Six complete sets of readings were taken. These included readings of three different temperatures and of both directions of the field. In all of the runs the currents to the two edge plates dropped rapidly to zero, and were negligible for fields greater than 300 gauss.

In three sets of readings no variation in total plate current was noted, except for the random error of 1.5% or less, as the magnetic field was increased from zero to 6000 gauss and then decreased to zero. Other runs showed a slight decrease of 1.5 to 4% in total current. This decrease was caused by losses of electrons after emission. The overlapping of the plates was sufficient to insure collection of all electrons only when the filament remained in or near its equilibrium position on the tube axis. But under the force exerted on it by the applied magnetic field, the ribbon was displaced from its axial position, and a small fraction of the emitted electrons escaped through the gaps between the plates. This displacement problem is difficult to avoid when a filament is heated in a strong magnetic field.

The negative result that was obtained in several sets of measurements taken under very favorable conditions is interpreted as sufficient evidence that no magnetic-field effect on the saturation electron emission exists, at values up to the maximum of 6000 gauss which were attainable in this experiment. We therefore conclude that the discrepancy between Shelton's result and the results of the earlier observations of Nottingham and Hutson was not caused by any modification of the probability of emission by the magnetic field.

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Random-Phase Approximation in the Theory of Superconductivity*

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A generalization of the random-phase approximation of the theory of Coulomb correlation energy is applied to the theory of superconductivity. With no further approximations it is shown that most of the elementary excitations have the Bardeen-Cooper-Schrieffer energy gap spectrum, but that there are collective excitations also. The most important of these are the longitudinal waves which have a velocity $v_F\{\frac{1}{3}[1-4N(0)\,|\,V\,|\,]\}^{\frac{1}{2}}$ in the neutral Fermi gas, and are essentially unperturbed plasma oscillations in the charged case. Other collective excitations resembling higher bound pair states may or may not exist but do not seriously affect the energy gap. The theory obeys the sum rules and is gauge invariant to an adequate degree throughout.

I. INTRODUCTION

RECENTLY Bardeen, Cooper, and Schrieffer proposed a theory of superconductivity[1] which has been successful in explaining experimental results of many kinds. The theory is founded on the idea that in superconductors there is a net attraction between electrons caused by the phonons. The ground-state wave function used is a product function designed to have the maximum number of pairs of electrons of zero total momentum taking advantage of this attraction. This ground-state function is

$$\Psi_g=\prod_{\mathbf{k}}[(1-h_k)^{\frac{1}{2}}+h_k^{\frac{1}{2}}c_{\mathbf{k}\uparrow}^{*}c_{-\mathbf{k}\downarrow}^{*}]\Psi_v, \qquad (1)$$

where Ψ_v is the vacuum, and $c_{\mathbf{k}\sigma}^{*}$ is the creation operator for electrons of momentum $\mathbf{k}$ and spin σ. h_k is a number determined so as to minimize the energy. The approximations most necessary to the theory are the use of screened Coulomb and screened second-order phonon interactions according to the scheme of Bardeen and Pines[2]; and the neglect of all interactions except those between pairs with zero total momentum and spin.

Bogoliubov[3] arrived at practically the same result by an apparently different method, using from B.C.S. only the zero-momentum pairing idea. He formed pairs by introducing a new set of fermions, composed partly of an electron with ($\mathbf{k}$, spin up) and partly of a hole with ($-\mathbf{k}$, spin down):

$$\begin{aligned}\alpha_{\mathbf{k}0}&=u_kc_{\mathbf{k}\uparrow}-v_kc_{-\mathbf{k}\downarrow}^{*},\\ \alpha_{\mathbf{k}1}&=u_kc_{-\mathbf{k}\downarrow}+v_kc_{\mathbf{k}\uparrow}^{*},\end{aligned} \qquad (2)$$

and defined the ground state as the "vacuum" with

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[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957). We abbreviate this reference B.C.S. hereafter.

[2] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).

[3] N. N. Bogoliubov, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 65 (1958)[translation: Soviet Phys. JETP **34(7)**, 41 (1958)]; J. G. Valatin, Nuovo cimento **7**, 843 (1958).

respect to the new set of "particles":

$$\alpha_{k0}^*\alpha_{k0}\Psi_g=\alpha_{k1}^*\alpha_{k1}\Psi_g=0. \quad (3)$$

By this trick he was able to eliminate certain terms in perturbation theory which diverged, essentially because of the presence of the attractive interaction which is responsible for the binding of pairs.

It is easy to show that the B.C.S. state (1) and the Bogoliubov state (3) are identical if

$$v_k=h_k^{\frac{1}{2}}, \quad u_k=(1-h_k)^{\frac{1}{2}}. \quad (4)$$

Less obviously, the basic assumptions of the two theories are actually very close, as are the "energy gap" spectra of elementary excitations.[4] Bogoliubov treats the phonons only in second order, and his limitations to the lowest-order energy and to the "most divergent terms" (meaning those which cause binding of Cooper pairs[5]) are equivalent to the B.C.S. reduced-Hamiltonian assumption.

It was demonstrated[6] that the most serious question in regard to these theories is that the sum rules and gauge invariance are not obeyed, so that a consistent explanation of the Meissner effect requires, at the very least, that the whole interaction Hamiltonian be taken into account. If this is done and the long-range Coulomb forces are neglected, a new set of states of nonzero momentum must be present in the energy gap, which not only would have experimental effects but might well lead to large perturbations in the ground state. When long-range Coulomb forces are included, the gap again becomes empty, and one is led to the conclusion that the success of the zero-momentum pair theories may only be a consequence of plasma effects.

Both for this reason, and because it seems optimistic to assume that the collective and screening effects (which are vital even in determining the phonon spectrum) will be necessarily unaffected by the radical changes in the Fermi sea embodied in (1) or (3) (and vice versa), it is desirable to have a theory of the ground state of a superconductor which can simultaneously handle these collective effects in the best available approximation, that of Gell-Mann, Brueckner, Sawada, and Brout,[7,8] and yet lead to (1) or (3) and the Bardeen energy-gap excitation spectrum. Such a theory is the subject of this paper.

This theory also has a few by-products which commend it as an alternative to the earlier ones. First, it shows in a natural way why the restriction to a fixed number of electrons must be relaxed and how to handle the projection back onto $N=\text{const}$; second, it is capable of computing the correlation corrections to superconductivity—or vice versa—and showing that they are small; third, it gives a good account of other collective effects such as phonons and higher bound pair states; and finally, it seems to give a simpler and more physical picture of the nature of the superconducting state. The method we use may also have more general interest as an approach to the many-body problem, in particular in reconciling collective and individual-particle behavior.

The basic, and almost the only, assumption is a generalized form of the random-phase approximation.[9] Sawada, Sawada *et al.*, and Brout[8] have shown that the R.P.A. of Bohm and Pines and the diagram-summing method of Brueckner and Gell-Mann[7] both lead to a certain set of linear eigenvalue equations or "equations of motion" for the elementary excitations in terms of the quantities

$$\rho_{k,\sigma}^{Q}=c_{k+Q,\sigma}^*c_{k,\sigma}. \quad (5)$$

Sawada and Brout[8] showed that the Gell-Mann and Brueckner "high-density" assumption, that excited particles interact with the unperturbed Fermi sea only, is equivalent to a certain effective Hamiltonian together with altered commutation relations for the ρ's of (5), which then lead to the equations of motion; they then showed how to derive the energy and other results from these equations. The same equations were arrived at—without realizing how nearly full a solution to the correlation problem they were—by Bohm and Pines,[9] first by physical reasoning and then by a method of direct linearization of the full equations of motion. The method of this paper is a natural generalization of this last way of arriving at the "equations of motion" to the case in which the unperturbed state is not the Fermi sea but the B.C.S. state (1).

The high-density ("weak coupling") limit is the only proven domain of validity of the R.P.A., and even in that limit a consistent theory of superconductivity is interesting. However, the R.P.A. gives results in the correlation problem which appear to be satisfactory even in the intermediate range,[10] and other entirely different domains.[11] A full discussion of the domains of validity of these methods is, however, beyond the scope of this paper; in any case they represent the only approach known to be effective in studying collective effects in the many-body problem.

Briefly, the Bohm-Pines technique linearizes the full equations of motion by observing that in the Fermi sea the quantities $n_k=c_k^*c_k$ and $1-n_k=c_kc_k^*$ may have finite "c-number" average values. These average values

[4] N. N. Bogoliubov, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 73 (1958) [translation: Soviet Phys. JETP **34(7)**, 51 (1958)].
[5] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).
[6] P. W. Anderson, Phys. Rev. **110**, 827 (1958).
[7] K. A. Brueckner and M. Gell-Mann, Phys. Rev. **106**, 364 (1957).
[8] K. Sawada, Phys. Rev. **106**, 372 (1957); Sawada, Brueckner, Fukuda, and Brout, Phys. Rev. **108**, 507 (1957); see also J. Hubbard, Proc. Roy. Soc. (London) **A243**, 336 (1958).
[9] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953). We shall abbreviate "random-phase approximation" as R.P.A. hereafter.
[10] When supplemented by a treatment of exchange terms which falls rather naturally out of the more general method of the present paper. See P. Nozières and D. Pines, Phys. Rev. **109**, 1009 (1958), and Nuovo cimento (to be published); and especially Hubbard, reference 8.
[11] E. Montroll and J. C. Ward, J. Phys. Fluids **1**, 55 (1958).

are taken as zeroth order, while the $\rho_k^Q = c_{k+Q}^* c_k$ act like first-order infinitesimals. In the B.C.S. state (1) not only n_k but $b_k^* = c_k^* c_{-k}^*$ and b_k have c-number averages, so that correspondingly we must derive equations of motion for first-order quantities $c_{k+Q}^* c_{-k}^* = \bar{b}_k^Q$, etc. The unperturbed B.C.S. state is itself determined, in our method, by finding equations of motion for the b_k^*'s and n_k's themselves and taking their stable solution; this is then equivalent to both the B.C.S. and Bogoliubov definitions. Physically it means that we construct the zeroth-order state so as to be stable against the formation of any more zero-momentum bound electron pairs.

The most direct results of the theory are the solutions of the equations of motion. These are the elementary excitations, and they fall into two groups: individual-particle-like excitations, the spectrum of which is practically the same as the B.C.S. energy-gap spectrum, but which include many of the effects of scattering; and collective solutions.

The collective solutions are calculated for two cases. The usual models implicitly ignore the long-range Coulomb forces; we may call this the "neutral" case. In this case there is, strictly speaking, no gap: we find a collective excitation of longitudinal type, which has resemblances to a longitudinal wave as well as to a bound pair of electrons of nonzero momentum, with a velocity

$$v = v_F\{\tfrac{1}{3}[1 - 4N(0)|V|]\}^{\frac{1}{2}}.$$

The velocity $3^{-\frac{1}{2}}v_F$ was obtained by Bogoliubov[12] and is a kinematical effect; the $N(0)|V|$ term represents the effect of interaction in this weak-coupling limit. Other collective excitations, describable physically as other bound-pair excitations orthogonal to the Cooper bound pairs, may or may not exist depending on the form of the interaction, but lie near or above the top of the gap in any case.

In the physical case of the charged Fermi gas, the longitudinal excitations have a spectrum identical with the plasmons of the normal Fermi gas, and thus the gap really exists; we see that the gap in a strict sense is enforced only by the long-range Coulomb effects. Fundamentally, also, in this case the R.P.A., with its automatic separation of the equations of excitations of different total momentum, may be expected to be more accurate, because these long-range forces single out momentum zero as having special properties. In an appendix we go on to calculate the effect of superconductivity on the long-wave phonons, and show that their spectrum is changed only to the order of the ratio of electron to ion mass.

[12] Bogoliubov, Tolmachev, and Shirkov, *New Method in the Theory of Superconductivity*, (Academy of Sciences of the U.S.S.R., Moscow, 1958). See also V. M. Galitskii, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 1011 (1958) [translation: Soviet Phys. JETP **34** (7), 698 (1958)].

II. DISCUSSION OF THE INTERACTION HAMILTONIAN

If the interaction between electrons is given, the only approximation of the method is the R.P.A. The interaction responsible for superconductivity is, however, the rather complicated interaction through the lattice phonons, and fundamentally the calculation of this interaction is outside the scope of the R.P.A. In this paper we shall assume the essential correctness of the results of Bardeen and Pines for this interaction.[13]

In a later paper this assumption will be justified by means of the renormalization methods of Hubbard.[14] Although this justification is not a part of the present paper, in view of its importance a brief qualitative description will be given here. Hubbard shows, in the free electron gas problem, that the effect of certain apparently divergent terms of perturbation theory is to replace each Coulomb interaction between electrons by a new, "effective" interaction, which may be thought of as having been modified and screened by the frequency- and wave-number-dependent dielectric constant of the electron gas.[15]

The effect of superconductivity, like that of the Coulomb correlations, can be expected to be merely a "smearing" of the properties of the surface of the Fermi sea, while (as the present paper will show) there is no serious effect on the collective modes, which primarily determine the dielectric constant in the long-wave region where the smearing might be important. Thus in interactions of secondary importance to the R.P.A., like exchange and phonon exchange, it is a good approximation to screen using a dielectric constant computed according to the unperturbed R.P.A. (including of course the phonon contribution[16]). The direct Coulomb interactions, which have the major effect on the collective modes and thus on the dielectric constant, must on the other hand be left in explicitly in deriving the excitation modes.

The procedure is thus a kind of successive approximation method, checked by demonstrating its self-consistency. In stage (1), we imagine that we have calculated the dielectric constant of the free electron gas with phonons, using only the direct interactions in the R.P.A. In stage (2), we recalculate the properties of the free electron gas including the phonon and exchange terms. These must be screened by the stage (1) dielectric constant because the corrections in stage (1) are large (in fact formally divergent). Stage (2) is the present theory of superconductivity, but also includes Hubbard's[10] method for the second-order exchange correction. In stage (3) we might recalculate the dielectric constant from stage (2) and insert it in the

[13] See Bardeen and Pines, reference 2. See also D. Pines, Phys. Rev. **109**, 280 (1958), for a probably more accurate expression.

[14] J. Hubbard, Proc. Roy. Soc. (London) **A240**, 539 (1957).

[15] See also Nozières and Pines, reference 10.

[16] See G. Wentzel, Phys. Rev. **108**, 1593 (1957) for the calculation of phonons by the method of Sawada and Brout.

screened terms; however, this correction is small and stage (3) is unnecessary. The resulting interaction is similar to that of reference 2, although it must be slightly complex to allow for real inelastic scattering; the screening is not by a cutoff as in Bardeen and Pines but smoothly decreasing, and the subsidiary condition is irrelevant to the present scheme.

We shall, then, write down as our Hamiltonian the Bardeen-Pines result. This already implies the relatively minor (at this stage) assumptions of the free-electron gas model for Bloch electrons and of the neglect of transverse phonons, as well as of the residual, real scattering interaction of electrons and phonons. It is

$$\mathcal{H}=\mathcal{H}_K+\mathcal{H}_V+\mathcal{H}_C. \tag{6}$$

Here ($\hbar=1$)

$$\mathcal{H}_K=\sum_{\mathbf{k},\sigma}\epsilon_k c_{\mathbf{k},\sigma}{}^* c_{\mathbf{k},\sigma}, \tag{7}$$

$$\mathcal{H}_V=-\tfrac{1}{2}\sum_{\mathbf{k}\neq\mathbf{k}',\mathbf{q}}\sum_{\sigma,\sigma'}\frac{\omega_{\mathbf{k}-\mathbf{k}'}M_{\mathbf{k}-\mathbf{k}'}{}^2}{(\omega_{\mathbf{k}-\mathbf{k}'})^2-(\epsilon_k-\epsilon_{k'})^2}\times c_{\mathbf{k}',\sigma'}{}^* c_{-\mathbf{k}'+\mathbf{q},\sigma}{}^* c_{-\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma'}. \tag{8}$$

$\mathcal{H}_C$ is a somewhat more complicated thing, because (according to the above discussion) we must understand its terms differently depending on whether we use them as direct or exchange terms. We shall write down the full Hamiltonian and simply understand that the exchange terms are to be screened:

$$\mathcal{H}_C=\sum_{\mathbf{k}\neq\mathbf{k}',\mathbf{q}}\sum_{\sigma,\sigma'}2\pi e^2(\mathbf{k}-\mathbf{k}')^{-2}\times c_{\mathbf{k}',\sigma'}{}^* c_{-\mathbf{k}'+\mathbf{q},\sigma}{}^* c_{-\mathbf{k}'+\mathbf{q},\sigma} c_{\mathbf{k},\sigma'}. \tag{9}$$

In (8), Bardeen and Pines give

$$M_k=v_k{}^i(1+6\pi ne^2/k^2\epsilon_F)^{-1}, \tag{10}$$

where $v_k{}^i$ is the true electron-phonon interaction without screening, approximately

$$v_k{}^i\cong-2\pi Ze^2ik^{-1}(n/M)^{\frac{1}{2}}. \tag{11}$$

The actual values of these constants will be of little further importance to us except in Appendix I.

In principle the description of our method implies that the direct *phonon* interactions should also be included, so that we calculate correctly both collective modes, the phonon and the plasmon. To avoid complication we do not do this in the main paper, but only in an appendix. In that case for direct interactions one must include the following two terms in the Hamiltonian:

$$\mathcal{H}_{\rm ph}=\sum_{\mathbf{s}}\tfrac{1}{2}(p_{\mathbf{s}}p_{-\mathbf{s}}+f_s{}^2q_{\mathbf{s}}q_{-\mathbf{s}}), \tag{12}$$

and

$$\mathcal{H}_i=-\sum_{\mathbf{s}}(q_{\mathbf{s}}\rho^{-\mathbf{s}}v_s{}^i+q_{-\mathbf{s}}\rho^{\mathbf{s}}v_s{}^{i*}), \tag{13}$$

where p and q are the phonon coordinates, $\rho^{\mathbf{s}}$ is the $\mathbf{s}$ Fourier component of electron density fluctuation, and

$$f_s{}^2=4\pi ne^2Z^2M^{-1}+\omega_s{}^2,\quad \omega_s\to 0 \text{ as } s\to 0. \tag{14}$$

(ω_s includes any interactions between the ion cores.)

One further point must be made with regard to (8). An instantaneous space interaction is a function of $\mathbf{k}-\mathbf{k}'$ only; but the phonon (and actually even the screened Coulomb) interaction is not instantaneous, and must in some sense depend on the time difference between the creation and destruction of the longitudinal photon or phonon, or in Fourier-analyzed form on a frequency variable.

Our approach is to understand the particles as being embedded in a medium with a certain frequency- and wavelength-dependent dielectric constant. The frequency which enters is the energy difference between the initial state and the intermediate state of the particle system "after" the phonon-photon is emitted. This is usually well enough approximated by $\epsilon_k-\epsilon_{k'}$ in (8). As we shall see in Sec. IV, the important thing for gauge invariance and the sum rules is that there be no q dependence in (8), which is ensured by the fact that the initial and final states always have the same energy, so that the difference from the intermediate state, whatever it be, is fixed. We shall take advantage of this fact by discarding any apparent q dependence wherever it may appear.

Thus our equations will always automatically satisfy the basic sum rule $[\mathcal{H}_V,\rho^{\mathbf{Q}}]=0$, which ensures that the usual perturbation theory will give gauge-invariant results.[6]

III. IMPROVED TREATMENT OF THE B.C.S. REDUCED HAMILTONIAN

The R.P.A. theory which we use might be thought of as a generalization of the Sawada-Brout[8] theory to the superconductor. It also follows naturally as a generalization of a certain slightly improved description of the B.C.S. theory of the "reduced" Hamiltonian. There exist in the literature elegant treatments of the Sawada-Brout theory[8,17]; since the alternative calculation of B.C.S. is new, and is of some interest in itself, we present it here in full.[18]

The first step of the B.C.S. theory is to neglect in (8) and (9) all terms involving σ and σ' parallel, as well as all terms with $q\neq 0$. This step may be justified in terms of the worst instability of the Fermi sea being caused by binding of zero-momentum pairs, and the considerations of reference 6 reinforce this point; but since we intend to relax this assumption later no extensive discussion is necessary. Then the Hamiltonian

[17] See Wentzel, reference 16.

[18] A similar treatment has recently appeared in N. N. Bogoliubov, J. Exptl. Theoret. Phys. U.S.S.R. 34, 73 (1958) [translation: Soviet Phys. JETP 34(7), 51 (1958)], although different in detail and interpretation.

becomes

$$\mathcal{H}_{\mathrm{RED}}=\sum_{\mathbf{k}}\epsilon_k(n_{\mathbf{k}}+n_{-\mathbf{k}})-\sum_{\mathbf{k}\neq\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}c_{\mathbf{k}'}{}^*c_{-\mathbf{k}'}{}^*c_{-\mathbf{k}}c_{\mathbf{k}}, \quad (15)$$

where we have used the convention that an explicitly negative k has a down spin and vice versa. (This convention will be used hereafter in this paper.) $V_{\mathbf{k}\mathbf{k}'}$ is the resultant interaction obtained by subtracting the screened Coulomb exchange matrix element from the phonon exchange:

$$V_{\mathbf{k}\mathbf{k}'}=\frac{\omega_{\mathbf{k}-\mathbf{k}'}M_{\mathbf{k}-\mathbf{k}'}{}^2}{(\omega_{\mathbf{k}-\mathbf{k}'})^2-(\epsilon_k-\epsilon_{k'})^2}-\frac{4\pi e^2}{(\mathbf{k}-\mathbf{k}')^2}\times(\text{screening factor}). \quad (16)$$

Basic to the whole theory of superconductivity is the idea that, for small enough $(\epsilon_k-\epsilon_{k'})^2$, $V_{\mathbf{k}\mathbf{k}'}$ is positive on the average. Pines[19] has discussed qualitatively whether and when this may be true.

It is convenient to use as the zero of energy that of a particular Fermi sea of N^0 electrons, Fermi level $\epsilon_F{}^0$; and to sum only over k's with $\epsilon_k<2\epsilon_F{}^0$. Then (15) becomes

$$\mathcal{H}_{\mathrm{RED}}=-\sum_{\mathbf{k}}(\epsilon_k-\epsilon_F{}^0)(1-n_{\mathbf{k}}-n_{-\mathbf{k}})-\sum_{\mathbf{k}\neq\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}c_{\mathbf{k}'}{}^*c_{-\mathbf{k}'}{}^*c_{-\mathbf{k}}c_{\mathbf{k}}+\epsilon_F{}^0(N-N^0), \quad (17)$$

where

$$N=\textstyle\sum_{\mathbf{k}}(n_{\mathbf{k}}+n_{-\mathbf{k}}). \quad (18)$$

B.C.S. pointed out that the appropriate algebra for dealing with (17) involved the operators

$$1,\quad b_{\mathbf{k}}=c_{-\mathbf{k}}c_{\mathbf{k}},\quad b_{\mathbf{k}}{}^*=c_{\mathbf{k}}{}^*c_{-\mathbf{k}}{}^*,\quad\text{and}\quad 1-(n_{\mathbf{k}}+n_{-\mathbf{k}}). \quad (19)$$

Their importance lies in the fact that in the subspace defined by

$$n_{\mathbf{k}}-n_{-\mathbf{k}}\equiv 0 \quad (20)$$

they are a complete set, while it is easy to show that the lowest eigenstate of (17) is in this subspace.

The properties of these operators become clear by writing them [in the subspace (20)] in the representation in which the basis functions are ($\mathbf{k}$ and $-\mathbf{k}$ empty) and ($\mathbf{k}$ and $-\mathbf{k}$ full). The operators (19) commute for different k, so only the $\mathbf{k}$, $-\mathbf{k}$ subspace need be written down:

$$1-n_{\mathbf{k}}-n_{-\mathbf{k}}=\begin{array}{c}\\ \text{empty}\\ \text{full}\end{array}\begin{array}{c}\begin{array}{cc}\text{empty} & \text{full}\end{array}\\ \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{array},$$
$$1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad b_{\mathbf{k}}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad b_{\mathbf{k}}{}^*=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \quad (21)$$

The further manipulation will be much clearer using the following set of Pauli spin matrices which are fully equivalent to the b's:

$$2s_{z\mathbf{k}}=1-n_{\mathbf{k}}-n_{-\mathbf{k}},\qquad s_{x\mathbf{k}}+is_{y\mathbf{k}}=b_{\mathbf{k}}{}^*,\quad s_{x\mathbf{k}}-is_{y\mathbf{k}}=b_{\mathbf{k}}. \quad (22)$$

These s operators are not to be confused with real physical spin-operators; they act in an imaginary space, where z component of spin up means "empty," spin down means "full," and spin sidewise simply implies a certain phased linear combination of up and down.

In terms of the pseudospins (22), the Hamiltonian is

$$\mathcal{H}_{\mathrm{RED}}=-2\sum_{\mathbf{k}}(\epsilon_k-\epsilon_F{}^0)s_{z\mathbf{k}}-2\epsilon_F{}^0\sum_{\mathbf{k}}s_{z\mathbf{k}}-\sum_{\mathbf{k}\neq\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}(s_{x\mathbf{k}}s_{x\mathbf{k}'}+s_{y\mathbf{k}}s_{y\mathbf{k}'}). \quad (23)$$

The second term on the right changes only when the total number of electrons changes, and may be ignored.

In the theory of magnetism such a spin problem is attacked by the so-called "semiclassical method,"[20] which is actually a perfectly well defined quantum-mechanical approximation scheme. The similarity of this scheme to the "intermediate coupling" methods of field theory is not widely appreciated but has been mentioned by Gross.[21] Later in this section we shall discuss the scheme from a fully quantum-mechanical point of view, but in the meantime we shall describe it more or less from the semiclassical viewpoint. The first approximation is to take the spin vectors $\mathbf{s}_{\mathbf{k}}$ and rotate them into the best possible classical arrangement, i.e., parallel to the field acting upon them. This, we shall see, is the same as taking the optimum product wave function. In the next approximation one finds the small oscillations about this classical equilibrium and quantizes them; then the ground-state energy and wave function are corrected for the zero-point motion of the small oscillations, the basic assumption being that these are actually small.

The field $\mathbf{H}_{\mathbf{k}}$ which $\mathbf{s}_{\mathbf{k}}$ sees is, from (23),

$$\mathbf{H}_{\mathbf{k}}=2(\epsilon_k-\epsilon_F{}^0)\hat{z}+2\textstyle\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\mathbf{s}_{\perp\mathbf{k}'}, \quad (24)$$

where $\mathbf{s}_{\perp\mathbf{k}}$ is that portion of s perpendicular to $\hat{z}$.

In the unperturbed Fermi sea, only the z component is present, and each spin is either up or down, with a sharp break at ϵ_F [see Fig. 1(a)]. It is easy to see that because of the small fields at ϵ_F, only a small V is necessary to turn a few spins sidewise and make the configuration of Fig. 1(b) more stable: a "domain wall" in k space with states rotating smoothly from "full" to "empty."

This configuration is determined in terms of the angle θ_k between the new direction of the spin $\mathbf{k}$ and the z axis by $\mathbf{H}_{\mathbf{k}}\|\mathbf{s}_{\mathbf{k}}$:

$$s_{x\mathbf{k}}/s_{z\mathbf{k}}=\tan\theta_k=\tfrac{1}{2}(\epsilon_k-\epsilon_F{}^0)^{-1}\textstyle\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\sin\theta_{k'}. \quad (25)$$

[19] D. Pines, reference 13.

[20] G. Heller and H. A. Kramers, Proc. Acad. Sci. Amsterdam **37**, 378 (1934); M. J. Klein and R. S. Smith, Phys. Rev. **80**, 1111 (1951); P. W. Anderson, Phys. Rev. **86**, 694 (1952).

[21] E. P. Gross, Phys. Rev. **100**, 1571 (1955).

This may easily be shown to be the same as the B.C.S. integral equation; the correspondence is

$$\sin\theta_k = 2[h_k(1-h_k)]^{\frac{1}{2}}. \tag{26}$$

It is also obvious that the product wave function of B.C.S. is of the same form as our wave function: a product of rotated factors, one for each k vector.

It is also interesting to make contact with the Bogoliubov theory. To do this we consider the spin components in the new direction:

$$\begin{aligned} s_{z'\mathbf{k}} &= \cos\theta_k s_{z\mathbf{k}} + \sin\theta_k s_{x\mathbf{k}}, \\ s_{x'\mathbf{k}} &= -\sin\theta_k s_{z\mathbf{k}} + \cos\theta_k s_{x\mathbf{k}}. \end{aligned} \tag{27}$$

On the other hand, if we set

$$2s_{z'\mathbf{k}} = 1 - (\alpha_{\mathbf{k}0}^*\alpha_{\mathbf{k}0} + \alpha_{\mathbf{k}1}^*\alpha_{\mathbf{k}1}), \tag{28}$$

in correspondence with the definition of $s_{z\mathbf{k}}$, and use (2), we get

$$2s_{z'\mathbf{k}} = (u_k^2 - v_k^2)(1 - n_{\mathbf{k}} - n_{-\mathbf{k}}) + 2u_k v_k (b_{\mathbf{k}}^* + b_{\mathbf{k}}).$$

This is the same as (27) if

$$\cos\theta_k = u_k^2 - v_k^2, \quad \sin\theta_k = 2u_k v_k. \tag{29}$$

Thus the first approximation gives exactly the B.C.S.-Bogoliubov results. The first approximation to the excitation spectrum is obtained by taking the energy to turn over the "spins" in the effective fields H_k:

$$2E_k = |H_k| = 2[(\epsilon_k - \epsilon_F^0)^2 + \tfrac{1}{4}(\textstyle\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \sin\theta_{k'})^2]^{\frac{1}{2}}, \tag{30}$$

which is the energy of excitation of "real pairs" in the B.C.S. theory.[22]

An improvement on the B.C.S. theory comes when we study the true excitation spectrum modified by the interaction between "spins." First, however, we observe that this improvement can be expected to change the result *for* $\mathcal{H}_{\text{RED}}$ *only* to order only $1/N$. The reason is that the field $\mathbf{H}_{\mathbf{k}}$, insofar as it involves the other spins, is a sum over a number of the order of N other spins. Thus we expect the quantum fluctuations to average out, and the semiclassical theory to be nearly valid, in contrast with the theories of reference 20.

In order to study the modified excitations we must write down the equations of motion. This is most simply done by observing that since the $\mathbf{s}_{\mathbf{k}}$'s obey the usual spin commutation relations,

$$\mathbf{s}\times\mathbf{s} = i\mathbf{s}, \tag{31}$$

the usual spin equations of motion are valid:

$$[\mathcal{H}, \mathbf{s}_{\mathbf{k}}] = i(d\mathbf{s}_{\mathbf{k}}/dt) = i[\mathbf{H}_{\mathbf{k}}\times\mathbf{s}_{\mathbf{k}}]. \tag{32}$$

Now we allow each spin $\mathbf{s}_{\mathbf{k}}$ to have, besides its static component $\mathbf{s}_{\mathbf{k}}^0$, an increment $\delta\mathbf{s}_{\mathbf{k}}$. Then (32) is

$$\delta(d\mathbf{s}_{\mathbf{k}}/dt) = [\mathbf{H}_k^0\times\delta\mathbf{s}_{\mathbf{k}}] + [\delta\mathbf{H}_{\mathbf{k}}\times\mathbf{s}_k^0] + [\delta\mathbf{H}_{\mathbf{k}}\times\delta\mathbf{s}_{\mathbf{k}}], \tag{33}$$

and the last term is neglected as nonlinear. Here

$$\delta\mathbf{H}_{\mathbf{k}} = 2\textstyle\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\delta\mathbf{s}_{\perp\mathbf{k}'}. \tag{34}$$

Let $\mathbf{s}_k^0$ and $\mathbf{H}_k^0$ be in the x-z plane; the components of $\delta\mathbf{s}$ will be δs_y and $\delta s_{||}$, the latter meaning the component perpendicular to $\mathbf{s}^0$ in the x-z plane. Then (33) is

$$\begin{aligned} \delta(ds_{\mathbf{k}y}/dt) &= H_k^0\delta s_{\mathbf{k}||} - \tfrac{1}{2}\delta H_{\mathbf{k}x}\cos\theta_k, \\ \delta(ds_{\mathbf{k}||}/dt) &= -H_k^0\delta s_{\mathbf{k}y} + \tfrac{1}{2}\delta H_{\mathbf{k}y}. \end{aligned} \tag{35}$$

Equations (35) are easily solved only if we make the B.C.S. assumption that $V_{\mathbf{k}\mathbf{k}'}$ is a constant over a region in k space, and otherwise zero. That case has been worked out in detail by Suhl.[23] Using Suhl's solution as a guide, one can see some rather general properties of (33) or (35).

The unperturbed, "individual particle" solutions result from neglecting the $\delta\mathbf{H}_{\mathbf{k}}$ term. Then $\delta\mathbf{s}_{\mathbf{k}}$ simply precesses about $\mathbf{H}_k^0$ at the frequency H_k^0, which is the energy of the "real pair" excitation. The great majority of excitations are of approximately this form, because, from (34), if only a few $\delta\mathbf{s}_{\mathbf{k}}$ are large ($V_{\mathbf{k}\mathbf{k}'}$ is of order N^{-1}), $\delta\mathbf{H}_{\mathbf{k}}$ is indeed of order N^{-1} and may be neglected. Actually, there is a solution of (33) between every pair of unperturbed solutions, for which $\delta\mathbf{s}$ for some few particular k's is of order N larger than all other $\delta\mathbf{s}$'s.

Collective solutions may be defined as solutions for which the sum (34) is replaced by an integral, which is understood in a principal part sense; they will usually lie outside the unperturbed spectrum. A collective solution always comes at the frequency $\nu = 0$. Such a

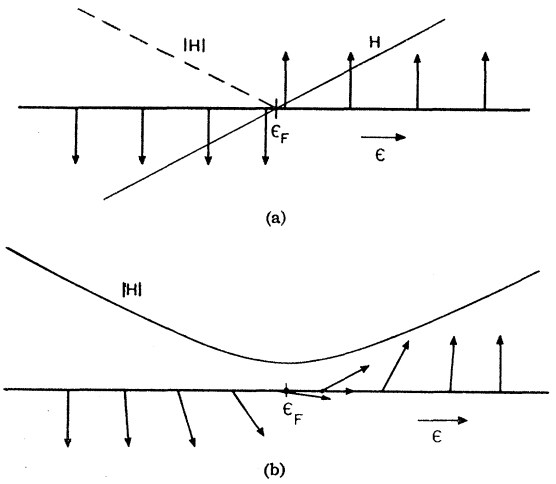

FIG. 1. Configurations in the pseudospin analogy to the electron gas. (a) Normal Fermi sea. Spin arrows up represent empty states. Also given are the effective field H_k acting on the spins, and its absolute value. (b) The superconducting ground state, showing the gradual rotation, like a domain wall, of the pseudospin vectors.

[22] Bogoliubov's fermions give a more complete excitation spectrum for practical calculations. It is easily shown that in this approximation $[\mathcal{H}_{\text{RED}}, \alpha_{k0}^*] = \frac{1}{2}H_k\alpha_{k0}^*$, etc. The α's are the "singles" of the B.C.S. theory, and all thermal etc. effects may be calculated using them alone.

[23] H. Suhl (private communication).

solution has, from (35),

$$2\delta s_{\mathbf{k}y}=\delta H_{\mathbf{k}y}(H_k^0)^{-1}, \quad \delta s_{\mathbf{k}||}=0. \tag{36}$$

To check this we must be sure (36) is compatible with (34). To do this we try

$$2\delta s_{\mathbf{k}y}=\epsilon \sin\theta_k. \tag{37}$$

Then (34) becomes

$$\delta H_{\mathbf{k}y}=2\epsilon \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \sin\theta_{k'}, \tag{38}$$

or

$$2H_k^0\epsilon \sin\theta_k=4\epsilon(\epsilon_k-\epsilon_F^0)\tan\theta_k,$$

by (25). Use of the value of H_k^0 checks (36).

With the B.C.S. assumption that $V_{\mathbf{k}\mathbf{k}'}$ is a constant, there are no other collective solutions of any interest. More realistic V's may have other solutions, but these will not lie low in the gap, at least in the weak-coupling case in which V varies much only over an energy range large compared with the gap, and thus with the energy range of variation of the wave function. The demonstration of these statements follows.

It is a good approximation to assume the problem symmetrical about ϵ_F, and then $\delta s_{\mathbf{k}y}$ is either an even or an odd function of $\epsilon_k-\epsilon_F$. If $\delta s_{\mathbf{k}y}$ is even, $\delta H_{\mathbf{k}y}$ is finite and even, the second equation of (35) makes $\delta s_{\mathbf{k}||}$ even, thus $\delta s_{\mathbf{k}x}$ odd and $\delta H_{\mathbf{k}x}=0$. Similarly the odd solutions have $\delta H_{\mathbf{k}y}=0$.

Two types of solutions may occur: angle-dependent solutions, physically like bound pairs of p, d, etc., symmetry; or s-like solutions orthogonal to the $\nu=0$ one. The lowest of the latter is necessarily the lowest odd solution; because of (35) and (34) these obey the equation

$$\delta s_{\mathbf{k}y}=[(H_k^0)^2-\nu^2]^{-1}\cos\theta_k \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}H_{k'z}^0\delta s_{\mathbf{k}'y}.$$

When V is roughly constant this is equivalent to

$$1=\sum_{\mathbf{k}} \frac{V_{\mathbf{k}\mathbf{k}'}H_k^0}{(H_k^0)^2-\nu^2}\cos^2\theta_k,$$

which may be quickly verified to have its lowest solution precisely at the top of the gap.

We may expect the lowest angle-dependent solution to be an even one. The fundamental equation for the even solutions is

$$[(H_k^0)^2-\nu^2]\delta s_{\mathbf{k}y}=\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\delta s_{\mathbf{k}'y}.$$

If V does not vary rapidly with angle the sum on the right will be quite small, and ν must approach H_k^0 closely to allow a solution.

It is physically obvious (also from the discussion of reference 6) that any such solutions which actually occur in the gap are simply bound pairs of Cooper type in excited states. We shall discuss the corresponding $Q\neq 0$ excitations in a later section.

The $\nu=0$ solution could have been expected from the first, and serves a very useful and important purpose. Equations (36) and (37) show that it represents a uniform rotation of the whole "wall" about the pseudo-z axis. Now the original $\mathcal{H}_{\text{RED}}$ [Eq. (23)] was axially symmetric about z. The reason for this symmetry is of course that

$$S_z^{\text{tot}}=N^0-N, \tag{39}$$

must be a constant of the motion, as particles are not really created or destroyed.

On the other hand, the product solution in the semi-classical, "naive" approximation does not have a definite value of S_z^{tot}, because $s_{z\mathbf{k}}$ does not commute with the other components of $\mathbf{s}_{\mathbf{k}}$, which are assumed constants of the motion. Correspondingly, the product solution is not axially symmetric but picks a particular direction in the x-y plane, which must be unphysical. The $\nu=0$ mode is the free rotation of this solution about the z axis. We can expect that proper inclusion of the zero-point motion of the $\nu=0$ mode will repair this situation by projecting our solution onto the space of $S_z^{\text{tot}}=$ constant. In fact this can be verified using the B.C.S. representation. An eigenfunction of the $\nu=0$ rotation about the z axis is just

$$\int d\varphi\, \Psi_{\text{B.C.S.}}(\varphi)\exp(in\varphi), \tag{40}$$

where $\Psi_{\text{B.C.S.}}(\varphi)$ is the B.C.S. solution rotated to a new direction φ in the x-y plane. It can be shown that (40) is

$$\Psi_n=\int d\varphi\, e^{in\varphi}\prod_{\mathbf{k}}[(1-h_k)^{\frac{1}{2}}e^{-i\varphi/2}+h_k^{\frac{1}{2}}e^{i\varphi/2}b_{\mathbf{k}}^*]\Psi_v, \tag{41}$$

which quite clearly is just such a projection.

A few final remarks will close this section. First, the correction to the energy could be calculated,

$$\Delta E=\tfrac{1}{2}\sum \nu-\tfrac{1}{2}\sum H_k^0, \tag{42}$$

just the difference of the perturbed and unperturbed zero-point energies (a special case of a relationship we will prove later). The largest part of (42) will be an amount $-H_k^0$ from the mode $\nu=0$; but the whole correction is only of order N^{-1} relative to the total energy so need not be calculated.

Second, note the existence of two formal solutions of the equations of motion:

$$[\mathcal{H}_{\text{RED}}, n_{\mathbf{k}}-n_{-\mathbf{k}}]\equiv 0, \quad [\mathcal{H}_{\text{RED}},\delta s_{z'\mathbf{k}}]=0, \tag{43}$$

where z' is the direction along which $\mathbf{s}_k^0$ points. The assumed ground state satisfies

$$\begin{aligned}(n_{\mathbf{k}}-n_{-\mathbf{k}})\Psi_0&\equiv 0,\\ \delta s_{z'\mathbf{k}}\Psi_0&\equiv 0 \quad (\text{i.e., } \delta\mathbf{s}_{\mathbf{k}}\perp\mathbf{s}_k^0),\end{aligned} \tag{44}$$

and (43) assures that these conditions remain satisfied throughout the zero-point motion.

These conditions are the conditions that essentially

nonphysical excited states do not enter the problem. Our calculation can be thought of in the following way[17]: starting from the exact ground state Ψ_0^e, we make all possible zero-momentum excitations of pairs of electrons, such as $\delta b_k \Psi_0^e$. Certain linear combinations

$$x^\nu = \sum_k (\alpha_k \delta n_k + \beta_k \delta n_{-k} + \gamma_k \delta b_k + \delta_k \delta b_k^*),$$

satisfy commutation rules

$$[\mathcal{H}, x^\nu] = \nu x^\nu,$$

so that

$$\mathcal{H}(x^\nu \Psi_0^e) = (\nu + E_0)(x^\nu \Psi_0^e),$$

and our attempt is to calculate the ν and the x^ν approximately by assuming the commutators to have the values appropriate to some zeroth order Ψ_0^0. There are, however, certain pair excitations which are not possible from our unperturbed Ψ_0^0: we cannot destroy electrons in states which are empty, or create them in full states. The procedure is consistent only if the equations of motion are compatible with these restrictions, i.e., if the conditions (44) are indeed solutions of the equations, as we see from (43). We will show in the next section that this is more generally true.

This point of view may be clearer if we see that (44) may be expressed in Bogoliubov notation as

$$\begin{aligned} (\alpha_{k0}^* \alpha_{k0} - \alpha_{k1}^* \alpha_{k1}) \Psi_0 &= 0, \\ (\alpha_{k0}^* \alpha_{k0} + \alpha_{k1}^* \alpha_{k1}) \Psi_0 &= 0. \end{aligned} \tag{43'}$$

A little algebra with (28) and (2) shows that this rather convenient expression is the same as (43).

IV. RANDOM-PHASE APPROXIMATION TREATMENT OF THE FULL HAMILTONIAN: EQUATIONS OF MOTION

What we shall now show is that there is a single approximation scheme, starting from the full Hamiltonian and not paying undue attention to the $q=0$ part, which leads to the equations of the last section as an integral part, while reducing to the usual R.P.A. treatment of correlation energy and of plasmons in appropriate limits. In fact, in the absence of phonon attractive forces it is the same in principle as Hubbard's[10] inclusion of Coulomb exchange.

In this scheme, as in the second approximation of the last section, we calculate equations of motion of quantities which are bilinear in the original fermion operators. We generalize in two ways: we calculate the full equations of motion, by commuting with the full Hamiltonian; and we calculate equations of motion for quantities with momentum $\mathbf{Q}$ as well as momentum zero:

$$\begin{aligned} b_k^Q &= c_{-k-Q} c_k, \quad \bar{b}_k^Q = c_{k+Q}^* c_{-k}^*, \\ \rho_k^Q &= c_{k+Q}^* c_k, \quad \bar{\rho}_k^Q = c_{-k}^* c_{-k-Q}, \end{aligned} \tag{45}$$

as well as b_k, b_k^*, n_k, and n_{-k}. Note that

$$(b_k^Q)^* = \bar{b}_{k+Q}^{-Q}, \quad (\rho_k^Q)^* = \bar{\rho}_{-k}^{-Q};$$

these Hermitian conjugates of our quantities have momentum $-\mathbf{Q}$.

These full equations of motion are of course useless. However, what can be done is to linearize them, as we have done in the preceding section for $\mathcal{H}_{\text{RED}}$, and as Bohm and Pines,[9] and Sawada and Brout,[8] have done for the Coulomb problem. The linearization used in the usual R.P.A. ignores the exchange terms, which are of the form of the terms we discussed in Sec. III; but there is no need to do so, as Hubbard has shown in a slightly different way, so long as the exchange terms are correctly screened. The method of linearization we use is straightforward in the extreme: each term in the interaction part of the equations of motion is a product of four fermions, and thus a product of two bilinear combinations b or ρ (or n) in a number of ways. We assume that the state about which we linearize is a B.C.S.-Bogoliubov product state, so that it may have finite zero-order values of b_k, b_k^*, or $[1-(n_k+n_{-k})]$. Then we keep only terms which contain one of these quantities.

When b and b^* are zero, the resulting equations are those of Bohm and Pines with exchange added.[24] The equations of motion for b and n themselves are just (32) itself. We shall go on to discuss the solutions of various kinds and to show that the individual-particle solutions give just the B.C.S. spectrum, while the longitudinal collective solutions are such as to insure the validity of the sum rules and of gauge invariance, as suggested in reference 6.

The full nonlinear equations of motion are of almost no interest in themselves. We shall write down the one for ρ_K^Q and then show briefly how the various terms of the linear approximation follow from it. Then we shall give the linearized equations of motion for all of the quantities (45).

Let us lump together the Coulomb interaction (9) and the phonon term (8) with a common matrix element $V(\mathbf{k},\mathbf{k}')$ (which is a function of $\mathbf{k}$ and $\mathbf{k}'$ only, for the reasons put forth at the end of Sec. II). Then for the Hamiltonian we have

$$\mathcal{H} = \mathcal{H}_K + \tfrac{1}{2} \sum_{\mathbf{k} \neq \mathbf{k}', q} \sum_{\sigma,\sigma'} V(\mathbf{k},\mathbf{k}') \times c_{k',\sigma'}^* c_{-k'+q,\sigma}^* c_{-k+q,\sigma} c_{k,\sigma'}. \tag{46}$$

The full equation of motion of ρ_K^Q is

$$\begin{aligned} [\mathcal{H}, \rho_K^Q] = (\epsilon_{K+Q} - \epsilon_K) \rho_K^Q \\ + \sum_{\mathbf{k},\sigma,q} [V(\mathbf{K}+\mathbf{Q}, \mathbf{k}) c_{-k+q,\sigma}^* c_{-K-Q+q,\sigma} c_k^* c_K \\ - V(\mathbf{K},\mathbf{k}) c_{-K+q,\sigma}^* c_{-k+q,\sigma} c_{K+Q}^* c_k], \end{aligned} \tag{47}$$

or

$$\begin{aligned} [\mathcal{H}, \rho_K^Q] = (\epsilon_{K+Q} - \epsilon_K) \rho_K^Q + \sum_k [V(\mathbf{K}+\mathbf{Q}, \mathbf{k}) \\ \times \rho^{K-k+Q} c_k^* c_K - V(\mathbf{K},\mathbf{k}) \rho^{k-K} c_{K+Q}^* c_k]. \end{aligned}$$

[24] The equations for ρ_k^Q are the Bohm-Pines equations; those for b and $\bar{b}$, on the other hand, are the Bethe-Goldstone equations [H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1957)].

We have used $\rho^{\mathbf{Q}}$ for the $\mathbf{Q}$th Fourier component of density:

$$\rho^{\mathbf{Q}}=\sum_{\mathbf{k},\sigma}\rho_{\mathbf{k},\sigma}{}^{\mathbf{Q}}.$$

(The second form above, though brief, is actually quite inconvenient for calculations of all but the "direct" term.)

If we sum (47) over $\mathbf{K}$, and in the first interaction term replace $\mathbf{k}$ by $\mathbf{K}+\mathbf{Q}$ and $\mathbf{K}$ by $\mathbf{k}$ (both are now dummy variables), we see that the interaction terms in $[\mathcal{H},\rho^{\mathbf{Q}}]$ vanish. Thus our equations of motion are such as to automatically satisfy the sum rules (this is maintained throughout) and the most important consequences of gauge invariance. The equation of motion (47) contains five types of linear terms:

$$[\mathcal{H},\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=\text{kinetic}+\text{direct}+\text{exchange self-energy}+\text{exchange scattering}+\text{superconductivity}. \quad (48)$$

The kinetic-energy term is simply the first term:

$$\text{Kinetic:}\quad [\mathcal{H}_K,\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=(\epsilon_{\mathbf{K}+\mathbf{Q}}-\epsilon_K)\rho_{\mathbf{K}}{}^{\mathbf{Q}}. \quad (49)$$

The direct terms are those terms which result directly from the interaction with the components of fluctuation of electron density of wave number $\mathbf{Q}$, i.e., from the term $V(\mathbf{Q})\rho^{\mathbf{Q}}\rho^{-\mathbf{Q}}+\text{cc}$. The linear part is, from the second half of (47):

$$\text{Direct:}\quad [\mathcal{H}_D,\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=V_D(\mathbf{Q})\rho^{\mathbf{Q}}(n_{\mathbf{K}}-n_{\mathbf{K}+\mathbf{Q}}). \quad (50)$$

(We put the subscript D on V to indicate that only in this term should the "direct," unscreened interaction enter; this is primarily a function of $\mathbf{k}-\mathbf{k}'=\mathbf{Q}$ alone.)

The exchange self-energy terms come from the usual exchange terms,

$$\mathcal{H}_{\text{ex}}{}^{\text{self}}=\tfrac{1}{2}\sum_{\mathbf{k},\mathbf{q},\sigma}V(\mathbf{k},-\mathbf{k}+\mathbf{q})n_{\mathbf{k},\sigma}n_{-\mathbf{k}+\mathbf{q},\sigma}, \quad (51)$$

in (46). There is a phonon contribution here to $V(q)$, and it was this contribution which was the basis of the old superconductivity theories of Fröhlich and Bardeen[25]; the assumption now is, however, that this coupling is far too weak to have a visible effect. The appropriate terms are as follows:

$$\text{Self-energy:}\quad [\mathcal{H}_{\text{ex}}{}^{\text{self}},\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=-\rho_{\mathbf{K}}{}^{\mathbf{Q}}\sum_{\mathbf{q}}[n_{\mathbf{q}-\mathbf{K}-\mathbf{Q}}V(\mathbf{K}+\mathbf{Q},\mathbf{q}-\mathbf{K}-\mathbf{Q})-n_{\mathbf{q}-\mathbf{K}}V(\mathbf{K},\mathbf{q}-\mathbf{K})]. \quad (52)$$

The "exchange scattering" terms are the exchange terms which Hubbard includes by a device similar to ours, and follow from those exchange terms (with parallel spin) which contain $\rho^{\mathbf{Q}}$'s:

$$\text{Exchange scattering:}\quad [\mathcal{H}_{\text{ex}},\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=-(n_{\mathbf{K}}-n_{\mathbf{K}+\mathbf{Q}})\sum_{\mathbf{q}}\rho_{\mathbf{q}-\mathbf{K}-\mathbf{Q}}{}^{\mathbf{Q}}V(\mathbf{K}+\mathbf{Q},\mathbf{q}-\mathbf{K}). \quad (53)$$

Note the formal similarity to the direct terms (50), just as the exchange self-energy is similar to the kinetic term (49). For small Q, (53) is negligible compared to (50), and we shall neglect it because it has no relevance to the superconductivity problem. However, for consistency we must then also neglect the exchange self-energy terms (52). That is justifiable as a "weak-coupling" approximation.

Finally we come to the superconductivity terms. These are of two types: "individual-particle" terms, rather like self-energy terms, which come from the B.C.S. reduced Hamiltonian [$q=0$, $\sigma'=-\sigma$ in (46)]; and "collective" terms which follow from $\mathbf{q}=-\mathbf{Q}$, $\sigma'=-\sigma$ in (46). The terms are:

$$\text{Supercond.:}\quad [\mathcal{H}_S,\rho_{\mathbf{K}}{}^{\mathbf{Q}}]=\sum_{\mathbf{k}}V(\mathbf{k},\mathbf{K})(b_{\mathbf{k}+\mathbf{Q}}{}^{*}b_{\mathbf{K}}{}^{\mathbf{Q}}-b_{\mathbf{k}}\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}+b_{\mathbf{K}}\bar{b}_{\mathbf{k}}{}^{\mathbf{Q}}-b_{\mathbf{K}+\mathbf{Q}}{}^{*}b_{\mathbf{k}}{}^{\mathbf{Q}}). \quad (54)$$

[We shall normally take Q small, so we neglect the small difference of $V(\mathbf{K}+\mathbf{Q},\mathbf{k}+\mathbf{Q})$ and $V(\mathbf{K},\mathbf{k})$.]

This method of presentation shows why the "superconductivity" terms have not appeared in previous types of theories: they come in only when b and b^* are treated as number operators are in usual theories. The relationship—or lack thereof—of superconductivity and ferromagnetism is also rather clear in this scheme. Exchange interactions involving a *repulsive* interelectronic potential act like an *attraction* between parallel spins—compare (53) and (50)—and the resulting self-energy-like terms (52) are responsible for ferromagnetism, if they are big enough to outweigh the kinetic energy. The corresponding "superconductivity" terms are repulsive, whether between parallel or antiparallel spins (as could be verified by writing down equations of motion of $c_{-\mathbf{k}-\mathbf{Q}\uparrow}c_{\mathbf{k}\uparrow}$). The interelectronic *attraction* caused by the phonons is thus detrimental to ferromagnetism, and the interactions responsible for ferromagnetism are correspondingly detrimental to superconductivity.

Now we shall write down without further explanation the linearized equations of motion for the remainder of the pair quantities (45), classifying the terms as before.

$$\text{Kinetic:}\quad \begin{aligned}[\mathcal{H}_K,\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}]&=-(\epsilon_{\mathbf{K}+\mathbf{Q}}-\epsilon_K)\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}},\\ [\mathcal{H}_K,b_{\mathbf{K}}{}^{\mathbf{Q}}]&=-(\epsilon_K+\epsilon_{\mathbf{K}+\mathbf{Q}})b_{\mathbf{K}}{}^{\mathbf{Q}},\\ [\mathcal{H}_K,\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}]&=(\epsilon_K+\epsilon_{\mathbf{K}+\mathbf{Q}})\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}.\end{aligned} \quad (55)$$

$$\text{Exchange self-energy:}\quad [\mathcal{H}_{\text{ex}}{}^{\text{self}},\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}]=\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}[\sum_{\mathbf{q}}V(\mathbf{K}+\mathbf{Q},\mathbf{q}-\mathbf{K}-\mathbf{Q})n_{-\mathbf{q}+\mathbf{K}+\mathbf{Q}}-V(\mathbf{K},\mathbf{q}-\mathbf{K})n_{-\mathbf{q}+\mathbf{K}}]. \quad (56)$$

These terms may be simply taken into account by inserting into (55) an exchange self-energy

$$\delta\epsilon_{\mathbf{K},\sigma}=-\sum_{\mathbf{q}}V(\mathbf{K},\mathbf{q}-\mathbf{K})n_{\mathbf{K}-\mathbf{q},\sigma}. \quad (57)$$

$$\text{Direct:}\quad \begin{aligned}[\mathcal{H}_D,\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}]&=V_D(Q)\rho^{\mathbf{Q}}(n_{-\mathbf{K}-\mathbf{Q}}-n_{-\mathbf{K}}),\\ [\mathcal{H}_D,b_{\mathbf{K}}{}^{\mathbf{Q}}]&=-V_D(Q)\rho^{\mathbf{Q}}(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}}),\\ [\mathcal{H}_D,\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}]&=V_D(Q)\rho^{\mathbf{Q}}(b_{\mathbf{K}}{}^{*}+b_{\mathbf{K}+\mathbf{Q}}{}^{*}).\end{aligned} \quad (58)$$

[25] H. Fröhlich, Phys. Rev. **79**, 845 (1950); and J. Bardeen, Phys. Rev. **80**, 567 (1950).

Exchange scattering:

$$\begin{aligned}
[\mathcal{H}_{ex},\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}]&=-(n_{-\mathbf{K}-\mathbf{Q}}-n_{-\mathbf{K}})\\
&\quad\times\textstyle\sum_{\mathbf{q}} V(\mathbf{K}+\mathbf{Q},\mathbf{q}-\mathbf{K})\bar{\rho}_{\mathbf{q}-\mathbf{K}-\mathbf{Q}}{}^{\mathbf{Q}},\\
[\mathcal{H}_{ex},b_{\mathbf{K}}{}^{\mathbf{Q}}]&=b_{\mathbf{K}}\textstyle\sum_{\mathbf{q}}\bar{\rho}_{-\mathbf{K}-\mathbf{Q}-\mathbf{q}}{}^{\mathbf{Q}}V(\mathbf{K},-\mathbf{q}-\mathbf{K}-\mathbf{Q})\\
&\quad+b_{\mathbf{K}+\mathbf{Q}}\textstyle\sum_{\mathbf{q}} V(\mathbf{K}+\mathbf{Q},-\mathbf{q}-\mathbf{K})\\
&\quad\times\rho_{-\mathbf{K}-\mathbf{Q}-\mathbf{q}}{}^{\mathbf{Q}},\\
[\mathcal{H}_{ex},\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}]&=-b_{\mathbf{K}}{}^{*}\textstyle\sum_{\mathbf{q}} V(\mathbf{K},-\mathbf{q}-\mathbf{K}-\mathbf{Q})\rho_{-\mathbf{K}-\mathbf{Q}-\mathbf{q}}{}^{\mathbf{Q}}\\
&\quad-b_{\mathbf{K}+\mathbf{Q}}{}^{*}\textstyle\sum_{\mathbf{q}} V(\mathbf{K}+\mathbf{Q},-\mathbf{q}-\mathbf{K})\bar{\rho}_{-\mathbf{K}-\mathbf{Q}-\mathbf{q}}{}^{\mathbf{Q}}.
\end{aligned}\tag{59}$$

Superconductivity:

$$\begin{aligned}
[\mathcal{H}_{S},\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}]&=\textstyle\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})(b_{\mathbf{k}}{}^{*}b_{\mathbf{K}}{}^{\mathbf{Q}}-b_{\mathbf{k}+\mathbf{Q}}\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}\\
&\quad+b_{\mathbf{K}+\mathbf{Q}}\bar{b}_{\mathbf{k}}{}^{\mathbf{Q}}-b_{K}{}^{*}b_{k}{}^{\mathbf{Q}}),\\
[\mathcal{H}_{S},b_{\mathbf{K}}{}^{\mathbf{Q}}]&=\textstyle\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})[\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}b_{\mathbf{k}}+\rho_{\mathbf{K}}{}^{\mathbf{Q}}b_{\mathbf{k}+\mathbf{Q}}\\
&\quad-(1-n_{\mathbf{K}}-n_{-\mathbf{K}-\mathbf{Q}})b_{\mathbf{k}}{}^{\mathbf{Q}}],\\
[\mathcal{H}_{S},\bar{b}_{\mathbf{K}}{}^{\mathbf{Q}}]&=-\textstyle\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})[b_{\mathbf{k}+\mathbf{Q}}{}^{*}\bar{\rho}_{\mathbf{K}}{}^{\mathbf{Q}}+b_{\mathbf{k}}{}^{*}\rho_{\mathbf{K}}{}^{\mathbf{Q}}\\
&\quad-\bar{b}_{\mathbf{k}}{}^{\mathbf{Q}}(1-n_{-\mathbf{K}}-n_{\mathbf{K}+\mathbf{Q}})].
\end{aligned}\tag{60}$$

It is clear why this method can be called a "random-phase" method: it has the effect of decoupling excitations of different momentum $\mathbf{Q}$, because the only zeroth-order quantities have zero momentum. For this reason the $Q=0$ equations are decoupled from the rest, and are in fact almost exactly those of Sec. III. We can thus write down the $Q=0$ equations without distinguishing zeroth- and first-order quantities.

$$\begin{aligned}
&[\mathcal{H}_{K},n_{\mathbf{k}}]=[\mathcal{H}_{K},n_{-\mathbf{k}}]=0,\\
&[\mathcal{H}_{K},b_{\mathbf{k}}]=-2\epsilon_{k}b_{\mathbf{k}};\quad[\mathcal{H}_{K},b_{\mathbf{k}}{}^{*}]=2\epsilon_{k}b_{\mathbf{k}}{}^{*},
\end{aligned}\tag{61}$$

$$\begin{aligned}
&[\mathcal{H}_{ex}{}^{\mathrm{self}},n_{\mathbf{k}}]=[\mathcal{H}_{ex}{}^{\mathrm{self}},n_{-\mathbf{k}}]=0,\\
&[\mathcal{H}_{ex}{}^{\mathrm{self}},b_{\mathbf{k}}]=-2\delta\epsilon_{k}b_{\mathbf{k}};\quad[\mathcal{H}_{ex}{}^{\mathrm{self}},b_{\mathbf{k}}{}^{*}]=2\delta\epsilon_{k}b_{\mathbf{k}}{}^{*}.
\end{aligned}\tag{62}$$

The "direct" terms all vanish because $V_D(0)=0$. The exchange scattering terms have coalesced with the exchange self-energy (62); they would reappear, if we separated out first-order effects, as terms involving $\sigma_{zk}{}^{0}\delta(\delta\epsilon_{\mathbf{k}})$ (this is why a consistent treatment must include both or neither exchange terms). Finally, the two parts of the superconducting terms coalesce, also to reappear upon applying δ's as in Sec. III:

$$\begin{aligned}
&[\mathcal{H}_{S},n_{\mathbf{K}}]=\textstyle\sum_{\mathbf{k}} V(\mathbf{k},\mathbf{K})(b_{\mathbf{k}}{}^{*}b_{\mathbf{K}}-b_{\mathbf{k}}b_{\mathbf{K}}{}^{*})=[\mathcal{H}_{S},n_{-\mathbf{K}}],\\
&[\mathcal{H}_{S},b_{\mathbf{K}}]=-\textstyle\sum_{\mathbf{k}} V(\mathbf{k},\mathbf{K})(1-n_{\mathbf{K}}-n_{-\mathbf{K}})b_{\mathbf{k}},\\
&[\mathcal{H}_{S},b_{\mathbf{K}}{}^{*}]=\textstyle\sum_{\mathbf{k}} V(\mathbf{k},\mathbf{K})(1-n_{\mathbf{K}}-n_{-\mathbf{K}})b_{\mathbf{k}}{}^{*}.
\end{aligned}\tag{63}$$

Using the definitions (22) we find, as we expect, that (63) may be written

$$\begin{aligned}
[\mathcal{H}_{S},s_{z\mathbf{K}}]&=-i(H_{y\mathbf{K}}s_{x\mathbf{K}}-H_{x\mathbf{K}}s_{y\mathbf{K}}),\\
[\mathcal{H}_{S},\,s_{x\mathbf{K}}+is_{y\mathbf{K}}]&=s_{z\mathbf{K}}(H_{x\mathbf{K}}+iH_{y\mathbf{K}}).
\end{aligned}\tag{63'}$$

These equations may be thought of as determining the zero-order values of the quantities b, b^{*}, and n:

$$|b_{k}{}^{0}|=\tfrac{1}{2}\sin\theta_{k},\quad(1-n_{k}-n_{-k})^{0}=\cos\theta_{k}.$$

Henceforth we write b_k for $b_k{}^0$, etc., so that the b's and n's appearing in the equations without superscripts are just numbers.

Thus, except for the exchange self-energy terms, which we explicitly neglected in Sec. III, (61)–(63) are identical with the equations of the spin model of the B.C.S. theory.

Equations (61)–(63) demonstrate the first feature of our method: that the equations of the B.C.S. theory separate out automatically as the zero-momentum component of a random-phase approximation. The B.C.S. equations have also the effect of determining the stablest ground state about which to linearize the rest of the theory.

This connection with the B.C.S. theory of the last section shows us the meaning of the equations we have called "equations of motion." Our attempt is to find a complete set of "elementary excitations" $x_\nu{}^{\mathbf{Q}}$ involving pairs of particles, and having momentum $\mathbf{Q}$, analogously to the zero-momentum excitations of Sec. III:

$$x_\nu{}^{\mathbf{Q}}=\textstyle\sum_{\mathbf{k}}(\alpha_{\mathbf{k}}\rho_{\mathbf{k}}{}^{\mathbf{Q}}+\beta_{\mathbf{k}}\bar{\rho}_{\mathbf{k}}{}^{\mathbf{Q}}+\gamma_{\mathbf{k}}b_{\mathbf{k}}{}^{\mathbf{Q}}+\delta_{\mathbf{k}}\bar{b}_{\mathbf{k}}{}^{\mathbf{Q}}),\tag{64}$$

such that

$$[\mathcal{H},x_\nu{}^{\mathbf{Q}}]=\nu x_\nu{}^{\mathbf{Q}},\tag{65}$$

because then

$$\Psi_\nu{}^{\mathbf{Q}}=x_\nu{}^{\mathbf{Q}}\Psi_0,\tag{66}$$

will be an excited eigenstate of energy $E_0+\nu$. To find the ν and $x_\nu{}^{\mathbf{Q}}$ it is a valid scheme to replace the commutators by time derivatives, and assume all quantities have frequency ν.

The ν give immediately the most important observable phenomenon, the excitation spectrum. The properties of the ground state must be found more indirectly. Since the system is one with time-reversal symmetry, we can expect—and do find—that the secular equation is a function only of ν^2, so there are eigenfrequencies $\pm\nu$. If Ψ_0 is the true ground state,

$$x_{-\nu}{}^{\mathbf{Q}}\Psi_0\equiv 0\tag{67}$$

[it will be useful later to note that $x_{-\nu}{}^{\mathbf{Q}}=(x_\nu{}^{-\mathbf{Q}})^{*}$], and this is the simplest expression of the modification of our assumed ground state $\Psi_0{}^0$, the product wave function, by the zero-point motion. In the absence of scattering terms (67) is trivially satisfied, being simply the condition that particles cannot be destroyed in the vacuum, or created within the Fermi sea; but the coupling terms make (67) a definite modification of the product function.

In Appendix II we will show how to compute the energy using the $x_\nu{}^{\mathbf{Q}}$'s, and particularly the condition (67). In the simple cases of the pure plasmon theory, or the theory of the B.C.S. $\mathcal{H}_{\mathrm{RED}}$, the energy is corrected simply by the sum of the zero-point shifts of the frequencies ν; but that requires that the Hamiltonian as well as the equations of motion be separable into parts identifiable with the separate momenta Q, which is not in general so. The more general expression for the energy correction which we give in the Appendix is very com-

plicated and usually can be only approximately computed.

In the next section we will discuss the actual solutions of Eqs. (48)–(60); here we are concerned with various generalities about them, and particularly their connection with other theories. For this purpose we might examine their structure more closely.

Let $f_{\mathbf{K}}^{Q}$, $(f')_{\mathbf{K}}^{Q}$ be any of the quantities $\rho_{\mathbf{K}}^{Q}$, $\bar{\rho}_{\mathbf{K}}^{Q}$, $b_{\mathbf{K}}^{Q}$, or $\bar{b}_{\mathbf{K}}^{Q}$, and f^0 be any of the zeroth-order, zero-momentum quantities. The equation for $f_{\mathbf{K}}^{Q}$ contains two general types of terms: self-energy-like terms like (55), (56), and the first two terms on the right in (60), which have the form

$$(f')_{\mathbf{K}}^{Q}\sum_{\mathbf{k}} V f_{k}^{0}; \tag{68}$$

and scattering (or collective) terms such as (58), (59), and the last terms on the right of (60), which have the form

$$f_{K}^{0}\sum_{\mathbf{k}} V(f')_{\mathbf{k}}^{Q}. \tag{69}$$

These two types of terms come from two types of terms in the Hamiltonian: the self-energy terms (68) come from terms

$$\sum_{\mathbf{k},\mathbf{k}'} f_{k}^{0} f_{k'}^{0}, \tag{70}$$

which are (aside from kinetic energy) the exchange-type self-energy and the B.C.S. reduced Hamiltonian. That is, the term $(f')_{\mathbf{K}}^{Q}$ in (68) comes from commuting $f_{\mathbf{K}}^{Q}$ with one of the two f's in (70) having $\mathbf{k}$ or $\mathbf{k}'=\pm\mathbf{K}$ or $\pm(\mathbf{K}+\mathbf{Q})$, the remaining sum giving the self-energy sum of (68). Thus the reduced Hamiltonian gives all terms (68) correctly (aside from exchange self-energy, which was neglected in B.C.S. as well as Bogoliubov, as a weak-coupling assumption). On the other hand, the scattering terms (69) come from terms in the Hamiltonian

$$\sum_{\mathbf{k},\mathbf{k}'} f_{\mathbf{k}}^{Q}(f_{\mathbf{k}'}^{Q})^{*}, \tag{71}$$

and result from commuting $f_{\mathbf{k}'}^{Q*}$ ($\mathbf{k}'=\pm\mathbf{K}$ or $\pm(\mathbf{K}+\mathbf{Q})$) with $f_{\mathbf{K}}^{Q}$ to give the zero-order f_{K}^{0}, the remaining $\mathbf{k}$-sum being the sum in (69).

Just as in the last section, there are two possible types of solutions: "individual-particle" and "collective." Individual-particle (i.p.) solutions have only one or a few $f_{\mathbf{k}}^{Q}$ finite, so that terms of type (69) are of order $1/N$ relative to terms (68). Thus the frequencies of all individual-particle modes are correctly given by $\mathcal{H}_{\mathrm{RED}}$, which is the basic reason behind the success of the B.C.S.-Bogoliubov theories. As Bogoliubov has shown, energies of type (70) may be rewritten in terms of self-energies of the transformed Fermions $\alpha_{\mathbf{K}0}$, $\alpha_{\mathbf{K}1}$, and in that scheme the α's act like independent particles. Thus also for independent-particle modes with $Q\neq 0$ our scheme is fully equivalent to B.C.S.-Bogoliubov. On the other hand, the scattering as well as any collective modes require the inclusion (at least) of all terms of type (71).

There is one last point to be made about the formal structure of the theory. To understand the difficulty, let us take the simplest case of the unperturbed Fermi sea, and for definiteness take $K<k_F$, $|\mathbf{K}+\mathbf{Q}|>k_F$. We have equations for four excitations (we give also the corresponding excitation for the general case in terms of Bogoliubov operators):

$$\begin{aligned}
&\rho_{\mathbf{K}}^{Q}: \quad \nu=\epsilon_{\mathbf{K}+\mathbf{Q}}-\epsilon_{K}>0 \leftrightarrow \alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}1}^{*},\\
&b_{\mathbf{K}}^{Q}: \quad \nu=-(\epsilon_{\mathbf{K}+\mathbf{Q}}+\epsilon_{K})<0 \leftrightarrow \alpha_{\mathbf{K}1}^{*}\alpha_{\mathbf{K}+\mathbf{Q}1},\\
&\bar{\rho}_{\mathbf{K}}^{Q}: \quad \nu=\epsilon_{K}-\epsilon_{\mathbf{K}+\mathbf{Q}}<0 \leftrightarrow \alpha_{\mathbf{K}0}\alpha_{\mathbf{K}+\mathbf{Q}1},\\
&\bar{b}_{\mathbf{K}}^{Q}: \quad \nu=\epsilon_{\mathbf{K}+\mathbf{Q}}+\epsilon_{K}>0 \leftrightarrow \alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}0}.
\end{aligned}$$

The ν's of course come in $\pm$ pairs, and obviously (67) is satisfied for b and $\bar{\rho}$. In the coupled theory, there will still be two conditions (67) on the ground state eliminating the new coupled version of b and $\bar{\rho}$.

There is as yet no condition built into the theory to eliminate the nonphysical *positive*-frequency quantity $\bar{b}$. The product wave function, of course, automatically satisfied

$$\bar{b}_{\mathbf{K}}^{Q}\Psi_{0}^{0}=0,$$

but we have no guarantee that this condition is maintained throughout the motion, once coupling is introduced. This means that we have to prove the theorem that even in the presence of coupling the "unphysical modes"

$$(x_{\nu}^{Q})_{u}=\alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}0} \tag{72}$$

are eigensolutions. We know this to be identically so as far as the self-energy terms (68) are concerned, but must prove the theorem in regard to the coupling terms.

This is easily done. In terms of the α's, the only finite zero-order quantities are

$$\langle\alpha_{\mathbf{K}0}\alpha_{\mathbf{K}0}^{*}\rangle^{0}=\langle\alpha_{\mathbf{K}1}\alpha_{\mathbf{K}1}^{*}\rangle^{0}=1. \tag{73}$$

The collective terms come [see (71)] from commutators

$$\langle[(f_{\mathbf{K}}^{Q})^{*},\alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}0}]\rangle^{0}; \tag{74}$$

but by simple enumeration we find that none of these commutators can result in quantities (73), so that (74)$\equiv 0$. Then there are no coupling terms in the equations of motion of $\alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}0}$, or for that matter the corresponding $-\nu$ quantity $\alpha_{\mathbf{K}+\mathbf{Q}1}^{*}\alpha_{\mathbf{K}1}$. This proves the theorem: the equations of motion are compatible with the requirement

$$(x_{\pm\nu}^{Q})_{u}\Psi_{0}=\left\{\begin{matrix}\alpha_{\mathbf{K}+\mathbf{Q}0}^{*}\alpha_{\mathbf{K}0}\\ \alpha_{\mathbf{K}+\mathbf{Q}1}^{*}\alpha_{\mathbf{K}1}\end{matrix}\right\}\Psi_{0}\equiv 0, \tag{75}$$

because

$$[\mathcal{H},(x_{\pm\nu}^{Q})_{u}]=\pm(\nu_{\mathbf{K}}^{Q})_{u}(x_{\pm\nu}^{Q})_{u}. \tag{76}$$

The analogous requirement in the spin theory was of course (43′), and that analogy shows us the nature of (75) as opposed to the similar (67). (75) is a linearized version of a *kinematical* condition, automatically satisfied by the equations of motion; we do not know the nonlinear equivalent, although it must exist. On the other hand, (67) is simply a defining condition for the

ground state, which is not satisfied even in infinitesimally excited states, and also differs in that it defines a true change in the wave function from the simple product result. Equation (75) says kinematically that the motion involves *only* the simultaneous excitation of pairs; while (67) expresses the *dynamical* extent to which this occurs in the ground state. (75) will be a useful result in finding the solution of our equations.

Now let us briefly note the second limiting case: that (48)–(60) reduce to the R.P.A. theory of Coulomb correlation in the normal case in which $b_k^0=b_k^{0*}=0$. The Sawada-Brout theory corresponds to keeping only (49) and (50) [(55) and (58)], and to including the effect of exchange only by perturbation theory. As Sawada shows, in terms of diagrams this amounts to summing, besides exchange diagrams to second order, all diagrams of the form of Fig. 2(a).

Our theory in its complete form, including the screened exchange terms (51)–(52) [(56) and (59)], is an equation-of-motion equivalent of the diagram method of Hubbard, which is more accurate than Sawada-Brout.

This method sums automatically also all the diagrams of the form of Fig. 2(b), in which the interaction lines for the exchange scatterings themselves imply complete sums of terms like Fig. 2(a). A more complete discussion of the relationship of this method to other treatments of Coulomb correlation and other many-body problems[26] will be given in a later publication.

V. PARTIAL SOLUTION OF EQUATIONS OF MOTION

In this section we shall attempt a discussion of the solutions, and particularly the collective ones, of the equations of motion (49)–(60). As we pointed out in the last section, the individual-particle modes will automatically agree with the B.C.S.-Bogoliubov theory; we shall, however, verify that also. Let us first write down the equations, making use of the following abbreviations, and neglecting the exchange terms (which have the effect fundamentally of simply altering slightly the kinetic energy and the direct scattering terms, without changing their character) throughout:

$$\omega_{\mathbf{KQ}}=\epsilon_{\mathbf{K+Q}}-\epsilon_{\mathbf{K}},\quad \Omega_{\mathbf{KQ}}=\epsilon_{\mathbf{K+Q}}+\epsilon_{\mathbf{K}},$$
$$-I_{\mathbf{K}}=\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})b_{\mathbf{k}}=\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})b_{\mathbf{k}}^*, \qquad (77)$$
$$n_{\mathbf{KQ}}=n_{\mathbf{K+Q}}-n_{\mathbf{K}},\quad z_{\mathbf{KQ}}=1-n_{\mathbf{K}}-n_{\mathbf{K+Q}}.$$

Here we have chosen, with no loss in generality, $b_k=b_k^*$ (the domain wall in the $+x$ direction). I_K is then half of the x component of H_K, and is defined to be positive

[26] Note the presence in Eqs. (60) of terms like
$$(1-n_{\mathbf{K}}-n_{-\mathbf{K}-\mathbf{Q}})b_{\mathbf{k}}^{\mathbf{Q}},$$
by means of which the scattering of excited pairs of electrons or holes in the presence of the Fermi sea may be calculated. Thus these equations reduce to the Bethe-Goldstone ones for the case of the normal Fermi sea.

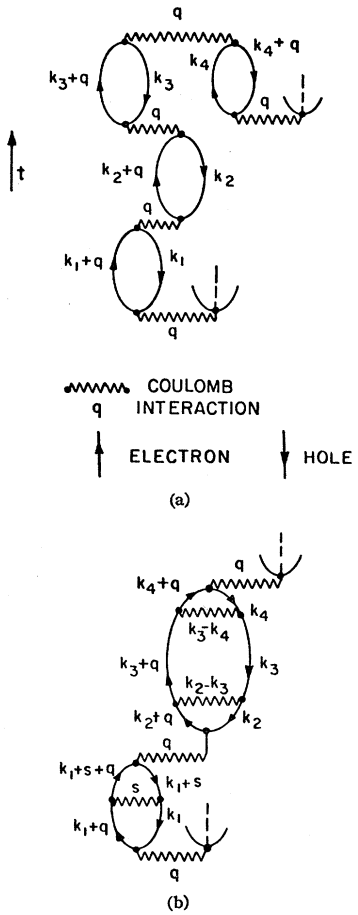

FIG. 2. Perturbation-theoretic diagrams summed by the random-phase method. Plain arcs represent electrons and holes, wavy lines Coulomb interactions ("longitudinal photons"); the momentum and momentum transfer, respectively, are given next to the lines. (a) Typical chain-type diagrams summed by the Sawada *et al.*-Brout equations. (b) More general diagrams summed by full method. Use of dielectric constant screening causes exchange interactions (like that labeled s) to imply full sums of diagrams like 2(a).

(V is negative). With (77), we get

$$[\mathcal{H},\rho_{\mathbf{K}}^{\mathbf{Q}}]=\omega_{\mathbf{KQ}}\rho_{\mathbf{K}}^{\mathbf{Q}}-V_D\rho^{\mathbf{Q}}n_{\mathbf{KQ}}-I_{\mathbf{K+Q}}b_{\mathbf{K}}^{\mathbf{Q}}+I_{\mathbf{K}}\bar{b}_{\mathbf{K}}^{\mathbf{Q}}$$
$$+b_{\mathbf{K}}\sum_{\mathbf{k}} V\bar{b}_{\mathbf{k}}^{\mathbf{Q}}-b_{\mathbf{K+Q}}\sum_{\mathbf{k}} Vb_{\mathbf{k}}^{\mathbf{Q}}, \qquad (78a)$$

$$[\mathcal{H},\bar{\rho}_{\mathbf{K}}^{\mathbf{Q}}]=-\omega_{\mathbf{KQ}}\bar{\rho}_{\mathbf{K}}^{\mathbf{Q}}+V_D\rho^{\mathbf{Q}}n_{\mathbf{KQ}}-I_{\mathbf{K}}b_{\mathbf{K}}^{\mathbf{Q}}+I_{\mathbf{K+Q}}\bar{b}_{\mathbf{K}}^{\mathbf{Q}}$$
$$+b_{\mathbf{K+Q}}\sum_{\mathbf{k}} V\bar{b}_{\mathbf{k}}^{\mathbf{Q}}-b_{\mathbf{K}}\sum_{\mathbf{k}} Vb_{\mathbf{k}}^{\mathbf{Q}}, \qquad (78b)$$

$$[\mathcal{H},b_{\mathbf{K}}^{\mathbf{Q}}]=-\Omega_{\mathbf{KQ}}b_{\mathbf{K}}^{\mathbf{Q}}-V_D\rho^{\mathbf{Q}}(b_{\mathbf{K}}+b_{\mathbf{K+Q}})-I_{\mathbf{K}}\bar{\rho}_{\mathbf{K}}^{\mathbf{Q}}$$
$$-I_{\mathbf{K+Q}}\rho_{\mathbf{K}}^{\mathbf{Q}}-z_{\mathbf{KQ}}\sum_{\mathbf{k}} Vb_{\mathbf{k}}^{\mathbf{Q}}, \qquad (78c)$$

$$[\mathcal{H},\bar{b}_{\mathbf{K}}^{\mathbf{Q}}]=\Omega_{\mathbf{KQ}}\bar{b}_{\mathbf{K}}^{\mathbf{Q}}+V_D\rho^{\mathbf{Q}}(b_{\mathbf{K}}+b_{\mathbf{K+Q}})$$
$$+(I_{\mathbf{K+Q}}\bar{\rho}_{\mathbf{K}}^{\mathbf{Q}}+I_{\mathbf{K}}\rho_{\mathbf{K}}^{\mathbf{Q}})+z_{\mathbf{KQ}}\sum_{\mathbf{k}} V\bar{b}_{\mathbf{k}}^{\mathbf{Q}}. \qquad (78d)$$

Here we have $V=V(\mathbf{K},\mathbf{k})$ for brevity, and $V_D=V_D(Q)$.

The time-reversal feature of (78) may be seen by noticing that if we take $\bar{\rho}_{\mathbf{k}}^{\mathbf{Q}}=(\rho_{\mathbf{k}}^{\mathbf{Q}})^*$, $\bar{b}=(b)^*$, (78b) is (78a) and (78d) is (78c). Interpreting $[\mathcal{H},f]$ as $i\dot{f}$, this means that (78) connects the real parts with time

derivatives of imaginary parts and vice versa; thus it is basically an equation in $(d/dt)^2$. Our method of solution will be to derive the two second-order equations for the (say) imaginary parts. For instance, we write down

$$[\mathcal{H}, \rho_{\mathbf{K}}^{Q}+\bar{\rho}_{\mathbf{K}}^{Q}]=\omega_{KQ}(\rho-\bar{\rho})_{\mathbf{K}}^{Q}-(I_{\mathbf{K}}+I_{\mathbf{K}+\mathbf{Q}}) \times(b-\bar{b})_{\mathbf{K}}^{Q}-(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})\sum_{\mathbf{k}} V(b-\bar{b})_{\mathbf{k}}^{Q}, \quad (79)$$

and

$$[\mathcal{H}, [\mathcal{H}, (\rho-\bar{\rho})_{\mathbf{K}}^{Q}]]=\omega_{KQ}[\mathcal{H}, (\rho+\bar{\rho})_{\mathbf{K}}^{Q}] -2V_D n_{KQ}[\mathcal{H},\rho^{Q}]-(I_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}})[\mathcal{H}, (b+\bar{b})_{\mathbf{K}}^{Q}] +(b_{\mathbf{K}}-b_{\mathbf{K}+\mathbf{Q}})[\mathcal{H}, \sum_{\mathbf{k}} V(b+\bar{b})_{\mathbf{k}}^{Q}]. \quad (80)$$

In (80) we substitute from (79) and the corresponding equation for $b+\bar{b}$, except in the last sum, which is most conveniently left in its present form:

$$[\mathcal{H},[\mathcal{H},w_{\mathbf{K}}^{Q}]]=\nu^2 w_{\mathbf{K}}^{Q}=[\omega_{KQ}^2+(I_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}})^2]w_{\mathbf{K}}^{Q} -2(\epsilon_{\mathbf{K}+\mathbf{Q}}I_{\mathbf{K}}-\epsilon_{\mathbf{K}}I_{\mathbf{K}+\mathbf{Q}})y_{\mathbf{K}}^{Q}-2V_D n_{KQ}\Pi^{Q} -[\omega_{KQ}(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})+z_{KQ}(I_{\mathbf{K}}-I_{\mathbf{K}+\mathbf{Q}})]\sum_{\mathbf{k}} V y_{\mathbf{k}}^{Q} +(b_{\mathbf{K}}-b_{\mathbf{K}+\mathbf{Q}})B_{\mathbf{K}}^{Q}. \quad (81)$$

Here we have made some further abbreviations:

$$y_{\mathbf{k}}^{Q}=b_{\mathbf{k}}^{Q}-\bar{b}_{\mathbf{k}}^{Q}; \quad w_{\mathbf{k}}^{Q}=\rho_{\mathbf{k}}^{Q}-\bar{\rho}_{\mathbf{k}}^{Q}; \quad B_{\mathbf{K}}^{Q}=[\mathcal{H}, \sum_{\mathbf{k}} V(b_{\mathbf{k}}^{Q}+\bar{b}_{\mathbf{k}}^{Q})]; \quad \Pi^{Q}=[\mathcal{H},\rho^{Q}]=\sum_{\mathbf{k}}\omega_{\mathbf{kQ}}w_{\mathbf{k}}^{Q}. \quad (82)$$

This last equality is the all-important sum rule.[6]

The equation for $y_{\mathbf{K}}^{Q}$ is obtained similarly:

$$\nu^2 y_{\mathbf{K}}^{Q}=[\Omega_{KQ}^2+(I_{\mathbf{K}}+I_{\mathbf{K}+\mathbf{Q}})^2]y_{\mathbf{K}}^{Q} -2(I_{\mathbf{K}}\epsilon_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}+\mathbf{Q}}\epsilon_{\mathbf{K}})w_{\mathbf{K}}^{Q}-2V_D(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})\Pi^{Q} +[\Omega_{KQ}z_{KQ}+(I_{\mathbf{K}}+I_{\mathbf{K}+\mathbf{Q}})(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})] \times\sum_{\mathbf{k}} V y_{\mathbf{k}}^{Q}-z_{KQ}B_{\mathbf{K}}^{Q}. \quad (83)$$

First note that, because

$$-\frac{n_{KQ}}{b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}}}=\frac{b_{\mathbf{K}}-b_{\mathbf{K}+\mathbf{Q}}}{z_{KQ}} =\frac{\omega_{KQ}(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})-z_{KQ}(I_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}})}{\Omega_{KQ}z_{KQ}+(I_{\mathbf{K}}+I_{\mathbf{K}+\mathbf{Q}})(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})}, \quad (84)$$

the collective parts of the two Eqs. (81) and (83) are simply proportional. This is the result of the condition (74)–(75): one solution factors from the secular equation.

Second, we derive the spectrum of the individual-particle solutions. This too we know must come out right, but we shall do it as a check on our reasoning. Leaving out the collective terms, the secular equation is

$$0=\begin{vmatrix} \Omega_{KQ}^2+(I_{\mathbf{K}}+I_{\mathbf{K}+\mathbf{Q}})^2-\nu^2 & -2(I_{\mathbf{K}}\epsilon_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}+\mathbf{Q}}\epsilon_{\mathbf{K}}) \\ -2(I_{\mathbf{K}}\epsilon_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}+\mathbf{Q}}\epsilon_{\mathbf{K}}) & \omega_{KQ}^2+(I_{\mathbf{K}+\mathbf{Q}}-I_{\mathbf{K}})^2-\nu^2 \end{vmatrix},$$

which indeed has the solutions

$$(\nu_{\mathbf{K}}^{Q})^2=[(\epsilon_{\mathbf{K}}^2+I_{\mathbf{K}}^2)^{\frac{1}{2}}\pm(\epsilon_{\mathbf{K}+\mathbf{Q}}^2+I_{\mathbf{K}+\mathbf{Q}}^2)^{\frac{1}{2}}]^2, \quad (85)$$

of which those with the $-$ sign are exact solutions because of (76), but must be discarded because of (75).

Now we shall go on to discuss collective solutions. To avoid complications we make some assumptions which are valid in the weak-coupling limit or in the "cutoff" theory of B.C.S. in which V is a constant. In some cases we assume small Q, the case we study primarily (for large Q the overwhelming majority of pair excitations of momentum Q are obviously unaffected by superconductivity).

We assume $I_{\mathbf{K}}=I$ independent of $\mathbf{K}$, as it will nearly be if $\epsilon_0 \ll \omega_D$. $y_{\mathbf{K}}^{Q}$ will be small except within ϵ_0 of the Fermi surface; then we can assume

$$\sum_{\mathbf{k}} V(\mathbf{K},\mathbf{k})y_{\mathbf{k}}^{Q}=A^{Q} \text{ independent of } \mathbf{K}. \quad (86)$$

We shall find that the most important collective modes have y even; in that case

$$[\mathcal{H}, \sum_{\mathbf{K}} V(b+\bar{b})_{\mathbf{K}}^{Q}]\cong-\sum_{\mathbf{K}} V\Omega_{KQ}y_{\mathbf{K}}^{Q}=B^{Q} \quad (87)$$

is also zero. However, for the sake of completeness we shall retain B^{Q}, also assuming it to be a constant with varying $\mathbf{K}$. Then the Eqs. (81) and (83) become

$$(\nu^2-\omega_{KQ}^2)w_{\mathbf{K}}^{Q}=-2\omega_{KQ}I y_{\mathbf{K}}^{Q}-2V_D n_{KQ}\Pi^{Q} -\omega_{KQ}(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})A^{Q}+(b_{\mathbf{K}}-b_{\mathbf{K}+\mathbf{Q}})B^{Q}, \quad (81')$$

and

$$[\nu^2-(\Omega_{KQ}^2+4I^2)]y_{\mathbf{K}}^{Q}=-2\omega_{KQ}I w_{\mathbf{K}}^{Q} -2V_D(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})\Pi^{Q}+[\Omega_{KQ}z_{KQ} +2I(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})]A^{Q}-z_{KQ}B^{Q}. \quad (83')$$

To take advantage of (84), let us define

$$\Phi_{\mathbf{K}}^{Q}=2V_D\Pi^{Q}+\frac{\omega_{KQ}}{n_{KQ}}(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})A^{Q}-\frac{(b_{\mathbf{K}}-b_{\mathbf{K}+\mathbf{Q}})}{n_{KQ}}B^{Q}. \quad (88)$$

Then (81′) and (83′) may be rewritten

$$\begin{aligned}(\nu^2-\omega_{KQ}^2)w_{\mathbf{K}}^{Q}+2\omega_{KQ}I y_{\mathbf{K}}^{Q}&=-n_{KQ}\Phi_{\mathbf{K}}^{Q},\\ [\nu^2-(\Omega_{KQ}^2+4I^2)]y_{\mathbf{K}}^{Q}+2\omega_{KQ}I w_{\mathbf{K}}^{Q}&\\ =-(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})\Phi_{\mathbf{K}}^{Q}.&\end{aligned} \quad (89)$$

We may solve by multiplying the first by $2\omega_{KQ}I$, the second by $(\nu^2-\omega_{KQ}^2)$, and subtracting, or vice versa. In that case we get

$$\{[\nu^2-(\Omega_{KQ}^2+4I^2)][\nu^2-\omega_{KQ}^2]-4\omega_{KQ}^2I^2\}y_{\mathbf{K}}^{Q} =-[(\nu^2-\omega_{KQ}^2)-2n_{KQ}\omega_{KQ}I(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})^{-1}] \times(b_{\mathbf{K}}+b_{\mathbf{K}+\mathbf{Q}})\Phi_{\mathbf{K}}^{Q}.$$

With the collective terms (the right-hand side) left out, this equation must have the solutions (85), and so the left-hand side must simply be

$$[\nu^2-(\nu_{\mathbf{K}}^{Q})^2][\nu^2-(\nu_{\mathbf{K}}^{Q})_u^2]y_{\mathbf{K}}^{Q},$$

where $\nu_{\mathbf{K}}^{Q}$ is the physical (positive-sign) root of (85) and $(\nu_{\mathbf{K}}^{Q})_u$ the unphysical, negative-sign one. The only way, then, to satisfy the requirement (76) is for $[\nu^2-(\nu_{\mathbf{K}}^{Q})_u^2]$ to be a factor on *both* sides; canceling

this factor out, we obtain

$$[\nu^2-(\nu_K{}^Q)^2]y_K{}^Q=-(b_K+b_{K+Q})\Phi_K{}^Q. \quad (90)$$

By a similar procedure we can also arrive at

$$[\nu^2-(\nu_K{}^Q)^2]w_K{}^Q=-n_{KQ}\Phi_K{}^Q. \quad (91)$$

[So far we have not used our simplifying assumptions in any essential way, and (90) and (91) could be used to obtain complete spectra of collective modes if the resulting integral equations could be solved.]

From (90), (91), and the definition (88) of Φ, we can form the sums Π, A, and B, and arrive at the integral equations which determine the frequencies:

$$\Pi^Q=\{\textstyle\sum_k \omega_{kQ}n_{kQ}[(\nu_k{}^Q)^2-\nu^2]^{-1}\}2V_D\Pi^Q +\{\sum_k \omega_{kQ}{}^2(b_k+b_{k+Q})[(\nu_k{}^Q)^2-\nu^2]^{-1}\}A^Q -\{\sum_k \omega_{kQ}(b_k-b_{k+Q})[(\nu_k{}^Q)^2-\nu^2]^{-1}\}B^Q. \quad (92)$$

$$A^Q=\{\textstyle\sum_k V(b_k+b_{k+Q})[(\nu_k{}^Q)^2-\nu^2]^{-1}\}2V_D\Pi^Q +\{\sum_k V\omega_{kQ}(b_k+b_{k+Q})^2n_{kQ}{}^{-1}[(\nu_k{}^Q)^2-\nu^2]^{-1}\}A^Q -\{\sum_k V(b_k{}^2-b_{k+Q}{}^2)n_{kQ}{}^{-1}[(\nu_k{}^Q)^2-\nu^2]^{-1}\}B^Q, \quad (93)$$

$$B^Q=-\{\textstyle\sum_k V\Omega_{kQ}(b_k+b_{k+Q})[(\nu_k{}^Q)^2-\nu^2]^{-1}\}2V_D\Pi^Q -\{\sum_k V\omega_{kQ}\Omega_{kQ}(b_k+b_{k+Q})^2n_{kQ}{}^{-1} \times[(\nu_k{}^Q)^2-\nu^2]^{-1}\}A^Q+\{\sum_k V\Omega_{kQ} \times(b_k{}^2-b_{k+Q}{}^2)n_{kQ}{}^{-1}[(\nu_k{}^Q)^2-\nu^2]^{-1}\}B^Q. \quad (94)$$

(92)–(94) are a set of three simultaneous linear equations in Π, A, and B. Symmetry about the sphere $|\mathbf{k}+(\mathbf{Q}/2)|=k_F$ makes the cross-terms coupling B to A and Π vanish. Thus there are, as in the $Q=0$ case, two independent types of collective solutions: the odd ones involving B^Q, corresponding to δH_x in (33) finite ($\lim_{Q\to 0}B^Q\propto\delta H_x$), and the even ones, with finite A^Q and Π^Q (like the δH_y solutions). The odd solutions obey (94), which by use of (84) is

$$1=\textstyle\sum_k (-V)\Omega_{kQ}z_{kQ}[(\nu_k{}^Q)^2-\nu^2]^{-1}. \quad (95)$$

As in the $Q=0$ case, all solutions lie in or close to the continuum, and approach as $Q\to 0$ the corresponding solution of (34)–(35).

As for the "even" solutions, our assumptions have been equivalent to the $V=$constant assumption which eliminates any higher bound states for $Q=0$. These higher states may also exist for $Q\neq 0$, for physical V's, but we shall ignore them, simply observing that p, d, etc. states, or if you like, transverse and more complicated collective waves, may exist but will be near the continuum, and their energies will approach the corresponding $Q=0$ energies as $Q\to 0$.

This leaves the coupled Eqs. (92)–(93) to determine the physically important longitudinal collective modes. The dispersion equation is a determinant

$$\begin{vmatrix} 1-2V_Df & l \\ 2V_Dh & 1-g \end{vmatrix}=0, \quad (96)$$

where after some algebra we define

$$\begin{aligned} f&=\textstyle\sum_k \omega_{kQ}n_{kQ}[(\nu_k{}^Q)^2-\nu^2]^{-1},\\ g&=\textstyle\sum_k (-V)\nu_k{}^Q\cos^2[\tfrac{1}{2}(\theta_k-\theta_{k+Q})][(\nu_k{}^Q)^2-\nu^2]^{-1},\\ h&=\textstyle\sum_k (-V)(b_k+b_{k+Q})[(\nu_k{}^Q)^2-\nu^2]^{-1},\\ l&=\textstyle\sum_k \omega_{kQ}{}^2(b_k+b_{k+Q})[\nu^2-(\nu_k{}^Q)^2]^{-1}. \end{aligned} \quad (97)$$

The collective modes have entirely different behavior depending on whether we consider the charged or neutral cases. In the charged case, V_D is singular and large, Π is the important variable, and f determines the frequencies; in the neutral case A^Q (a variable which closely resembles the "rotation" of Sec. III) is the important variable and g mostly determines the frequency. We shall analyze the two cases separately.

Case I. Neutral Fermi Gas

This case is defined by $V_D=$const as $Q\to 0$ (we might as well let $V_D=V$, the ordinary interaction). Since $f\propto Q^2$, $1-2V_Df\cong 1$ in the long-wave limit except very near a $\nu_k{}^Q$; in a principal-value sense, f is small everywhere. This limit of small Q is the interesting region, since in reference 6 we proved that in this case states with $Q\to 0$ lie at the bottom of the energy gap; thus we use a perturbation procedure suited to this case. Since also $l\propto Q^2$, in all terms except $1-g$ we can make approximations; in particular, we neglect f and get

$$1-g=2Vlh. \quad (98)$$

In the limit $Q=0$, this has, aside from the individual-particle solutions, only the $\nu=0$ collective mode which we discussed in Sec. III; $g=1$, $\nu=0$ leads to precisely the equilibrium condition for the ground state. Our task here is to get the dispersion of this mode to lowest order in Q. For this purpose it is adequate to expand h and l to lowest nonvanishing order in Q^2 and ν^2, and g to first order:

$$\begin{aligned} h&\cong\textstyle\sum_k (-V)\sin\theta_k(\nu_k{}^0)^{-2}=2I\sum_k (-V)(\nu_k{}^0)^{-3};\\ l&\cong-\textstyle\sum_k \omega_{kQ}{}^2\sin\theta_k(\nu_k{}^0)^{-2}\\ &\qquad=-\tfrac{1}{3}k_F{}^2Q^2m^{-2}\textstyle\sum_k \sin\theta_k(\nu_k{}^0)^{-2};\\ g&\cong 1+\nu^2\textstyle\sum_k (-V)(\nu_k{}^0)^{-3}\\ &\quad+\{\textstyle\sum_k (-V)(\nu_k{}^Q)^{-1}\cos^2[\tfrac{1}{2}(\theta_{k+Q}-\theta_k)]\\ &\quad-\tfrac{1}{2}[\textstyle\sum_k (-V)(\nu_k{}^0)^{-1}+\sum_k (-V)(\nu_{k+Q}{}^0)^{-1}]\}. \end{aligned} \quad (99)$$

The calculation of this last difference follows:

$$\begin{aligned} \{\ \ \}&=\textstyle\sum_k (-V)I^{-1}(\sin\theta_k+\sin\theta_{k+Q})^{-1}[\cos^2(\tfrac{1}{2}(\theta_{k+Q}-\theta_k))\\ &\qquad\times\sin\theta_k\sin\theta_{k+Q}-\tfrac{1}{4}(\sin\theta_k+\sin\theta_{k+Q})^2]\\ &=-\textstyle\sum_k (-V)(4I)^{-1}(\sin\theta_k+\sin\theta_{k+Q})^{-1}\\ &\qquad\times(\sin\theta_k\cos\theta_{k+Q}-\sin\theta_{k+Q}\cos\theta_k)^2\\ &\qquad\cong-\textstyle\sum_k (-V)\omega_{kQ}{}^2(\nu_k{}^0)^{-3}. \end{aligned}$$

Thus the dispersion of this mode is given by

$$1-[1+\nu^2\textstyle\sum_k (-V)(\nu_k{}^0)^{-3}-\sum_k (-V)(\nu_k{}^0)^{-3}\omega_{kQ}{}^2] =2V[-2I\textstyle\sum_k (-V)(\nu_k{}^0)^{-3}][\sum_k \omega_{kQ}{}^2\sin\theta_k(\nu_k{}^0)^{-2}],$$

or

$$\nu^2=\tfrac{1}{3}v_F^2Q^2[1+4IV\sum_k \sin\theta_k(\nu_k{}^0)^{-2}]$$
$$=\tfrac{1}{3}v_F^2Q^2[1-4N(0)|V|]. \quad (100)$$

Apparently the "phonon" velocity $3^{-\frac{1}{2}}v_F$ is strictly a kinematical "ideal gas" effect, the $N(0)V$ correction being a result of the coupling to the direct interaction in the present weak coupling case. The kinematical term has been obtained in a different way by Bogoliubov.[12]

Case II. Charged Fermi Gas

In this case

$$V_D(Q)=2\pi e^2Q^{-2}, \quad (101)$$

and in the important $Q\to 0$ limit, the cross-coupling $2V_Dlh$ as well as the term involving f are rather large constants unless ν is large. First let us satisfy ourselves that there are no low-lying collective modes. For such modes ν could be neglected in estimating l and h; we calculated in Case I that $lh(\nu=0)\cong I^{-2}N^2(0)V\langle\omega_{kQ}{}^2\rangle$ so we find

$$2V_Dlh(\nu=0)\cong\omega_p{}^2I^{-2}N(0)V\gg 1.$$

Thus the collective mode with $\nu=0$ near $1-g=0$ disappears without trace; we can only hope for a solution at very large ν. We expect such a solution to lie near ω_p; we shall find there that now l, h, and g are small and in them we assume $\nu=\omega_p$. What we then seek are the corrections to ω_p of lowest order in the energy gap $\epsilon_0=2I$.

Near ω_p, g is small, if we assume that the exchange-phonon interaction, when averaged over attractive and repulsive regions, is small compared to the direct interaction. In any case $1-g$ is only coupled in by small terms, so we can neglect g and write

$$1-2V_Df=2V_Dlh. \quad (102)$$

Near $\nu=\omega_p$, $Q=0$, we have

$$2V_Dlh=-2V_D(\sum_k \omega_{kQ}{}^2\omega_p{}^{-2}\sin\theta_k)\times(\sum_k|V|\omega_p{}^{-2}\sin\theta_k)$$
$$=-2V_D\langle\omega_{kQ}{}^2\rangle_{Av}\omega_p{}^{-4}4I^2|V|^{-1}. \quad (103)$$

The correction to f may be calculated as follows:

$$f\cong-\nu^{-2}\sum_k \omega_{kQ}n_{kQ}-\omega_p{}^{-4}\sum_k \omega_{kQ}n_{kQ}(\nu_k{}^Q)^2,$$

so that

$$2V_Df=\omega_p{}^2\nu^{-2}-\langle\omega_{kQ}{}^2\rangle_{Av}2V_D\omega_p{}^{-4}\times\sum_k n_{kQ}\omega_{kQ}{}^{-1}(\nu_k{}^Q)^2$$
$$=\omega_p{}^2\nu^{-2}+2V_D\langle\omega_{kQ}{}^2\rangle_{Av}\omega_p{}^{-4}4I^2|V|^{-1}. \quad (104)$$

We see that the corrections in (103) and (104) cancel, leaving the plasma frequency as $Q\to 0$ unchanged, even to the very small terms which we are calculating. It seems likely that the dispersion of the plasma mode is also unchanged, since ω_p must be the same also for large Q.

Thus, in conclusion, the predictions of reference 6 are in large part borne out: that the charged Fermi gas has no low-lying collective modes because of the strong plasma effect, while the neutral gas has a low-lying branch. The present weak-coupling theory gives no correction to the plasma mode, and derives a phonon-like mode for the neutral case with a small interaction correction. The presence of the interaction correction represents a definite improvement of the present method.

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APPENDIX I. INCLUSION OF DIRECT INTERACTIONS WITH PHONONS

In the actual physical superconductor there always remain a group of excitations in the energy gap: namely, the lattice phonons. According to our discussion of Sec. II, it should be feasible to study the effect of superconductivity on the phonons by introducing explicitly only the direct electron-phonon interaction; if the phonons are not seriously perturbed by the difference between normal and superconducting states, then their contribution to the dielectric constant, and thus to the superconducting interaction V, can be studied as though the electron wave function were normal.

Clearly it is only important to include phonons in the physical, charged case; the neutral case is of possible physical interest only in such problems as He^3 or the nucleus, where there is no lattice.

The inclusion of phonons simply involves including among the excitations of momentum $\mathbf{Q}$ the coordinate and momentum q_Q and p_Q, and the terms of (12) and (13) involving these:

$$(\mathcal{H}_{ph}+\mathcal{H}_i)^Q=(p_Qp_{-Q}+f_Q{}^2q_Qq_{-Q})-(q_Q\rho^{-Q}v_Q{}^i+q_{-Q}\rho^Qv_Q{}^{i*}). \quad (A1)$$

The equations of motion for the new variables lead rather simply to

$$[\mathcal{H},[\mathcal{H},p_Q]]=f_Q{}^2p_Q+2iv_Q{}^{i*}\Pi^Q. \quad (A2)$$

Now we must calculate the effect of $\mathcal{H}_i$ in (A1) on the various electron coordinates. The results are

$$\begin{aligned}[\mathcal{H}_i,\rho_K{}^Q]&=-q_Qv_Q{}^i(n_K-n_{K+Q}),\\ [\mathcal{H}_i,\bar\rho_K{}^Q]&=q_Qv_Q{}^i(n_K-n_{K+Q}),\\ [\mathcal{H}_i,b_K{}^Q]&=q_Qv_Q{}^i(b_K+b_{K+Q}),\\ [\mathcal{H}_i,\bar b_K{}^Q]&=-q_Qv_Q{}^i(b_K+b_{K+Q}).\end{aligned} \quad (A3)$$

The extra terms (A3) lead to an extra term on the right of Eqs. (81) and (82) for w and y:

$$\begin{aligned}[\mathcal{H}_i+\mathcal{H}_{ph},[\mathcal{H}_i,w_K{}^Q]]&=2ip_Qv_Q{}^in_{KQ},\\ [\mathcal{H}_i+\mathcal{H}_{ph},[\mathcal{H}_i,y_K{}^Q]]&=2ip_Qv_Q{}^i(b_K+b_{K+Q}).\end{aligned} \quad (A4)$$

Thus the effect of including the phonons appears (appropriately, since the phonons are collective variables) entirely as an extra term in the collective part $\Phi_K{}^Q$ of Eqs. (90) and (91); we must define a new $\Phi_K{}^Q$ which is given by

$$\Phi_K{}^Q=2V_D\Pi^Q+\frac{\omega_{KQ}}{n_{KQ}}(b_K+b_{K+Q})A^Q-\frac{(b_K-b_{K+Q})}{n_{KQ}}B^Q -2iv_Q{}^i p_Q, \quad \text{(A5)}$$

while p_Q is related to Π^Q by (A2).

Again we find that symmetry about the Fermi surface allows us to decouple B^Q from the rest of the equations, so we may assume B^Q zero. Then we are left with a three-by-three set of equations for Π^Q, A^Q, and p^Q instead of the two-by-two determinant (96). The calculation is so similar that one needs only to write down the final determinant:

$$\begin{vmatrix} f_Q{}^2-\nu^2 & 2iv_Q{}^{i*} & 0 \\ 2iv_Q{}^i f & 1-2V_D f & l \\ -2iv_Q{}^i h & 2V_D h & 1-g \end{vmatrix}=0, \quad \text{(A6)}$$

where the symbols f, g, h, and l are the same as they were in (97).

The result for the normal metal would be the limiting case of (A6) in which h and l are zero, decoupling A^Q, and $\nu_k{}^Q\rightarrow\omega_{kQ}$ in f. Then the secular equation is

$$\nu_{\text{norm}}{}^2=f_Q{}^2+\frac{4|v_Q{}^i|^2 f}{1-2V_D f}. \quad \text{(A7)}$$

This is exactly the same as the secular equation [his (7a)] of Wentzel,[16] if we note that our f is his f_q; our $v_Q{}^i$, his μ_q; our V_D, his λ_q. Using (14) and (11), we find

$$\nu^2=\omega_Q{}^2+(4\pi ne^2Z^2M^{-1})\frac{1}{1-2V_D f} \cong\omega_Q{}^2+\tfrac{1}{3}(m/M)Z^2v_f{}^2Q^2, \quad \text{(A8)}$$

using the fact that $f\cong-2N(0)$. This is a well-known result in this approximation.[2]

Now we shall study the effect of superconductivity on the phonons. Adding $iv_Q{}^iV_D{}^{-1}$ times the second column to the first simplifies the determinant to

$$\begin{vmatrix} f_Q{}^2-\nu^2-2|v_Q{}^i|^2V_D{}^{-1} & 2iv_Q{}^{i*} & 0 \\ iv_Q{}^i/V_D & 1-2V_D f & l \\ 0 & 2V_D h & 1-g \end{vmatrix}=0,$$

which, expanded, is

$$\nu^2-\omega_Q{}^2=2|v_Q{}^i|^2V_D{}^{-1}\{1-2V_D[f+lh/(1-g)]\}^{-1}. \quad \text{(A9)}$$

In this form the analogy with (A8) is clear: we have simply replaced the damping factor $(1-2V_Df)^{-1}$ by $\{1-2V_D[f+lh/(1-g)]\}^{-1}$. In the normal case, f is a constant and the singularity in V_D makes the second term the large one; now $f\propto Q^2$ and if the phonon frequency is to remain similar $lh/(1-g)$ must have the correct value. Fortunately, in the "neutral case" calculations of Sec. V we have the values of l, h, and $(1-g)$ in the appropriate limit, and we get

$$\frac{lh}{1-g}=-2N(0)\left(1+\frac{\nu^2}{\langle\omega_{kQ}{}^2\rangle_{Av}}\right). \quad \text{(A10)}$$

Since phonon frequencies are quite small relative to ω_{kQ}, this agrees well with the normal value (A8). The velocity change caused by superconductivity, given by the last term, is of order m/M or $\sim10^{-4}$. Whether this is physical depends on whether our calculation is really accurate for these very small terms.

Note added in proof.—Dr. A. W. Overhauser has pointed out to me that an effect agreeing with this correction in order of magnitude and sign was measured on Sn and Pb by B. Welber and S. L. Quimby, Acta Metallurgica **6**, 351 (1958).

APPENDIX II. TOTAL ENERGY CALCULATIONS

As Wentzel has pointed out, in the simple Sawada-Brout method the energy change caused by the interactions is given simply by the change in zero-point energy summed over all the excitations. Unfortunately, even the inclusion of phonon and photon exchange effects complicates this simple prescription very much. We shall show briefly here how the energy might be calculated on our method, but since it is relatively unimportant physically—representing only a small change from the B.C.S. result—we shall not make any attempt at evaluating it. This appendix is included primarily just to show that the equation of motion method is a complete and satisfactory substitute for the diagram method, for the energy as well as for the excitation spectrum which it exhibits so naturally. Since this is the purpose, we confine the calculation to the case of an ordinary space potential:

$$V=\textstyle\sum_Q V(Q)\rho^{Q*}\rho^Q, \quad \text{(A11)}$$

and then the potential energy (the kinetic can be obtained by the trick of integrating with respect to e^2)[8] is

$$\text{P.E.}=\textstyle\sum_Q V(Q)(\rho^Q\Psi_0,\rho^Q\Psi_0). \quad \text{(A12)}$$

The essential point of the method is to expand ρ^Q in terms of the eigenexcitations x_ν (64) (we shall work entirely with momentum $\mathbf{Q}$ alone, so we omit the $\mathbf{Q}$ index where possible hereafter) which we expand [instead of (64)] for convenience in Bogoliubov form:

$$\begin{aligned} x_\nu&=\textstyle\sum_k(\lambda_{\nu k}\alpha_{k+Q0}{}^*\alpha_{k1}{}^*+\mu_{\nu k}\alpha_{k+Q1}\alpha_{k0}), \\ x_{-\nu}&=\textstyle\sum_k(\mu_{\nu k}\alpha_{k+Q0}{}^*\alpha_{k1}{}^*-\lambda_{\nu k}\alpha_{k+Q1}\alpha_{k0}). \end{aligned} \quad \text{(A13)}$$

We also shall need the inverse transformation

$$\begin{aligned} \alpha_{k+Q0}{}^*\alpha_{k1}{}^*&=\textstyle\sum_\nu(l_{k\nu}x_\nu+m_{k\nu}x_{-\nu}), \\ \alpha_{k+Q1}\alpha_{k0}&=\textstyle\sum_\nu(m_{k\nu}x_\nu-l_{k\nu}x_{-\nu}). \end{aligned} \quad \text{(A14)}$$

Simply by substitution of (A14) in (A13) we get

$$\sum_{\mathbf{k}} (\lambda_{\nu\mathbf{k}} l_{\mathbf{k}\nu'} + \mu_{\nu\mathbf{k}} m_{\mathbf{k}\nu'}) = \delta_{\nu\nu'}. \tag{A15}$$

Let us define two quantities which relate ρ and x_ν, and which could if necessary be written out in terms of the expansion coefficients:

$$U_\nu = (x_\nu \Psi_0, \rho \Psi_0) = (\Psi_0, (x_\nu^* \rho - \rho x_\nu^*)\Psi_0); \qquad \sum_\nu Y_\nu x_\nu = \rho. \tag{A16}$$

Then clearly the term in the energy (A12) for momentum Q is

$$V(Q) \sum_{\nu>0} U_\nu Y_\nu. \tag{A17}$$

[U_ν is zero for $\nu<0$ by (67).]

We find that the equations of motion, together with (A15), give us a relationship for $U_\nu Y_\nu$. Suppose, for example, that we assume that (90) and (91) are the correct equations of motion. A little manipulation leads us to the following equations for the α's:

$$\nu \alpha_{\mathbf{k}+Q0}^* \alpha_{\mathbf{k}1}^* = \nu_{\mathbf{k}}^Q \alpha_{\mathbf{k}+Q0}^* \alpha_{\mathbf{k}1}^* + (1/2\nu) f_1(\mathbf{k}) \Phi_{\mathbf{k}}^Q, \tag{A18a}$$

$$\nu \alpha_{\mathbf{k}+Q1} \alpha_{\mathbf{k}0} = -\nu_{\mathbf{k}}^Q \alpha_{\mathbf{k}+Q1} \alpha_{\mathbf{k}0} - (1/2\nu) f_1(\mathbf{k}) \Phi_{\mathbf{k}}^Q, \tag{A18b}$$

where $f_1(\mathbf{k})$ is the complicated but known function

$$f_1(\mathbf{k}) = \cos[\tfrac{1}{2}(\theta_{\mathbf{k}+Q} - \theta_{\mathbf{k}})][b_{\mathbf{k}} + b_{\mathbf{k}+Q}] + \sin[\tfrac{1}{2}(\theta_{\mathbf{k}+Q} - \theta_{\mathbf{k}})] n_{\mathbf{k}Q}. \tag{A19}$$

Note added in proof.—G. Rickaysen has pointed out to me that Eqs. (A18) are in error, and that only their consequence

$$[\nu^2 - (\nu_{\mathbf{k}}^Q)^2](\alpha_{\mathbf{k}+Q0}^* \alpha_{\mathbf{k}1}^* - \alpha_{\mathbf{k}+Q1} \alpha_{\mathbf{k}0}) = f_1(\mathbf{k}) \Phi_{\mathbf{k}}^Q$$

is valid. The principle of the energy calculation is not affected by this error, and also (A24) is still correct.

Now we can derive equations both for the λ's and the l's from (A18). We take the average value in the ground state of the commutator of x_ν^* first with (A18a) and then (A18b). This gives

$$\lambda_{\nu\mathbf{k}}(\nu - \nu_{\mathbf{k}}^Q) = (2\nu)^{-1} f_1(\mathbf{k})(x_\nu \Psi_0, \Phi_{\mathbf{k}}^Q \Psi_0), \qquad \mu_{\nu\mathbf{k}}(\nu + \nu_{\mathbf{k}}^Q) = (2\nu)^{-1} f_1(\mathbf{k})(x_\nu \Psi_0, \Phi_{\mathbf{k}}^Q \Psi_0). \tag{A20}$$

Second, we expand (A18) by means of (A14). Then if

$$\Phi_{\mathbf{k}}^Q = \sum_\nu \varphi_{\mathbf{k}\nu} x_\nu, \tag{A21}$$

we have

$$(\nu - \nu_{\mathbf{k}}^Q) l_{\mathbf{k}\nu} = (2\nu)^{-1} f_1(\mathbf{k}) \varphi_{\mathbf{k}\nu} = -(\nu + \nu_{\mathbf{k}}^Q) m_{\mathbf{k}\nu}. \tag{A22}$$

The only remaining task is to study $\Phi_{\mathbf{k}}^Q$. To show what might be done with this let us take the simple case of Sawada-Brout, where

$$\Phi_{\mathbf{k}}^Q = 2V(Q) \Pi^Q.$$

Since

$$(x_\nu \Psi_0, \Pi^Q \Psi_0) = \nu U_\nu, \tag{A23}$$

(A20) becomes

$$\lambda_{\nu\mathbf{k}} = f_1(\mathbf{k})(\nu - \nu_{\mathbf{k}}^Q)^{-1} U_\nu V(Q), \qquad \mu_{\nu\mathbf{k}} = -f_1(\mathbf{k})(\nu + \nu_{\mathbf{k}}^Q)^{-1} U_\nu V(Q),$$

while, using (A16) and the definition of Π^Q,

$$\varphi_{\mathbf{k}\nu} = 2V \nu Y_\nu,$$

so that (A22) becomes

$$l_{\mathbf{k}\nu} = V Y_\nu (\nu - \nu_{\mathbf{k}}^Q)^{-1}, \qquad m_{\mathbf{k}\nu} = -V Y_\nu (\nu + \nu_{\mathbf{k}}^Q)^{-1}.$$

Then (A15) gives us our desired relation for $U_\nu Y_\nu$:

$$U_\nu Y_\nu \sum_{\mathbf{k}} V^2(Q) f_1^2(\mathbf{k}) [(\nu - \nu_{\mathbf{k}}^Q)^{-2} - (\nu + \nu_{\mathbf{k}}^Q)^{-2}] = 1. \tag{A24}$$

Eventually this leads to exactly the simple result of Sawada, when we insert the correct value of $f_1(\mathbf{k})$.

In the more general case $\Phi_{\mathbf{k}}^Q$ does not consist of Π^Q alone. In any soluble case, however, it consists of a finite number of sums like A^Q and B^Q which we must perforce consider as constants, and the solution of the problem involves finding a linear equation set like (92)–(94) connecting the parts of Φ. If this is so we can solve (92)–(94) for the A^Q, B^Q, and any further such sums in terms of Π^Q; then the only change in the theory is a redefinition of the $\mathbf{k}$- and ν-dependent quantity $f_1(\mathbf{k})$, which will have to contain factors coming from the solution for Φ in terms of Π.