for the use of a collective representation of light nuclei, in order to explain the preferential excitation of certain well-defined nuclear states by short-wavelength projectiles. As an example, the alpha-particle model is used to characterize Be^9 and the inelastic cross sections for excitation of rotational, vibrational, and single-particle states are calculated with an impulse approximation. The model, and results computed from it, are examined and compared to experimental findings.

I. INTRODUCTION

CONSIDERABLE attention has been directed in recent years to the experimental investigation of the angular distribution of particles with wavelengths of the order of 10^{-13} cm which are inelastically scattered by light nuclei to discrete final states. The experiments of interest include those carried out with incident beams of 10- to 200-Mev protons, 10- to 20-Mev deuterons, and 20- to 48-Mev alpha particles.

The general characteristics of the observed cross sections to definite final states are as follows: (1) The angular distributions display marked oscillations with angle. Such oscillations are particularly regular and sharp and persist to rather high "order" in the case of many alpha-particle experiments.[1] In these experiments the wavelength, $\lambda\!\!\!^{-}$, is quite small, being of the order of, or less than, 0.5×10^{-13} cm. (2) The inelastic scattering cross sections to some of the final states are surprisingly large. For example, the differential cross section, averaged over the range of center-of-mass scattered angles of 15° to 145°, for 43.5-Mev alpha particles which excite the 4.43-Mev level in C^{12} is estimated[2] to be 5.3 mb. This implies a total cross section to this one level alone of 7% of the geometric cross section. Indeed, it appears that the sum over final states of such inelastic scattering may comprise a major share of the nonelastic cross sections in light nuclei, in contradiction to the usual belief that "direct processes" form a small but interesting subset of nuclear reactions.[3] (3) The shape of the angular distribution and the magnitude of the cross section are strong functions of the final state. Some states, such as the 1.8- and 3.1-Mev levels of Be^9 and the 7.65-Mev level of C^{12}, are quite difficult to excite by inelastic scattering.

It is possible to draw several qualitative conclusions from the angular distributions of the cross sections. (1) They strongly suggest that we are dealing with simple optical interference phenomena, i.e., the scattered wave receives coherent contributions from different locations inside, or at the edge of, the nucleus and the angular distribution is a sensitive function of the phase, $\mathbf{K}\cdot\mathbf{r}$, suitably averaged over the spatial coordinate $\mathbf{r}$ ($\mathbf{K}$ is the momentum of the recoil nucleus). This statement is reinforced by the observation that, except in a few special cases, the maxima and minima of the alpha-particle angular distributions shift in a reasonable way as the wavelength is changed; as a specific example we cite the angular distributions when the 1.37-Mev state in Mg^{24} is excited with incident 48-Mev,[4] 42-Mev,[1] and[5] 31-Mev alpha particles. The variation of the angular distributions with wavelength is less regular in the case of inelastically scattered 10–20 Mev protons where the wavelength is larger than 10^{-13} cm and the number of oscillations is small. Even here the optical interference interpretation is supported by the observation that the distribution of gamma rays coincident with (p,p') is usually symmetric about the recoil direction[6]; this may be easily understood if $\mathbf{K}$ defines a preferred axis of quantization.[7]

(2) The observation of interference patterns implies, by analogy to physical optics, the existence of a well-defined and localized "radius" R; such a radius might

† Portions of this work have been reported previously: J. S. Blair and E. M. Henley, Bull. Am. Phys. Soc. Ser. II, 1, 20 (1956), and Physica 22, 1126 (1956).

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[1] This is strikingly illustrated by the excitation of the 1.37-Mev state of Mg^{24} by 42-Mev α particles, P. C. Gugelot and M. Rickey, Phys. Rev. **101**, 1613 (1956), and of the 1.63-Mev state of Ne^{20} by 18.9-Mev α particles, see Seidlitz, Bleuler, and Tendam, Phys. Rev. **110**, 682 (1958).

[2] G. Farwell and A. I. Yavin, Cyclotron Research, University of Washington, Annual Progress Report, 1957 (unpublished), and A. I. Yavin, Ph.D. thesis, University of Washington, 1958 (unpublished).

[3] For example, about $\frac{1}{3}$ the geometrical cross section is exhausted in Ne^{20} by alpha-particle inelastic excitation of discrete states with energy less than 8 Mev. See Seidlitz *et al.*, reference 1.

[4] F. J. Vaughn, University of California Radiation Laboratory Report, UCRL-3174, 1955 (unpublished).

[5] H. J. Watters, Phys. Rev. **103**, 1763 (1956).

[6] R. Sherr and W. F. Hornyak, Bull. Am. Phys. Soc. Ser. II, **1**, 197 (1956).

[7] G. R. Satchler, Proc. Phys. Soc. (London) **A68**, 1057 (1955).

be associated with the location of the scattering units of the nucleus which participate in the excitation or with the extent of homogeneous nuclear matter. The persistance of the interference patterns to high order indicates that the "radius" is extremely sharp. The "radii"[8] which one obtains by simple optical arguments are usually large compared to electromagnetic radii.

A simple formula for the inelastic differential cross section, $d\sigma/d\Omega$, incorporating the above conclusions has been developed by Austern, Butler, and McManus[9]:

$$d\sigma/d\Omega \sim \sum_l (l',l'',0,0|l,0)^2 j_l^2(KR), \quad (1)$$

where $(l',l'',0,0|l,0)$ is a Clebsch-Gordan coefficient, l' and l'' are initial and final angular momenta characteristic of the bound nuclear states, and j_l is a spherical Bessel function. The above angular distribution is a consequence of four drastic assumptions, which are independent of the choice of nuclear model: (1) The reaction proceeds through some "direct" (single-step) interaction in which angular momentum is conserved; the conservation of angular momentum is, of course, responsible for the occurrence of the Clebsch-Gordan coefficients and the order of the Bessel function. The assumption of a one-step interaction maximizes the interference oscillations obtainable since it guarantees that the coherence in inelastic events is not smeared out by subsequent secondary events. (2) There is only a slow variation of the direct interaction itself with angle and energy so that there is no dramatic modulation of the angular distribution given by Eq. (1). (3) The incident and final unbound particles are well represented by plane waves in the nuclear surface region, an assumption which guarantees that the angular distribution is a simple function of K. (4) The direct interaction occurs at the well-defined radius, R. We should not be surprised to find that Eq. (1) displays the desired interference oscillations since the assumptions necessary for its derivation essentially include our previous qualitative conclusions; what is surprising, however, is that many physical observations are so well described by a formula based on such crude assumptions, or equivalently, by the qualitative conclusions previously listed.

We have already noted that the derivation of the ABM formula does not depend explicitly on the choice of nuclear model[10]: Eq. (1) was first derived in the case where the direct interaction links the projectile and a target nucleus which is described by a shell model; it also follows, under similar approximations, for an interaction with a distorted nuclear surface[11] or with an alpha-particle substructure in the nucleus, a result which we shall derive later. (There are some differences in the appropriate Clebsch-Gordan coefficients and other angular momentum coupling factors for the three models above; however, in practice, it has been difficult or impossible to distinguish between these models on the basis of the different coupling factors.) Thus the primary questions which now face anyone attempting to construct a more complete theory of inelastic processes in light nuclei are: (α) Is it possible to obtain an angular distribution approximating Eq. (1) when the sweeping assumptions (1)–(4) are not explicitly introduced, and will this shed any light on which nuclear model is to be preferred? (β) Which model most naturally explains the further observations (2) and (3), that the magnitudes of the inelastic cross sections may be very large and are strong functions of the final state? (γ) In cases where there are marked deviations between observations and the ABM formula, is there a unique explanation for the deviation?

We shall not be able to answer completely and satisfactorily any of these questions in the present paper. Nonetheless, we believe that a partial solution is to be found in the use of some collective nuclear model for light nuclei, and especially for those states that are strongly excited by short-wavelength projectiles.[12] The recent success[13] of the collective model in the *d* shell particularly tempt us to apply similar ideas to the *p* shell.

The reasoning that leads us to advance the above suggestion can best be demonstrated by discussing a direct process with a target nucleus that is described by the independent-particle model. In the original derivation of the ABM formula and a subsequent derivation of a related expression,[14] the plane wave assumption and the occurrence of a definite radius are justified on the grounds that the direct interaction takes place when the target nucleon and projectile are both outside the main body of the nucleus; an extremely short mean free path of the projectile for compound-nucleus formation is presumed to concentrate the direct interaction at the surface. We note that without this surface assumption it is not possible to obtain the desired structure in the angular distribution; if the relevant radial integrals are evaluated throughout the nucleus, the diffraction structure is washed out because of the large spatial extent of the wave function of the struck nucleon or, in classical terms, there simply is not enough momentum in the initial and final states of the

[8] It is perhaps worth noting that, for incident alpha particles, these "radii" are of the same order as those obtained by extrapolation to small A of the formula for sharp cutoff radii obtained by analysis of elastic alpha scattering. See Kerlee, Blair, and Farwell, Phys. Rev. **107**, 1343 (1957).

[9] Austern, Butler, and McManus, Phys. Rev. **92**, 350 (1953). This paper will be referred to as ABM in the text. See also R. Huby and H. C. Newns, Phil. Mag. **42**, 1442 (1951).

[10] This point also has been made recently by H. Ui, Progr. Theoret. Phys. (Kyoto) **18**, 163 (1957).

[11] S. Hayakawa and S. Yoshida, Progr. Theoret. Phys. (Kyoto) **14**, 1 (1955); Proc. Phys. Soc. (London) **A68**, 656 (1955). S. Yoshida, Proc. Phys. Soc. (London) **A69**, 668 (1956).

[12] This point has also been made by K. Nishimura and M. Ruderman, Phys. Rev. **106**, 558 (1957), and by Th. A. J. Maris, Nuclear Phys. **3**, 213 (1957).

[13] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955); A. E. Litherland *et al.*, Can J. of Phys. **36**, 378 (1958).

[14] S. T. Butler, Phys. Rev. **106**, 272 (1957).

struck nucleon to absorb the large momentum transfer of the projectile. On the other hand, it is not clear that the short mean free path and plane wave assumptions are mutually compatible; if the mean free path is really so short that the interaction is effectively localized at the surface, then the projectile wave function is highly distorted from a plane wave. However, use of a reasonable mean free path may not lead to the desired localization.

Furthermore, the surface assumption leaves so little of the target nucleon visible to the projectile that it is difficult to explain the very large observed cross sections. The relative cross sections for excitation of definite final states have been discussed in terms of fractional parentage arguments[15]: the relative population of some states in Li^6 and Li^7 and the inhibition of the 7.65-Mev level in C^{12} can be explained with reasonable shell-model assignments.[16] An investigation of the relative cross sections to some of the low-lying states of Be^9 on the basis of the shell model is presently being made by Pinkston.[17]

Calculations performed by Levinson and Banerjee[16] bear on these points.[18] Cross sections for inelastic scattering of protons to the 4.43-Mev level of C^{12} were evaluated when the proton-nucleon interaction was treated in Born approximation but the projectile and target nucleons were described by complex optical- and shell-model wave functions, respectively; radial integrals were performed throughout the nuclear volume. Reasonable fits of the proton angular distribution[19] and gamma-ray angular correlation[6] could be obtained with realistic nuclear parameters for proton energies between 14 and 31 Mev, a range where the ABM formula fails to match experiment. For higher incident energies and large momentum transfer, however, the calculated cross sections became very small and lost their oscillatory behavior. Further, in all examples, the strength of the two-body potential needed (in Born approximation) to fit the magnitudes of the inelastic cross sections was more than a factor of two times reasonable strengths as estimated from effective range theory. We conjecture that these difficulties will persist when similar calculations are extended to inelastic alpha-particle scattering.

In contrast to the independent-particle model, both the ellipsoidally deformed nuclear surface and alpha-particle models introduce a sharp radius in a natural way, without essential need for a short mean free path. The interference pattern will be damped in the deformed-surface model by the extent of the taper in the nuclear potential[20] and the finite deformation,[21] and in the alpha-particle model by zero-point fluctuations about the equilibrium value of the distance between the centers of the alpha particles. Such damping should be no larger than that resulting in the independent-particle model from the use of finite mean free paths. In addition, large values for the inelastic cross sections can be obtained with known values of the nuclear well depth[1] and projectile-alpha-particle interactions, respectively. Physically, the large cross sections arise from the now highly correlated motion of the nucleons. The strongly excited states will be those belonging to the same rotational band as the initial state. But while these arguments make a collective mode of excitation more appealing to us, a thorough distorted-wave treatment of collective excitation has yet to be carried out in the regions of critical interest.[22,23]

In the preceding discussion, little mention has been made of the compound-nucleus theory of reactions. Where the dominant processes are qualitatively described in terms of the direct theory, we believe that the usual statistical theories have small relevance.[24] This is generally the case for inelastic scattering of high-energy projectiles to low-lying levels of light nuclei since the decay of the compound nucleus to such levels tends to be very improbable. In particular, we feel that it is then incorrect to attempt to separate such angular distributions into a compound-nucleus contribution, symmetric about 90°, and a direct contribution as given by simple Born-approximation arguments. For these cases, an understanding of the deviations from the simple direct theory should be sought in terms of all the neglected (higher order) corrections, such as effects due to nuclear binding forces, and multiple events. In other words, we believe that the true physical situation here converges towards the direct-interaction description and is not a linear combination of two diametrically opposite, highly simplified points of view. The phrase "compound-nucleus formation" should not be used to hide everything that is not understood about reactions in light nuclei.

The remainder of the paper will be devoted to the alpha-particle model, as an example of a model incorporating collective dynamical effects. Our discussion will be concentrated on the nucleus Be^9, which seems

[15] A. M. Lane and D. H. Wilkinson, Phys. Rev. **97**, 1199 (1955).

[16] C. A. Levinson and M. K. Banerjee, Ann. Phys. N. Y. **2**, 471, 499 (1957); **3**, 67 (1958).

[17] M. T. Pinkston, Bull. Am. Phys. Soc. Ser. II, **3**, 223 (1958).

[18] See also J. R. Lamarsh and H. Feshbach, Phys. Rev. **104**, 1633 (1957); Kajikawa, Sasakawa, and Watari, Progr. Theoret. Phys. (Kyoto) **16** (1956); and R. Kajikawa and W. Watari. Progr. Theoret. Phys. (Kyoto) **18**, 103 (1957).

[19] R. Peele, Phys. Rev. **105**, 1311 (1957).

[20] Chase, Wilets, and Edmonds, Phys. Rev. **110**, 1080 (1958).

[21] J. Sawicki, Nuclear Phys. **6**, 613 (1958).

[22] Yoshida, reference 11, has considered the inelastic scattering of 10-Mev protons from Mg^{24} due to a deformed surface without employing the Born approximation. Unfortunately, the experimentally observed rapid variation of the inelastic cross sections in this energy range indicates a rather complex situation, so that it is not surprising that his simple direct-interaction theory has difficulty in matching experiment.

[23] Chase, Wilets, and Edmonds, reference 20, have treated low-energy neutron inelastic scattering by heavy nuclei in a particularly careful fashion. Such computations are being extended to high-energy scattering from light nuclei. (L. Wilets, private communication.)

[24] See, however, G. E. Brown and C. T. De Dominicis, Proc. Phys. Soc. (London) **A70**, 686 (1957).

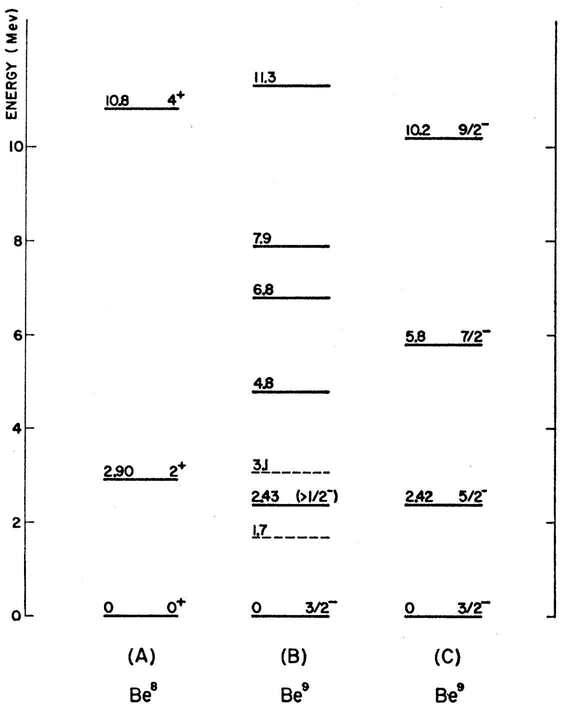

FIG. 1. Level structure of Be^8 and Be^9. The experimental levels[30] of Be^8 and[31] of Be^9 are given in Mev in the left-hand margins of parts (A) and (B), and the spins of the levels are given in the right-hand margins. The rotational band of Be^9 is shown in (C) for a moment of inertia that is taken to be equal to that of Be^8.

particularly appropriate to this model; arguments supporting this statement are presented in Sec. II. Cross sections for inelastic scattering into rotational, vibrational, and "single-particle" excited states are calculated in the impulse approximation in Sec. III. A critique of the model is made in Sec. IV. An appendix contains a general derivation of the inelastic scattering cross section in the impulse approximation.

II. ALPHA-PARTICLE MODEL FOR Be^9

The alpha-particle model for nuclei, proposed many years ago,[25] has not met with the success accorded the shell model. In recent years, however, it has been revived to discuss the energy levels of[26] O^{16} and[27] C^{12} as well as[28] Be^8 and Be^9. Further, theoretical investigations have shown similarities between shell-model and alpha-particle wave functions.[29]

Consideration of the energy levels of Fig. 1 and experimental results on inelastic scattering from Be^9 first suggested to us that it might be appropriate to picture Be^9 as a dumbbell of two alpha particles to which a neutron is strongly coupled. At the left are indicated the experimental levels of Be^8 as deduced from α-α scattering[30] and in the center are the experimental levels of Be^9.[31] On the right is the expected rotational band for Be^9 if the same moment of inertia is used for both Be^8 and Be^9, and if the projection of neutron angular momentum on the body axis, Ω, is assumed to be $\frac{3}{2}$. The fact that the relative spacing of states in Be^8 approximately fits the ordering of a rigid rotator cannot of itself be taken as evidence for the alpha-particle model since essentially the same sequence results from intermediate-coupling shell-model calculations.[32] However, if it is assumed that inelastic scattering preferentially excites states of the same rotational band and if the 2.43-, 6.8-, and 11.3-Mev levels are identified as those of angular momenta 5/2, 7/2, and 9/2 of the $\Omega=3/2$ band, one has a qualitative explanation for the large excitation of these levels[33] at moderate momentum transfer in contrast to the weak excitation of the 1.8-, 3.1-, and 4.8-Mev levels.

There is additional evidence favoring a collective approach: (1) Shell-model calculations have not been able to explain unambiguously the properties of the $A=9$ nuclei. The observed magnetic moment can be matched only for a value of the spin-orbit parameter, $\zeta=1.4$, which may not fit the energy levels[32]; further such a small value of ζ indicates a puzzling discontinuity between $A=9$ and $A=10$, where $\zeta\approx 4$. (2) The high-energy (p,d) reaction on Be^9 suggests that the strongly bound neutrons have a momentum distribution similar to neutrons from He^4 and C^{12} while the weakly bound neutron has a distribution weighted towards smaller momenta.[34]

We note that the above arguments can be applied equally well in favor of the Bohr-Mottelson model.[35] The preferred Nilsson configuration,[13]

$$(s,\,\Omega=\tfrac{1}{2})^4(p,\,\Omega=\tfrac{1}{2})^4(p,\,\Omega=\tfrac{3}{2}),$$

leads to a prolate nucleus with the same band structure as that which we have assumed in our alpha-particle model.

As an expression of the alpha-particle model, we adopt the following molecular-type wave function for Be^9:

$$\Phi(I,M,\kappa)=[(2I+1)/16\pi^2]^{\frac{1}{2}}\mathcal{R}(\xi)[\phi_\Omega(\varrho')D_{M,\kappa}{}^I(\Theta)+(-1)^{I-j}\phi_{-\Omega}(\varrho')D_{M,-\kappa}{}^I(\Theta)], \quad (2)$$

[25] J. A. Wheeler, Phys. Rev. **52**, 1083 and 1107 (1937); L. R. Hafstad and E. Teller, Phys. Rev. **54**, 681 (1938).
[26] D. M. Dennison, Phys. Rev. **96**, 318 (1954).
[27] A. E. Glassgold and A. Galonsky, Phys. Rev. **103**, 701 (1956).
[28] R. R. Haefner, Revs. Modern Phys. **23**, 228 (1951); D. R. Inglis, Revs. Modern Phys. **25**, 390 (1952); A. Herzenberg and A. S. Roberts, Nuclear Phys. **3**, 314 (1957); A. Herzenberg, Nuclear Phys. **3**, 1 (1957).
[29] J. K. Perring and T. H. R. Skyrme, Proc. Phys. Soc. (London) **A69**, 600 (1956).
[30] Nilson, Jentschke, Briggs, Kerman, and Snyder, Phys. Rev. **109**, 850 (1958).
[31] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[32] D. Kurath, Phys. Rev. **101**, 216 (1956).
[33] R. Summers-Gill, Phys. Rev. **109**, 1591 (1958).
[34] S. Glashow and W. Selove, Phys. Rev. **102**, 200 (1956); J. Dabrowski and J. Sawicki, Acta Phys. Polon. **14**, 143 (1955), and Nuovo cimento **12**, 293 (1954).
[35] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).

where I is the total spin, M the projection on an external z axis, κ the projection on the body axis (which joins the two alpha particles), j and Ω are the neutron angular momentum and its projection on the body axis. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the coordinates of the centers of mass of the alpha particles and $\mathbf{x}_n$ the neutron coordinate; then

$$\xi \equiv |\mathbf{x}_1 - \mathbf{x}_2|, \quad \boldsymbol{\varrho}' \equiv \mathbf{x}_n' - \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$$

(primed quantities signify that angles are referred to the body system), and Θ symbolizes the Eulerian angles for the alpha dumbbell. The wave function is separated into neutron motion (specified by ϕ_Ω), vibrational motion [specified by $\mathcal{R}(\xi)$] and rotational motion [indicated by the symmetric-top function $D_{M,\kappa}{}^I(\Theta)$]. The second term in (2) makes the wave function symmetrical under exchange of the alpha particles.

We shall assume that the vibrational wave function, $\mathcal{R}(\xi)$, is concentrated at the equilibrium separation of the alpha particles $\xi = 2R$. The equilibrium distance is estimated to be $\approx 4.6 \times 10^{-13}$ cm from comparison of the energy spacings in Be^8 and Be^9 to the familiar rotational formula,

$$E = \hbar^2 [I(I+1) - I_0(I_0+1)]/2\mathcal{I},$$

where $\mathcal{I} \equiv 2M_\alpha R^2$ is the moment of inertia. We assume that the neutron wave function may be written

$$\phi_\Omega(\boldsymbol{\varrho}') = \sum_n (l, \tfrac{1}{2}, \Omega - n, n | j, \Omega) f_l(\rho) Y_l{}^{\Omega - n}(\boldsymbol{\varrho}') \chi_{\frac{1}{2}}{}^n. \quad (3)$$

This may be easily generalized to the case of Nilsson wave functions,[13] where j is not a good quantum number, by insertion of the expansion coefficients $C_{j,l}{}^\Omega$ so that

$$\phi_\Omega(\boldsymbol{\varrho}') = \sum_{j,l} C_{j,l}{}^\Omega \sum_n (l, \tfrac{1}{2}, \Omega - n, n | j, \Omega) \times f_l(\rho) Y_l{}^{\Omega-n}(\boldsymbol{\varrho}') \chi_{\frac{1}{2}}{}^n. \quad (3a)$$

For the rotational band built on the ground state of Be^9, we have

$$j = l + \tfrac{1}{2} = \Omega = \kappa = \tfrac{3}{2}.$$

III. CALCULATION

The cross section for inelastic scattering from state a to b is[36]

$$\sigma_{ba} = (2\pi/\hbar v) |T_{ba}|^2 \omega_b, \quad (4)$$

where ω_b is the density of final states and v is the relative velocity of the incident particle and target nucleus. We derive in the Appendix expressions [Eqs. (I.17), (I.18), and (I.20)] for the scattering amplitude, T_{ba}, where in addition to the impulse approximation,[37] it is assumed that the two-body scattering amplitude may be factored outside the spatial overlap integral of initial and final states.[9] For a spinless projectile, T_{ba} is then

$$T_{ba} = \sum_i \langle t_i^{(+)} \rangle \langle \chi_b^{(-)} | \delta(\mathbf{x}_p - \mathbf{x}_i) | \chi_a^{(+)} \rangle. \quad (5)$$

Here, i labels constituent particles of the nucleus, p the incident particle, $\langle t_i^{(+)} \rangle$ is the appropriately symmetrized matrix element of the two-body scattering amplitude [see Eq. (I.20) for a precise definition]. $\chi_a^{(\pm)}$ represents a plane wave plus an outgoing (+) or incoming (−) solution of the scattering of the incident particle by those constituents of the nucleus not labeled by i [see Eq. (I.9)].

For the case of Be^9, described by an α-particle model, there are only two scattering amplitudes. These are t_α and t_n [we drop the bracket (+) for convenience of notation] which represent, respectively, the interaction of the projectile with an alpha particle and with the extra neutron of the nucleus. There are then no particles in the nucleus not labeled by the index i in Eq. (5), and consequently $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are the free-wave solutions for the projectile (the phenomenological potential, U, representing an interaction with a core, is zero):

$$\chi_a^{(+)} = e^{i\mathbf{k}_0 \cdot \mathbf{x}_p} \Phi_a, \quad \chi_b^{(-)} = e^{i\mathbf{k}_0' \cdot \mathbf{x}_p} \Phi_b. \quad (6)$$

Here $\Phi_{a,b}$ is given by Eq. (2) and

$$T_{ba} = \langle \Phi_b | t_\alpha \exp[i\mathbf{K} \cdot (\tfrac{1}{2}\boldsymbol{\xi} - \tfrac{1}{9}\boldsymbol{\varrho})] + t_\alpha \exp[-i\mathbf{K} \cdot (\tfrac{1}{2}\boldsymbol{\xi} + \tfrac{1}{9}\boldsymbol{\varrho})] + t_n \exp(8i\mathbf{K} \cdot \boldsymbol{\varrho}/9) | \Phi_a \rangle, \quad (7)$$

since $\mathbf{x}_1 = \mathbf{R}_{\text{c.m.}} + \frac{1}{2}\boldsymbol{\xi} - \frac{1}{9}\boldsymbol{\varrho}$, $\mathbf{x}_2 = \mathbf{R}_{\text{c.m.}} - \frac{1}{2}\boldsymbol{\xi} - \frac{1}{9}\boldsymbol{\varrho}$, and $\mathbf{x}_n = \mathbf{R}_{\text{c.m.}} + (8/9)\boldsymbol{\varrho}$, where $\mathbf{R}_{\text{c.m.}}$ is the center of mass of Be^9. In what follows, quantum numbers referring to state a will be unprimed, and those referring to state b will be primed.

To evaluate the matrix T_{ba}, the plane wave is expanded in spherical harmonics with $\mathbf{K}$ chosen as the direction of quantization in the fixed frame. Functions of the neutron relative coordinate, $\boldsymbol{\varrho}'$, referred to the body axis, are expressed in terms of functions of the coordinate, $\boldsymbol{\varrho}$, referred to the fixed frame, by means of the representations of the rotation group. The integrals are then evaluated; geometrical sums over magnetic quantum number are evaluated with the aid of Racah algebra.[38] The square of the scattering amplitude, summed over final spin and averaged over initial spin orientations, may be written as

$$\frac{1}{2I+1} \sum_{M,M'} |T_{ba}|^2 \equiv \sum_\Lambda |A_\Lambda + N_\Lambda|^2, \quad (8)$$

[36] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[37] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952); G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[38] We are indebted to Dr. G. R. Satchler for pointing out the simplification:

$$[(2l+1)(2l'+1)]^{\frac{1}{2}}(l,l',0,0|L,0)W(l,j,l',j';\tfrac{1}{2},L) = \tfrac{1}{2}[1+(-1)^{l+l'-L}](j,j',\tfrac{1}{2},-\tfrac{1}{2}|L,0),$$

where W is a Racah coefficient as defined, for example, by M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957), Chap. VI.

where

$$A_\Lambda=\sum_{\lambda,L}\left[\frac{(2I'+1)(2j+1)}{(2j'+1)}\right]^{\frac{1}{2}}\frac{1}{2\Lambda+1}\left[\frac{1+(-1)^{l+l'-L}}{2}\right]t_\alpha$$
$$\times(-i)^L(2L+1)(j,L,\tfrac{1}{2},0|j',\tfrac{1}{2})\langle f_{l'}|j_L(K\rho/9)|f_l\rangle$$
$$\times(i)^\lambda[1+(-1)^\lambda](2\lambda+1)(\lambda,L,0,0|\Lambda 0)$$
$$\times\langle\mathcal{R}'|j_\lambda(K\xi/2)|\mathcal{R}\rangle\{(I',I,-\kappa',\kappa|\Lambda,\Omega-\Omega')$$
$$\times(j,L,-\Omega,\Omega-\Omega'|j',-\Omega')$$
$$\times(\lambda,L,0,\Omega-\Omega'|\Lambda,\Omega-\Omega')$$
$$+(-1)^{j'-I'}(I',I,\kappa',\kappa|\Lambda,\Omega+\Omega')$$
$$\times(j,L,-\Omega,\Omega+\Omega'|j',\Omega')$$
$$\times(\lambda,L,0,\Omega+\Omega'|\Lambda,\Omega+\Omega')\},\quad(8a)$$

$$N_\Lambda=\delta_{\Lambda,L}\left[\frac{(2I'+1)(2j+1)}{(2j'+1)}\right]^{\frac{1}{2}}\left[\frac{1+(-1)^{l+l'-L}}{2}\right]t_n\langle\mathcal{R}'|\mathcal{R}\rangle$$
$$\times(i)^L(j,L,\tfrac{1}{2},0|j',\tfrac{1}{2})\langle f_{l'}|j_L(8K\rho/9)|f_l\rangle$$
$$\times\{(I',I,-\kappa',\kappa|L,\Omega-\Omega')$$
$$\times(j,L,-\Omega,\Omega-\Omega'|j',-\Omega')$$
$$+(-1)^{j'-I'}(I',I,\kappa',\kappa|L,\Omega+\Omega')$$
$$\times(j,L,-\Omega,\Omega+\Omega'|j',\Omega')\}.\quad(8b)$$

The term A_Λ arises from the interaction with the alpha-particle dumbbell; such an interaction also perturbs the neutron motion relative to the dumbbell and is evidenced by the second line of Eq. (8a). We remark here that such "shaking" of single nucleons through coupling with a collective mode of motion will be present whenever the projectile interacts with such collective modes; it is essentially a center-of-mass effect, however, and should become progressively less important with increasing nuclear mass. The term N_Λ, arising from the interaction with the neutron, can be easily generalized to several particles outside the core and Nilsson orbitals§; one merely multiplies (7b) by the product $C_{jl}^{\Omega}C_{j'l'}^{\Omega'}$, sums over j, l, j', l', and coherently adds the amplitude from each orbital. [The Clebsch-Gordan coefficients in Eq. (8) have been so chosen that no additional phase factors are required.] The argument $8K\rho/9$ must be appropriately modified according to the masses involved. The second term in the brackets of (8a) and (8b), multiplying the factor $(-)^{j'-I'}$, is due to the symmetry of the wave function, Eq. (2); only when the neutron angular momentum transfer, L, is larger than or equal to $\Omega+\Omega'$ will this term contribute.

The expression for the cross section, as given by Eq. (8), appears rather complex but simplifies when applied to specific types of excitation. In what follows, we consider excitation of (A) rotational, (B) vibrational, and (C) single-particle states.

(A) Excitation of Rotational States

Rotational excitation is characterized by the requirement that $\Omega=\Omega'$ and that there be no change in parity of the neutron wave function. We shall assume that the dumbbell is stiff enough to centrifugal stretching so that $\mathcal{R}'=\mathcal{R}$. To avoid unessential complications, we have restricted ourselves to the case where j and l are good quantum numbers (which is true for Be^9). Equations (8a) and (8b) now become

$$A_\Lambda=\sum_{\lambda,L\text{ even}}\frac{(2I'+1)^{\frac{1}{2}}}{2\Lambda+1}2t_\alpha$$
$$\times(-i)^L(2L+1)(j,L,\tfrac{1}{2},0|j,\tfrac{1}{2})\langle f_l|j_L(\tfrac{1}{9}K\rho)|f_l\rangle$$
$$\times(i)^\lambda(2\lambda+1)(\lambda,L,0,0|\Lambda 0)\langle\mathcal{R}|j_\lambda(\tfrac{1}{2}K\xi)|\mathcal{R}\rangle$$
$$\times\{(I',I,-\kappa,\kappa|\Lambda,0)$$
$$\times(j,L,-\Omega,0|j,-\Omega)(\lambda,L,0,0|\Lambda,0)$$
$$+(-1)^{j-I'}(I',I,\kappa,\kappa|\Lambda,2\Omega)$$
$$\times(j,L,-\Omega,2\Omega|j,\Omega)(\lambda,L,0,2\Omega|\Lambda,2\Omega)\},\quad(9a)$$

and

$$N_\Lambda=\delta_{\Lambda,L}(2I'+1)^{\frac{1}{2}}\left[\frac{1+(-1)^L}{2}\right]t_n$$
$$\times(i)^L(j,L,\tfrac{1}{2},0|j,\tfrac{1}{2})\langle f_l|j_L(8K\rho/9)|f_l\rangle$$
$$\times\{(I',I,-\kappa,\kappa|L,0)(j,L,-\Omega,0|j,-\Omega)$$
$$+(-1)^{j-I'}(I',I,\kappa,\kappa|L,2\Omega)$$
$$\times(j,L,-\Omega,2\Omega|j,\Omega)\}.\quad(9b)$$

Let us consider the cross section when the following additional conditions are true: (a) The "shaking" of the neutron has negligible effect, i.e.,

$$\langle f_l|j_L(K\rho/9)|f_l\rangle\approx\delta_{L,0}.$$

(b) The vibrational wave function is so concentrated around $\xi=2R$, so that

$$\langle\mathcal{R}|j_\lambda(\tfrac{1}{2}K\xi)|\mathcal{R}\rangle\approx j_\lambda(KR).$$

(c) N_Λ is small compared to A_Λ. We shall discuss later the validity of these conditions. The cross section then has a form similar to the simple ABM expression:

$$\frac{d\sigma}{d\Omega}=\frac{\text{“}d\sigma_\alpha\text{”}}{d\Omega}4\sum_{\lambda\text{ even}}(2I'+1)(I',I,-\kappa,\kappa|\lambda,0)^2\times j_\lambda^2(KR),\quad(10)$$

where

$$\frac{\text{“}d\sigma_\alpha\text{”}}{d\Omega}\equiv\frac{2\pi}{\hbar}\frac{\omega_b}{v}|t_\alpha|^2.\quad(11)$$

§ The contribution to the scattering amplitude from interaction with single nucleons in a deformed target has also been calculated recently by J. Sawicki, Nuclear Phys. **7**, 503 (1958) and by H. Matsunobu and S. Yoshida, Progr. Theoret. Phys. (Kyoto) **19**, 599 (1958).

We note that "$d\sigma_\alpha/d\Omega$" is not the elastic cross section for projectile-alpha scattering; ω_b is the energy density for the projectile-Be^9 collision and the matrix element of t_α is evaluated between plane wave states which do not conserve energy in the two-body collision [see Eqs. (I.18) and (I.20)]. Thus the experimental two-body cross section (laboratory system) at the same incident energy provides an estimate of "$d\sigma_\alpha/d\Omega$" which has merit only at small scattering angles. What "$d\sigma_\alpha/d\Omega$" is for large momentum transfers is a matter of speculation at the present time. We also observe that whereas there is a kinematical limitation placed on possible scattering angles in elastic projectile-alpha scattering, there is no such restriction implied in "$d\sigma_\alpha/d\Omega$."

The angular momenta for Be^9 are believed to be $j=\Omega=\kappa=\frac{3}{2}$, $l=1$. Let Eq. (10) be expressed in the form

$$\frac{d\sigma}{d\Omega}=\frac{\text{``}d\sigma_\alpha\text{''}}{d\Omega}\sum_\lambda B_\lambda j_\lambda^2(KR), \tag{12}$$

then the values of B_λ for various λ are given in Table I.

TABLE I. Coefficients B_λ occurring in Eq. (12).

I \ λ	2	4	6
5/2	72/7	12/7	...
7/2	40/7	72/7	...
9/2	...	180/11	40/11

We now consider the range of validity of our simplifying assumptions. (a) The "shaking" of the neutron is not important for small momentum transfer because of the factor $\frac{1}{9}$ occurring in the argument of j_L. To determine the detailed effect of neutron shaking, let us calculate the cross section for the specific rotational transition $I=\frac{3}{2}\rightarrow I'=\frac{5}{2}$ when only assumption (c) is adopted:

$$\frac{d\sigma}{d\Omega}=\frac{\text{``}d\sigma_\alpha\text{''}}{d\Omega}\cdot\frac{72}{7}\left\langle f_1\left|j_0\left(\frac{K\rho}{9}\right)\right|f_1\right\rangle^2$$
$$\times\{\langle\mathcal{R}|j_2(x)-\tfrac{1}{5}\gamma[j_0(x)-(10/7)j_2(x)+(18/7)j_4(x)]|\mathcal{R}\rangle^2$$
$$+\tfrac{1}{6}\langle\mathcal{R}|j_4(x)-(2/7)\gamma[j_2(x)-(10/11)j_4(x)+(35/22)j_6(x)]|\mathcal{R}\rangle^2\}, \tag{13}$$

where

$$x\equiv\tfrac{1}{2}K\xi \quad\text{and}\quad \gamma\equiv\langle f_1|j_2(K\rho/9)|f_1\rangle/\langle f_1|j_0(K\rho/9)|f_1\rangle.$$

The effect of neutron shaking will be of most interest when $K\rho\gtrsim 12$; for such values of K it is justifiable to use the large-argument approximation for j_λ, i.e., $j_\lambda(x)\approx x^{-1}\sin(x-\lambda\pi/2)$, in which case Eq. (13) becomes

$$\frac{d\sigma}{d\Omega}\approx\frac{\text{``}d\sigma_\alpha\text{''}}{d\Omega}\,12\left\langle\mathcal{R}\left|\frac{\sin x}{x}\right|\mathcal{R}\right\rangle^2\times\langle f_1|j_0(K\rho/9)|f_1\rangle^2(1+\gamma)^2. \tag{14}$$

Thus the rapidly oscillating diffraction structure from the term in x is modulated by slowly varying terms expressing the neutron "shaking." The magnitude of these latter terms may be easily estimated when three-dimensional harmonic-oscillator functions are used to describe the neutron motion. We obtain[39]

$$\langle f_1|j_0(K\rho/9)|f_1\rangle^2(1+\gamma)^2=\exp[-\tfrac{1}{2}(Kb/9)^2], \tag{15}$$

where b is defined through $|f_1\rangle\sim r\exp(-r^2/2b^2)$. For the specific case of 40-Mev alpha particles on Be^9, the center-of-mass value of $k\sim 2\times 10^{+13}$ cm^{-1}, while b should be of the order 2×10^{-13} cm; thus Eq. (15) is approximately $\exp[-(4/9)^2(1-\cos\theta)]$. The reduction of cross section due to neutron shaking is not negligible for angles larger than about 90°, but is not catastrophic even at the largest angles.

(b) To investigate effects due to the zero-point vibrations, let us assume that $\mathcal{R}(\xi)$ is represented by a simple harmonic-oscillator wave function centered about $\xi=2R$:

$$\mathcal{R}(\xi)=(\alpha^{\frac{1}{2}}/\pi^{\frac{1}{4}})\exp\{-\tfrac{1}{2}\alpha^2(\xi-2R)^2\}. \tag{16}$$

The values of α^2 that we choose will be large enough so that we can extend all integrals over the range $-\infty<\xi<\infty$. The vibrational matrix element is then

$$\langle\mathcal{R}|j_\lambda(K\xi/2)|\mathcal{R}\rangle=\frac{\alpha}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\{-\alpha^2(\xi-2R)^2\}j_\lambda(K\xi/2)d\xi$$
$$=\left(\frac{\hbar\omega}{2\pi\epsilon}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\frac{\hbar\omega}{\epsilon}(y-1)^2\right\}j_\lambda(KRy)dy, \tag{17}$$

where $y\equiv\xi/2R$, $\alpha^2=M_\alpha\hbar\omega/2\hbar^2=(2R)^{-2}(\hbar\omega/2\epsilon)$, $\hbar\omega$ is the energy spacing of vibrational states, ϵ is a rotational energy$=\hbar^2/(4M_\alpha R^2)\approx\frac{1}{2}$ Mev for Be^8 and Be^9. We have no reliable prescription for determining $\hbar\omega$; if we assume that the 6.13-Mev state in O^{16} and the 7.65-Mev state in C^{12} correspond to dilatational vibrations of an alpha-particle structure,[26,27] naive classical arguments lead to a two-particle vibrational energy, $\hbar\omega=4.3$ and 6.2 Mev, respectively. For $\hbar\omega=4$ and even 8 Mev, however, numerical integration of Eq. (17) shows that the matrix element is considerably damped below $j_\lambda(KR)$ for moderate momentum transfer. As an example the magnitudes of computed matrix element of $j_2(x)$ when $\hbar\omega=8$ Mev and 16 Mev, and $j_2(KR)$ are plotted as functions of KR in Fig. 2. From this graph we see that in order to explain the sharpness of the oscillatory

[39] There is a misprint in these matrix elements as given by Levinson and Banerjee[16] [Part 1, Eq. (47)]; the denominator of the exponential should be 4 instead of 2.

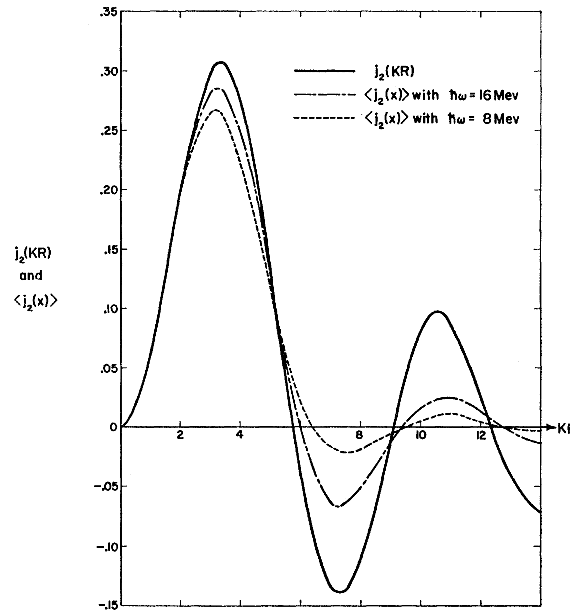

FIG. 2. Dependence of the expectation value of $j_2(x)$ on the strength of the zero-point vibrations of the alpha particles. In the figure, $\hbar\omega$ is the natural frequency of the harmonic oscillator in which the alpha particle is bound.

pattern in "high order," the bond joining the alpha particles must be much stiffer than usually assumed. This difficulty is not unique to the alpha-particle model; for any nuclear model, almost any realistic improvement over the assumptions necessary to derive the ABM formula tends to wash out the diffraction structure at higher orders.

It remains to consider assumption (c). The collisions with the neutron will contribute little to the cross section except at very small momentum transfers. For the $\frac{3}{2}\rightarrow\frac{5}{2}$ transition, only the matrix element of $j_2(8K\rho/9)$ enters; if the neutron motion is represented again by a three dimensional harmonic-oscillator wave function, we have

$$\left\langle f_1\left|j_2\left(\frac{8K\rho}{9}\right)\right|f_1\right\rangle=\frac{32}{243}b^2K^2\exp\left\{-\frac{16}{81}b^2K^2\right\}. \quad (18)$$

According to this estimate, the matrix element will have its maximum value ($\sim\frac{1}{4}$) when $K^2\sim5/b^2$ and will rapidly diminish for higher momentum transfers. Continuing our earlier example (40-Mev α and $b\sim2\times10^{-13}$ cm), we find the maximum to occur at $\theta\sim30°$.

The large reduction in the neutron radial integral occurs primarily because the neutron is not localized. The question can be raised, "Why not restrict the radial integration to ρ greater than some critical radius, $\rho_{\max}$, such as is usually done in direct process calculations, since then the damping is not so dramatic?" In answer to this, we make two comments: (1) Agreement between theory and experiment in the (p,d) Be^9 reaction,[34] involving the same single neutron, is definitely better for an over-all space integration than for the Butler formula. (2) If we do restrict the integration to $\rho\geqslant\rho_{\max}$, the matrix element is greatly reduced because the projectile sees only a small part of the wave function.

The direct neutron interaction is also inhibited by the angular momentum coupling factors. The cross section due solely to the neutron interaction would be, for the $\frac{3}{2}\rightarrow\frac{5}{2}$ transition,

$$\frac{d\sigma}{d\Omega}=\frac{``d\sigma_n"}{d\Omega}\frac{18}{175}\langle f_1|j_2(8K\rho/9)|f_1\rangle^2, \quad (19)$$

which is to be contrasted with the cross section with only alpha interaction, Eq. (12) and Table I. Thus, almost regardless of the value of "$d\sigma_n/d\Omega$" (which is drastically far from the energy shell and accordingly difficult to estimate), the direct neutron collision should make little contribution to the excitation of this state.

(B) Excitation of Vibrational States

Vibrational excitation is characterized by the properties of rotational excitation with the additional requirement that $\mathcal{R}'(\xi)$ be an excited wave function. There is no evidence for vibrational states in Be^9 but the 6.13-Mev state of O^{16} and the 7.65-Mev state of C^{12} have been so interpreted; the expressions derived below can be easily generalized to apply to dilatational excitation of these nuclei.

In consideration of the previous discussion, let us retain assumptions (a) and (c) of Sec. A. Then the cross section is given by Eq. (12) where now $j_\lambda(KR)$ is replaced by

$$\langle\mathcal{R}'(\xi)|j_\lambda(\tfrac{1}{2}K\xi)|\mathcal{R}(\xi)\rangle. \quad (20)$$

For a $\frac{3}{2}\rightarrow\frac{3}{2}$ transition, $B_0=B_2=4$; all other B_L are zero. The spherical Bessel function can be expanded about the equilibrium position so that, to first order in $(\xi-2R)$,

$$j_\lambda(\tfrac{1}{2}K\xi)=j_\lambda(KR)+\tfrac{1}{2}K(\xi-2R)\frac{d}{dx}j_\lambda(x)\bigg|_{x=KR}. \quad (21)$$

We again assume that the vibrational states are approximately represented by simple harmonic-oscillator wave functions centered about $\xi=2R$. The cross section for one-quantum excitation is then

$$\frac{d\sigma}{d\Omega}\approx\frac{``d\sigma_\alpha"}{d\Omega}\left(\frac{\epsilon}{\hbar\omega}\right)\sum_\lambda B_\lambda\left[x\frac{d}{dx}j_\lambda(x)\right]^2_{x=KR}, \quad (22)$$

where ϵ and $\hbar\omega$ have been defined following Eq. (17). The comments made in Sec. A (b) regarding damping of the angular distribution due to the zero-point motion also apply for vibrational excitation.

(C) Single-Particle Excitation

Transitions to states characterized by a neutron wave function, $\phi_{\Omega'}$, differing from that of the ground state may be induced by collisions with the neutron, Eq. (8b), or by collisions with the collective mode of motion, which couples to the neutron motion. In the former case, arguments similar to that of Sec. A(c) show that the cross section for single-particle excitation should be appreciable only in the forward direction (small K). In the latter case, the neutron "shaking" is effective only for large K as it arises from the noncoincidence of the center of mass of the collective motion and that of the nucleus. A considerable simplification can be achieved in the expression for A_Λ in the region of greatest interest, where $j_\lambda(x)$ can be represented by $x^{-1}\sin(x-\frac{1}{2}\lambda\pi)$. The portion of Eq. (8a) which involves λ is

$$\sum_\lambda (i)^\lambda[1+(-1)^\lambda](2\lambda+1)(\lambda,L,0,0|\Lambda,0) \times(\lambda, L, 0, \Omega-\Omega'|\Lambda, \Omega-\Omega')j_\lambda(x). \quad (23)$$

The factor $[1+(-1)^\lambda]$ guarantees that L and Λ are of the same parity; for large x, Eq. (23) becomes

$$\frac{\sin x}{x}\sum_\lambda[1+(-1)^\lambda](2\lambda+1)(\lambda,L,0,0|\Lambda,0) \times(\lambda, L, 0, \Omega-\Omega'|\Lambda, \Omega-\Omega') =\frac{\sin x}{x}(2\Lambda+1)[1+(-1)^{L+\Lambda}]\delta_{\Omega',\Omega} \quad (24)$$

by the completeness relation for Clebsch-Gordan coefficients. Similarly the symmetrization term in Eq. (8a) always vanishes in this approximation. Thus, we have the selection rule that Ω' must equal Ω in order to have appreciable single-particle excitation from the "shaking" mechanism. The cross section for single-particle excitation at large angles then has the following simple form (after summation over Λ):

$$\frac{d\sigma}{d\Omega}=\frac{``d\sigma_\alpha"}{d\Omega}\,2\frac{(2I'+1)(2j+1)}{(2j'+1)}\langle\mathcal{R}'|\sin x/x|\mathcal{R}\rangle^2 \times|\sum_L(-i)^L\tfrac{1}{2}[1+(-1)^{l+l'-L}](2L+1)(j,L,\tfrac{1}{2},0|j',\tfrac{1}{2}) \times(j,L,-\Omega,0|j',-\Omega)\langle f_{l'}|j_L(K\rho/9)|f_l\rangle|^2. \quad (25)$$

For Be^9 there is only one $\Omega=\frac{3}{2}$ p orbital and this presumably defines the band based on the ground state; therefore, the only other reasonable $\Omega'=\frac{3}{2}$ single-particle states will be d states. With the further assumption that $j'=\frac{5}{2}$, the large-angle cross section into such a state is given by

$$\frac{d\sigma}{d\Omega}=\frac{``d\sigma_\alpha"}{d\Omega}\frac{288}{25}\langle f_2|j_1(K\rho/9)+j_3(K\rho/9)|f_1\rangle^2 \times\langle\mathcal{R}'|\sin x/x|\mathcal{R}\rangle^2. \quad (26)$$

The cross section displays the rapid oscillations of the interaction with the collective mode, modulated by the square of single-particle matrix elements, which increase with increasing momentum transfer.

IV. DISCUSSION

Let us first review the experimental information concerning inelastic scattering from Be^9, which comes from studies with 48-, 43-, and 21.6-Mev alpha particles,[33,40,41] 24-, 10.8-, 15.1-, and 8.9-Mev deuterons,[33,41–43] and 31-, 12-, and 10-Mev protons.[44–46] In all cases the 2.43-Mev level is strongly excited; the 6.8-Mev level is prominent for bombardment by 48- and 43-Mev alpha particles, 24-Mev deuterons, 31- and 12-Mev protons; 48-Mev alpha particles and, less certainly, 31-Mev protons excite the 11.3-Mev state. The 1.7-Mev level, be it a final-state interaction or true state,[47] is difficult to excite; it has been seen with 21.6-Mev alpha particles, 10.8-Mev deuterons, and 12-Mev protons. Similarly the 3.1-Mev level has been weakly excited by 21.6-Mev alpha particles, 10.8-Mev deuterons, and possibly 43-Mev alpha particles and 24-Mev deuterons. For angles less than 75°, the data with 43-Mev alpha particles exclude cross sections for excitation of the 1.8- and 3.1-Mev levels greater than 10% of the 2.43-Mev cross sections; at larger angles there is some indication of excitation of the 3.1-Mev state, which suggests the possibility of single-particle excitation via the shaking mechanism (Sec. III C). There is no unambiguous evidence for a transition to the 4.8-Mev level.

Angular distributions have been measured only for 2.43- and 6.8-Mev excitation. The cross section to the 6.8-Mev state due to 31-Mev protons, $\sigma_{6.8}$ (31.3-Mev p), has a maximum around 50° (c.m.) but otherwise shows little structure; when this maximum is fitted to the first maximum of Eq. (12) (where I' is assumed $=\frac{7}{2}$), then R is found to be approximately 4.8×10^{-13} cm. Similarly the cross section to the 2.43-Mev state, $\sigma_{2.43}$(31.3-Mev p), shows a maximum at 45°; the corresponding R is 4.1×10^{-13} cm (if $I'=\frac{5}{2}$). Both of these radii are quite large. According to the simple Eq. (12), the ratio $\sigma_{2.43}/\sigma_{6.8}$ at the first maximum should be 1.64; in fact $\sigma_{2.43}$(31.3-Mev p)/$\sigma_{6.8}$(31.3-Mev p) is around 4. However, since the simple formula given by Eq. (12) does not fit the curve too well and since the radii needed

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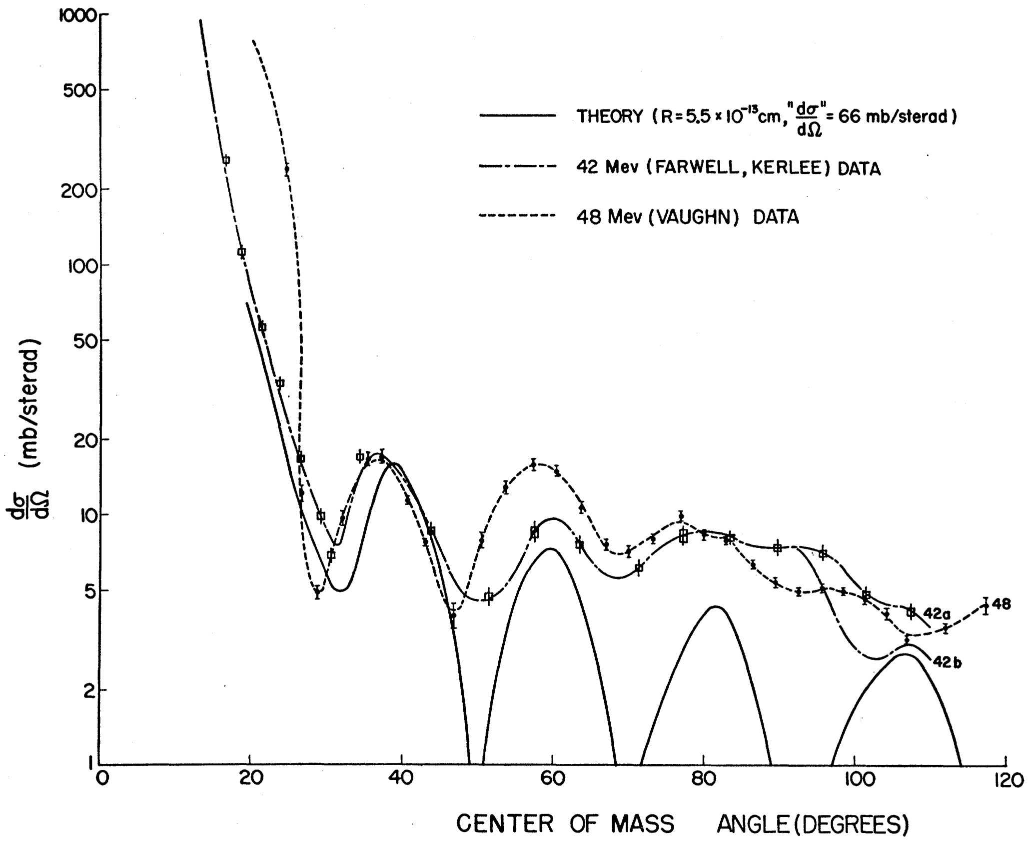

FIG. 3. Inelastic scattering of 48- and 42-Mev alpha particles to the 2.43-Mev level of Be^9. The experimental curve at 48 Mev has been multiplied by a constant of 2.4. The experimental curves at 42 Mev labeled 42*a* and 42*b* correspond, respectively, to no excitation of the 3.1-Mev level (unresolved) and maximum possible excitation of this level. The theoretical curve is that obtained from Eq. (12) with a constant "$d\sigma_\alpha/d\Omega$" = 66 mb/sterad and a radius of 5.5×10^{-13} cm. The 48-Mev curve should be credited to Summers-Gill, not Vaughn.

to match the experimental peaks for the two levels are different, this discrepancy is not surprising. We remark, nevertheless, that the discrepancy cannot be attributed to neglect of the neutron interaction, N_Λ, since the ratio

$$N_2^2(I'=\tfrac{5}{2})/N_2^2(I'=\tfrac{7}{2})=1.8.$$

The energy and angle dependence of "$d\sigma_\alpha/d\Omega$" may be partially responsible for the difference between experimental and theoretical results; we note that the elastic cross section, of protons on alphas (even though not directly related to "$d\sigma_\alpha/d\Omega$") is fairly energy- and angle-dependent.

The angular distribution to the 2.43-Mev state obtained with 12-Mev protons is strongly peaked forward and shows no fit to an ABM-type formula. Such behavior appears to be characteristic of protons with $\lambda\!\!\!^{-} > 10^{-13}$ cm. Banerjee and Levinson[16] find that distortion of the incident and final wave functions critically affects the angular distributions of protons in this energy region.

Cross sections to the same state due to 24-Mev deuteron bombardment show more angular structure. The locations of the first two maxima are given by Eq. (12) with a radius $R=5.6\times10^{-13}$ cm. The corresponding "$d\sigma_\alpha/d\Omega$" = 13 mb at the first maximum. 15.1-Mev deuterons lead to an angular distribution which has insufficient structure to analyze.

The most interesting angular structure is apparent for inelastic alpha-particle scattering. The differential cross section with 48-Mev incident alpha particles shows 4 maxima; there is an almost quantitative match to Eq. (12) (with constant "$d\sigma_\alpha/d\Omega$" ≈ 50 mb/steradian and for a radius $R=5.4\times10^{-13}$ cm). Alpha particles of 43 Mev lead to a similar pattern, shifted slightly to larger angles. A comparison of the experimental results at these two energies with Eq. (12) and a constant "$d\sigma_\alpha/d\Omega$" is shown in Fig. 3. It is not clear how to predict "$d\sigma_\alpha/d\Omega$" from measured alpha-alpha elastic cross sections. Not only is "$d\sigma_\alpha/d\Omega$" far "off the energy shell" except at small angles, but also the observed elastic cross sections for energy around 40 Mev are quite energy and angle sensitive.[2] The total elastic cross sections, which are of the order 500–1000 mb, lead to "average differential cross sections" which are consistent with "$d\sigma_\alpha/d\Omega$" ≅ 50 mb.

With respect to this collected data, the alpha-particle model has the following virtues: (1) The varying ease with which different levels are excited finds a natural explanation: strongly excited levels belong to the rotational band built on the ground state and are reached by collective transitions; weakly excited states have different neutron orbitals and are excited either through a direct interaction with this single particle or through a "shaking" via the core interaction. The meager experimental information leads us to refrain from speculating about possible single-particle orbitals.

(2) It is relatively easy to obtain the large magnitude for the cross section to the 2.43-Mev state. The relevant alpha-particle "cross sections," "$d\sigma_\alpha/d\Omega$," are uncertain but should be ample. We note that the constructive interference of the scattering from both alpha particles of the dumbbell contributes a factor 4 to the cross section.

(3) Sharp structure of the angular distribution is consistent with a large cross section; the possible sharpness is associated with amplitude of the oscillations about the equilibrium distance in the alpha bond rather than the penetration depth of the projectile.

There are, however, some serious discrepancies between our calculation and observation: (1) The equilibrium radius suggested by the energy levels of Be^8 and Be^9, $R\cong 2.3\times 10^{-13}$, is decidedly smaller than the radii appropriate to the angular distributions, $\sim(4$ to $5.5)\times 10^{-13}$ cm. This defect may well be a calculational deficiency. The size of the alpha particle should have considerable bearing on the elementary projectile-alpha particle interaction; this influence, however, has been disregarded in our calculation since the factorization of the scattering amplitude outside the overlap integral is equivalent, in a sense, to use of a zero-range projectile-alpha particle potential. We naively expect that introduction of a hard-core radius, R_{hc}, in the two-body interaction will lead to effective radii in the angular distributions of the order $R+R_{\mathrm{hc}}$, but we have not succeeded in formulating this modification.[48]

(2) The oscillations of the alpha bond must be very small [oscillator width $1/\alpha=2R(2\epsilon/\hbar\omega)^{\frac{1}{4}}\lesssim 1\times 10^{-13}$ cm] to avoid unwanted damping of the angular distribution.

By making the impulse approximation, we have neglected the influence of nuclear binding forces on the two-body interactions and also "multiple scattering" (or equivalently, distortion of the incident and final projectile waves by all particles in the nucleus, except the one it is interacting with). Distortion was not necessary to produce oscillatory angular distributions but nonetheless should modify our simple plane-wave results. In lieu of performing a distorted-wave or multiple-scattering calculation, we can only make the empirical observation that distortion effects appear less important with decreasing wavelength. The neglect of nuclear binding forces may not be justified for the alpha-particle model: A large, highly concentrated attractive potential, $W(\xi)$, is necessary to achieve a near-rigid dumbbell; because of the operator K_i in the Green's function ω_i, the commutator $[W,\omega_i^{(+)}]$ [see Eq. (I.12)] will not be small compared to W.

Inelastic scattering from other light nuclei, particularly the nuclei C^{12}, O^{16}, Ne^{20}, and possibly Mg^{24}, may be discussed in terms of the alpha-particle model. If the more complicated vibrational modes are neglected, rotational and dilatational excitation cross sections can be obtained as simple generalizations of the preceding work. We postpone detailed comment on these cases but would like to draw attention to one aspect of experimental angular distributions: Superimposed on the oscillating angular distribution of inelastic alpha particles is an increase at backward angles.[49] We would like to suggest that this is a manifestation of exchange scattering between the incident and target alpha particles. The relevant two-body exchange amplitude [see Eq. (I.20)] becomes, in Born approximation, $\int V(r)\exp(i\mathbf{k}_0'\cdot\mathbf{r})d^3r$. The final momentum, k_0', decreases slowly with increasing angle and thus should lead to an increasing exchange amplitude. A similar backward increase is observed in the elastic alpha cross section for Ne^{20} (with the oscillations now in phase with the inelastic angular distribution); such back-angle scattering may have the same origin, since the contribution from the usually rapidly decreasing diffraction cross section should be small at backward angles.

In conclusion we would like to point out that the calculations of this paper can also be applied to high-energy electron excitation; indeed, the use of plane waves is then a good approximation and the elementary interaction is a known quantity.[50]

APPENDIX. DERIVATION OF CROSS SECTION FOR DIRECT PROCESSES

Consider a nuclear system with constituent particles $1\cdots j\cdots N$ and an incident particle, p. In the absence of any interaction between the incident particle and the nucleus, the total Hamiltonian is assumed to be

$$H_0=K+W(\mathbf{x}_j)=K_p+\sum_{j=1}^{N}K_j+W, \qquad \text{(I.1)}$$

where K is kinetic energy and W is internal potential energy. Assume that the interaction between the incident particle and nucleus can be represented by the real potential

$$V+U=\sum_{i=1}^{M}V_i(\mathbf{x}_p,\mathbf{x}_i)+U(\mathbf{x}_p), \qquad \text{(I.2)}$$

where V_i is the potential between the incident particle and ith nuclear particle and U is a phenomenological

[48] An alternative explanation of the large radius is that the interaction is still collective but is better described by an ellipsoidally deformed optical potential.

[49] Seidlitz, Bleuler, and Tendam, Phys. Rev. **110**, 1080 (1958).

[50] L. I. Schiff, Phys. Rev. **92**, 988 (1953).

potential representing an interaction with all the other nuclear particles, $N \geqslant j > M$. For example, particles $1 \cdots M$ might be the "valence" nucleons while U represents an interaction with the core.

Denote the initial state of the system, with the incident particle well separated from the nucleus, by Φ_a, where

$$H_0\Phi_a = E\Phi_a, \tag{I.3a}$$

and

$$\Phi_a = e^{i\mathbf{k}_0 \cdot \mathbf{x}_p}\phi_a(x_j). \tag{I.3b}$$

The corresponding final state is Φ_b, where

$$H_0\Phi_b = E\Phi_b, \tag{I.4a}$$

and

$$\Phi_b = e^{i\mathbf{k}_0' \cdot \mathbf{x}_p}\phi_b(x_j)\exp(i\mathbf{K}\cdot\mathbf{X}), \tag{I.4b}$$

with $\mathbf{X}$ defined as the center-of-mass coordinate of the nucleus.

The cross section for a transition from state a to state b is[36]

$$\sigma_{ba} = \frac{2\pi}{\hbar}\frac{\omega_b}{v}|T_{ba}|^2, \tag{I.5}$$

where ω_b is the density in energy of final states and v is the relative velocity of the incident particle and nucleus. The transition amplitude, T_{ba}, is given by[36]

$$T_{ba} = \langle\Phi_b|V+U|\Psi_a^{(+)}\rangle, \tag{I.6}$$

where $\Psi_a^{(+)}$ is the solution of the integral equation

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E-K-W+i\epsilon}(V+U)\Psi_a^{(+)}. \tag{I.7}$$

In a straightforward manner, Gell-Mann and Goldberger[36] have shown that

$$T_{ba} = \langle\Phi_b|U|\chi_a^{(+)}\rangle + \langle\chi_b^{(-)}|V|\Psi_a^{(+)}\rangle$$
$$= \langle\Phi_b|U|\chi_a^{(+)}\rangle + \left\langle\chi_b^{(-)}\middle|V + V\frac{1}{E-K-W-U-V+i\epsilon}V\middle|\chi_a^{(+)}\right\rangle, \tag{I.8}$$

where $\chi^{(\pm)}$ represents the plane wave plus outgoing $(+)$ or incoming $(-)$ wave solution of the scattering problem when $V=0$; i.e.,

$$\chi_a^{(\pm)} = \Phi_a + \frac{1}{E-K-W\pm i\epsilon}U\chi_a^{(\pm)}. \tag{I.9}$$

We note that the phenomenological potential, U, is Hermitian in this derivation.

The first term of (I.8) is the exact transition amplitude when $V=0$. Such a term may contribute to inelastic scattering into states of collective nuclear motion through distortions of the nuclear surface. In the present work, however, we shall be discussing alternative mechanisms and hence drop this term from the transition amplitude.

The second term of T_{ba} can be separated into a term representing the impulse approximation plus additional terms representing the "error in the impulse approximation" and multiple scattering corrections. Let $\chi_a^{(+)}$ be expanded in plane-wave functions of particles p and i:

$$\chi_a^{(+)} = \sum_{\mathbf{k}_p,\mathbf{k}_i}|\mathbf{k}_p,\mathbf{k}_i\rangle\langle\mathbf{k}_p,\mathbf{k}_i|\chi_a^{(+)}\rangle. \tag{I.10}$$

With the notation $E_l = \hbar^2k_p^2/2m_p + \hbar^2k_i^2/2m_i$, we define

$$\omega_i^{(+)} = 1 + \frac{1}{E_l - K_p - K_i - V_i + i\epsilon}V_i, \tag{I.11a}$$

and

$$t_i^{(+)} = V_i\omega_i^{(+)}, \tag{I.11b}$$

where $t_i^{(+)}$ is now the two-body scattering operator. Then the second term of T_{ba} can be written

$$\left\langle\chi_b^{(-)}\middle|\sum_i\left\{t_i^{(+)} + V\frac{1}{E-K-W-U-V+i\epsilon}[U+W,\omega_i^{(+)}] + \left(1+V\frac{1}{E-K-W-U-V+i\epsilon}\right)(V-V_i)(\omega_i^{(+)}-1)\right\}\middle|\chi_a^{(+)}\right\rangle. \tag{I.12}$$

This equation is similar to one derived by Chew and Goldberger[37] and differs from Eq. (21) of this reference only through inclusion of effects due to *both* the internal potential, W, and the phenomenological potential, U.

The term in (I.12) containing the commutator gives "the error in the impulse approximation" and its size has been estimated by Chew and Wick[37] to be of the order of t_i times $\tau[U+W,\omega_i]$, where τ is the "average collision time." The occurrence of the commutator means that the correction depends on the change of the potentials during a two-body collision rather than on the magnitude of the potentials. This correction term for the impulse assumption and the last term in (I.12), which represents largely multiple-scattering corrections, will be neglected in following work. We shall consider in the discussion section whether such neglect is justified when intermediate-energy projectiles are scattered by a nucleus described in terms of the alpha-particle model.

With the above "impulse approximation," the transition amplitude becomes

$$T_{ba}=\sum_i \sum_{\mathbf{k}_p',\mathbf{k}_i'} \sum_{\mathbf{k}_p,\mathbf{k}_i} \langle \chi_b^{(-)} | \mathbf{k}_p',\mathbf{k}_i' \rangle \times \langle \mathbf{k}_p,\mathbf{k}_i | \chi_a^{(+)} \rangle \langle \mathbf{k}_p',\mathbf{k}_i' | t_i^{(+)} | \mathbf{k}_p,\mathbf{k}_i \rangle. \quad (\text{I.13})$$

The elementary two-body scattering conserves momentum so that

$$\langle \mathbf{k}_p',\mathbf{k}_i' | t_i^{(+)} | \mathbf{k}_p,\mathbf{k}_i \rangle = \delta(\mathbf{k}_p'+\mathbf{k}_i', \mathbf{k}_p+\mathbf{k}_i) \times \langle \mathbf{k}_p', \mathbf{k}_i+\mathbf{k}_p-\mathbf{k}_p' | t_i^{(+)} | \mathbf{k}_p, \mathbf{k}_i \rangle. \quad (\text{I.14})$$

Assume now (a) that the variation of the above scattering amplitude with the momenta contained in states $\chi_a^{(+)}$ and $\chi_b^{(-)}$ is small enough so that the scattering amplitude may be approximated by a mean amplitude,[9] and (b) that the appropriate mean amplitude is that obtained when the struck particle is initially at rest and the projectile is characterized by mean momenta, $\boldsymbol{\kappa}_p$ and $\boldsymbol{\kappa}_p'$, i.e.,

$$\langle \mathbf{k}_p', \mathbf{k}_i+\mathbf{k}_p-\mathbf{k}_p' | t_i^{(+)} | \mathbf{k}_p, \mathbf{k}_i \rangle \approx \langle \boldsymbol{\kappa}_p', \boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p' | t_i^{(+)} | \boldsymbol{\kappa}_p, 0 \rangle. \quad (\text{I.15})$$

In cases where the phenomenological potential U is zero, $\boldsymbol{\kappa}_p$ and $\boldsymbol{\kappa}_p'$ become the free momenta $\mathbf{k}_0$ and $\mathbf{k}_0'$, respectively. The rather strong assumption (a) is necessary to factor the two-body amplitude outside the overlap integral of initial and final states. In partial defense of this assumption is should be pointed out that in the limit of Born approximation the nonexchange scattering amplitude depends only on the momentum transfer, $\mathbf{k}_p-\mathbf{k}_p'$. Unfortunately we do not have direct experimental information about the two-body scattering amplitudes for momentum values of most interest since these amplitudes may lie far off the energy shell. This will be the case, particularly, if the scattering is backwards from a target particle of mass equal to or lighter than that of the projectile (e.g., α on n). Thus the amplitudes deduced from two-body elastic (α,α), (α,n), and (n,n) scattering, which lie on the energy shell, can be used as only very rough guides. Our ignorance therefore forces us to regard the factorization assumption (a) as one of the weaker links in the derivation.

With assumptions (a) and (b) the transition amplitude is

$$T_{ba}=\sum_i \langle \boldsymbol{\kappa}_p', \boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p' | t_i^{(+)} \boldsymbol{\kappa}_p, 0 \rangle \times \sum_{\mathbf{k}_p',\mathbf{k}_p,\mathbf{k}_i} \langle \chi_b^{(-)} | \mathbf{k}_p', \mathbf{k}_i+\mathbf{k}_p-\mathbf{k}_p' \rangle \langle \mathbf{k}_p, \mathbf{k}_i | \chi_a^{(+)} \rangle. \quad (\text{I.16})$$

Using the completeness of the momentum eigenfunctions, we obtain

$$T_{ba}=\sum_i \langle \boldsymbol{\kappa}_p', \boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p' | t_i^{(+)} | \boldsymbol{\kappa}_p, 0 \rangle \times \langle \chi_b^{(-)} | \delta(\mathbf{x}_p-\mathbf{x}_i) | \chi_a^{(+)} \rangle. \quad (\text{I.17})$$

The above equation is sometimes considered as the impulse approximation; we have seen, though, that some further nontrivial assumptions are needed in order to obtain such a simple expression.

It is pertinent to comment here on the relative virtues of the impulse approximation and the Born approximation, which has also been frequently employed in theoretical treatments of direct scattering. [The Born approximation in the distorted-wave formalism is simply obtained by dropping the last term of Eq. (I.8).] It is well known that the Born approximation may give unreliable estimates of the scattering amplitude for two-body collisions, particularly at low or moderate energies when the potential is such as to give a bound state, which is indeed the case for nuclear particles. In this respect, the impulse approximation is a definite improvement since it employs an exact two-body scattering operator. On the other hand, part of this gain is illusory: the relevant scattering amplitudes do not conserve energy in the two-body collision; thus they cannot be deduced reliably from measured two-body elastic collisions, as discussed earlier. To summarize: The Born approximation provides a calculable but probably erroneous value for the scattering amplitude; the exact two-body scattering amplitude appears in the impulse approximation, but it is difficult to evaluate.

When i and p are the same kind of particle [which can occur in (p,p') and (n,n') scattering when the nucleus is described by the shell model or (α,α') scattering when the alpha-particle model is adopted], modifications must be made in the above derivation. It is also desirable to include explicit dependence on spin and isotopic spin. The result is that Eq. (I.13) remains unchanged except that the two-body scattering amplitude is appropriately symmetrized. We find [see Eqs. (I.24) and (I.25)]

$$\langle t_i^{(+)} \rangle \rightarrow \langle \mathbf{k}_p',\mathbf{k}_i' | t_i^{(+)} | \mathbf{k}_p,\mathbf{k}_i \rangle \pm \langle \mathbf{k}_i',\mathbf{k}_p' | t_i^{(+)} | \mathbf{k}_p,\mathbf{k}_i \rangle. \quad (\text{I.18})$$

The $+$ sign holds for spin-0 Bose-Einstein particles, the $-$ sign for spin-$\frac{1}{2}$ Fermi-Dirac particles. In the latter case, to simplify notation, $\mathbf{k}$ designates the spin and isotopic spin as well as the momentum. If P^x, P^σ, P^τ are defined to be the space, spin, and isotopic spin exchange operators, Eq. (I.18) may be written more compactly in the spin-$\frac{1}{2}$ case as

$$\langle t_i^{(+)} \rangle \rightarrow \langle \mathbf{k}_p', \mathbf{k}_i' | (1-P^x P^\sigma P^\tau) t_i^{(+)} | \mathbf{k}_p, \mathbf{k}_i \rangle. \quad (\text{I.19})$$

The factorization of the exchange contribution to the two-body scattering amplitude is not as well justified as is the case for the direct term; in the limit of Born approximation, the exchange scattering amplitude is very definitely a function of k_i. If nonetheless we make the previous factorization, assumptions (a) and (b), and make use of the conservation of momentum in the two-body collision, we obtain

$$T_{ba}\approx\sum_i\{\langle\chi_b^{(-)}|\delta(\mathbf{x}_p-\mathbf{x}_i)\langle\boldsymbol{\kappa}_p',\boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p'|t_i^{(+)}|\boldsymbol{\kappa}_p,0\rangle|\chi_a^{(+)}\rangle \pm\langle\chi_b^{(-)}|\delta(\mathbf{x}_p-\mathbf{x}_i)P^\sigma P^\tau \times\langle\boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p',\boldsymbol{\kappa}_p'|t_i^{(+)}|\boldsymbol{\kappa}_p,0\rangle|\chi_a^{(+)}\rangle\}. \quad \text{(I.20)}$$

In this equation, we drop the notation of Eq. (I.18); $\mathbf{k}$ refers only to momentum so that the matrix element

$$\langle\boldsymbol{\kappa}_p',\boldsymbol{\kappa}_p-\boldsymbol{\kappa}_p'|t_i^{(+)}|\boldsymbol{\kappa}_p,0\rangle$$

may be an operator in spin and isotopic spin matrices of i and p. $\chi_a^{(+)}$ is unsymmetrized in projectile and target, i.e., $\chi_a^{(+)}=u_a^{(+)}(\mathbf{x}_p)\phi_a(\mathbf{x}_j)$.

The proof of (I.18) and (I.19) is straightforward: Assume that the nuclear wave function $\phi_a(\mathbf{x}_j)$ is already appropriately symmetrized or antisymmetrized in M equivalent particles. Then the symmetrized eigenfunction of H_0+U, that is $\chi_a^{(+)}$, can be written

$$\chi_a^{(+)}=(M+1)^{-\frac{1}{2}}\sum_{l=1}^{M}(-1)^{P_l}u_a^{(+)}(\mathbf{x}_l) \times\phi_a(\mathbf{x}_1\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1},\cdots\mathbf{x}_{N+1}) \equiv(M+1)^{-\frac{1}{2}}\sum_l\chi_{al}^{(+)}, \quad \text{(I.21)}$$

where $P_l=0$ for Bose-Einstein particles and $l-1$ for Fermi-Dirac particles; $\mathbf{x}$ denotes spin and isotopic spin coordinates as well as spatial coordinates. Since the particles are equivalent, it is not now correct to consider U and V as functions of just $\mathbf{x}_p$ and $\mathbf{x}_p$, $\mathbf{x}_i$, respectively; rather, for the lth term in $\chi_a^{(+)}$, U is to be considered a function of $\mathbf{x}_l$ and similarly

$$V=\sum_{j=1(\neq l)}^{j=M+1}V(\mathbf{x}_l,\mathbf{x}_j).$$

Thus in the impulse approximation, the contribution to the transition amplitude arising from collisions of the projectile with the equivalent particles in the nucleus is

$$(M+1)^{-1}\langle\sum_{l'}\chi_{bl'}^{(-)}|\sum_l\sum_{j\neq l}t(\mathbf{x}_l,\mathbf{x}_j)|\chi_{al}^{(+)}\rangle. \quad \text{(I.22)}$$

We can make the following expansion:

$$|\chi_{al}^{(+)}\rangle=|\mathbf{k}_p(\mathbf{x}_l)\rangle|\mathbf{k}(\mathbf{x}_j)\rangle\langle\mathbf{k}_p|u_a^{(+)}(\mathbf{x}_l)\rangle \times(-1)^{P_l}\langle\mathbf{k}(\mathbf{x}_j)|\phi_a(\mathbf{x}_1,\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1},\cdots\mathbf{x}_{N+1})\rangle,$$

which becomes, for Fermi-Dirac particles,

$$|\chi_{al}^{(+)}\rangle=|\mathbf{k}_p(\mathbf{x}_l)\rangle|\mathbf{k}(\mathbf{x}_j)\rangle\langle\mathbf{k}_p|u_a^{(+)}(\mathbf{x}_l)\rangle \times\langle\mathbf{k}(\mathbf{x}_j)|\phi_a(\mathbf{x}_j,\mathbf{x}_1,\cdots\mathbf{x}_{l-1}\cdots\mathbf{x}_{N+1})\rangle \times\begin{cases}(-1)^{l-1+j-1} & \text{if } j<l\\ (-1)^{l-1+j} & \text{if } j>l,\end{cases} \quad \text{(I.23)}$$

where for convenience $\mathbf{k}$ here designates spin as well as the momentum. For Bose-Einstein particles, all sign factors are $(+1)$. The final-state wave function, $\chi_b^{(-)}$, can be similarly expanded with the notational change $(l,j,\mathbf{k}_p,\mathbf{k}\rightarrow l',j',\mathbf{k}_p',\mathbf{k}')$.

If l' does not equal either l or j, the matrix element should be extremely small; when both bound and continuum particles are described by an independent-particle model with the same central potential and when small center-of-mass effects are ignored, the matrix element vanishes by the orthogonality of initial and final wave functions. If $l'=l$ and one chooses $j'=j$, the sign factors in initial and final wave functions are identical regardless of the order of l and j. If $l'=j$ and one chooses $j'=l$, the sign factors must differ by $(-)$ in the Fermi-Dirac case. Since

$$\langle\phi_b(\mathbf{x}_j,\mathbf{x}_1\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1}\cdots\mathbf{x}_{N+1}|\mathbf{k}'(\mathbf{x}_j)\rangle\langle u_b^{(-)}(\mathbf{x}_l)|\mathbf{k}'(\mathbf{x}_l)\rangle$$

is equivalent to

$$\langle\phi_b(\mathbf{x}_l,\mathbf{x}_1,\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1}\cdots\mathbf{x}_{N+1}|\mathbf{k}'(\mathbf{x}_l)\rangle\langle u_b^{(-)}(\mathbf{x}_j)|\mathbf{k}'(\mathbf{x}_j)\rangle,$$

Eq. (I.22) becomes

$$(M+1)^{-1}\sum_{l,j}\sum_{\mathbf{k}_p,\mathbf{k}_p'}\sum_{\mathbf{k},\mathbf{k}'} \times\langle\phi_b(\mathbf{x}_j,\mathbf{x}_1\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1}\cdots\mathbf{x}_{N+1})|\mathbf{k}'(\mathbf{x}_j)\rangle \times\langle\mathbf{k}(\mathbf{x}_j)|\phi_a(\mathbf{x}_j,\mathbf{x}_1\cdots\mathbf{x}_{l-1},\mathbf{x}_{l+1}\cdots\mathbf{x}_{N+1})\rangle \times\langle u_b^{(-)}|\mathbf{k}_p'\rangle\langle\mathbf{k}_p|u_a^{(+)}\rangle\{\langle\mathbf{k}_p',\mathbf{k}'|t(\mathbf{x}_l,\mathbf{x}_j)|\mathbf{k}_p,\mathbf{k}\rangle -\langle\mathbf{k}',\mathbf{k}_p'|t(\mathbf{x}_l,\mathbf{x}_j)|\mathbf{k}_p,\mathbf{k}\rangle\}. \quad \text{(I.24)}$$

Because of the symmetry of ϕ_a and ϕ_b, Eq. (I.24) can be written

$$\sum_{j=1}^{M}\sum_{\mathbf{k}_p,\mathbf{k}_p'}\sum_{\mathbf{k},\mathbf{k}'}\langle\phi_b(\mathbf{x}_1\cdots\mathbf{x}_N)|\mathbf{k}'(\mathbf{x}_j)\rangle\langle u_b^{(-)}|\mathbf{k}_p'\rangle \times\langle\mathbf{k}_p|u_a^{(+)}\rangle\langle\mathbf{k}(\mathbf{x}_j)|\phi_a(\mathbf{x}_1\cdots\mathbf{x}_N)\rangle \times\langle\mathbf{k}_p',\mathbf{k}'|(1-P^xP^\sigma P^\tau)t(\mathbf{x}_p,\mathbf{x}_j)|\mathbf{k}_p,\mathbf{k}\rangle. \quad \text{(I.25)}$$