Superpotentials for a Generally Covariant Field Theory

JAMES L. ANDERSON
Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey
(Received May 23, 1958)

An expression for the superpotentials corresponding to the energy-momentum complex of a generally covariant field theory is presented. It is somewhat more general than previous expressions in that it makes no special assumptions regarding the symmetry properties of the Lagrangian of the theory.

I. INTRODUCTION

It has long been known that one can construct an energy momentum complex, $T_\mu{}^\nu$, in general relativity which satisfies the conservation law

$$T_\mu{}^\nu{}_{,\nu}=0. \tag{1}$$

This law is a consequence of the covariance properties of the theory and represent a form of the Bianchi identities. Because these equations are identities, it follows that there must exist a superpotential complex $U_\mu{}^{[\nu\sigma]}$, antisymmetric in the indices ν and σ, such that

$$T_\mu{}^\nu=U_\mu{}^{[\nu\sigma]}{}_{,\sigma}. \tag{2}$$

Here the square braces indicate that the U's are antisymmetric in the indices ν and σ. An explicit statement of the form of the superpotentials for general relativity was first given by Freud.[1] However, his method was applicable only to this case and somewhat later Bergmann and Schiller[2] investigated the structure of the superpotentials for a much wider class of theories. While the approach employed by them was quite general, the applicability of their results was limited by certain assumptions concerning the nature of the Lagrangian of the theory.

In the course of an investigation of the secondary constraints in general relativity, we have had occasion to employ a Lagrangian which differs from the usual one by a complete divergence and does not satisfy the assumptions of the B. S. (Bergmann-Schiller) theory. Since it was necessary for our purposes to obtain the superpotentials for this form of the theory, we have had to generalize the B. S. results and this extention is presented below.

II. THE SUPERPOTENTIALS

In deriving their results, B. S. assumed that the Lagrangian density of the theory, L, differed from an invariant density by a known divergence, $S^\rho{}_{,\rho}$. As a consequence of the transformation law for L, they derived a set of identities satisfied by the field variables y_A which were somewhat more extensive than the Bianchi identities. After correcting for some typographical errors in the B. S. paper, they are of the form

$$[C_{A\mu}{}^\nu\partial^{A\rho}S^\sigma]_{(\nu\rho\sigma)}=0, \tag{3a}$$

$$L_\nu{}^{[\mu\sigma]}+L_\nu{}^{[\sigma\mu]}=0, \tag{3b}$$

$$T_\mu{}^\nu=V_\mu{}^{\nu\sigma}{}_{,\sigma}, \tag{3c}$$

where

$$L_\nu{}^{[\mu\rho]}=C_{A\nu}{}^\mu\partial^{A\rho}L+\delta_\nu{}^\mu S^\rho-\delta_\nu{}^\rho S^\mu+C_{A\nu}{}^\mu\partial^A S^\rho +C_{A\nu}{}^\mu{}_{,\sigma}\partial^{A\sigma}S^\rho-y_{A,\nu}\partial^{A\mu}S^\rho+(C_{A\nu}{}^\mu\partial^{A\rho}S^\sigma)_{,\sigma}, \tag{4}$$

and

$$V_\nu{}^{\mu\rho}=L_\nu{}^{[\mu\rho]}-(C_{A\nu}{}^\mu\partial^{A\rho}S^\sigma)_{,\sigma}. \tag{5}$$

In Eq. (3a) the notation $(\nu\rho\sigma)$ indicates that the expression within the brackets is to be summed over six terms that are completely symmetric with respect to the indices ν, ρ, σ. The quantities $C_{A\nu}{}^\mu$ are the "structure constants" which appear in the transformation law for the y_A, namely

$$\bar{\delta}y_A=C_{A\nu}{}^\mu\xi^\nu{}_{,\mu}-y_{A,\nu}\xi^\nu, \tag{6}$$

where the ξ^ν are the descriptors of the transformation.[3]

Equation (3c) gives us $T_\mu{}^\nu$ as the divergence of $V_\mu{}^{\nu\sigma}$. However, these latter quantities are not satisfactory superpotentials since, as we see from Eqs. (3b) and (5), they are not antisymmetric in ν and σ. In the B. S. paper, $V_\mu{}^{\nu\sigma}$ was antisymmetrized by adding to it the divergence of another skew-symmetric density. Although it is not explicitly stated there, their method is based on the assumption that

$$\partial^{A\rho}S^\sigma=\partial^{A\sigma}S^\rho, \tag{7}$$

which need not in general be true. If we restrict ourselves to Lagrangians of the usual type, then S^σ will have the form

$$S^\sigma=(\Gamma^{A(\rho\sigma)}+\Gamma^{A[\rho\sigma]})y_{A,\rho}, \tag{8}$$

with

$$\Gamma^{A(\rho\sigma)}=\Gamma^{A(\sigma\rho)};\quad \Gamma^{A[\rho\sigma]}=-\Gamma^{A[\sigma\rho]},$$

where $\Gamma^{A(\rho\sigma)}$ and $\Gamma^{A[\rho\sigma]}$ are functions of the undifferentiated field variables. Since the superpotentials depend linearly on S^σ, we can consider the symmetric and antisymmetric parts of S^σ separately. The symmetric part of S^σ can be treated exactly as in the B. S.

[1] Ph. von Freud, Ann. Math. 40, 417 (1939).

[2] P. G. Bergmann and R. Schiller, Phys. Rev. 89, 4 (1953).

[3] In the above equations, we have employed the following abbreviations: $\partial^A\equiv\partial/\partial y_A$; $\partial^{A\rho}\equiv\partial/\partial y_{A,\rho}$; the subscript μ denotes $\partial/\partial x^\mu$.

paper. We make use of the identity

$$(C_{A\nu}{}^{\mu}\partial^{A\rho}S^{\sigma})_{,\rho\sigma}=[C_{A\nu}{}^{\mu}\partial^{A\rho}S^{\sigma}+\tfrac{1}{3}(C_{A\nu}{}^{\sigma}\partial^{A\mu}S^{\rho}-C_{A\nu}{}^{\rho}\partial^{A\mu}S^{\sigma})]_{,\rho\sigma}.$$

The quantity in the square brackets is, however, antisymmetric in μ and ρ. In fact, by making use of the identities (3a) and the assumed symmetry properties of $\Gamma^{A(\rho\sigma)}$, it can be written as

$$\tfrac{2}{3}(C_{A\nu}{}^{\mu}\partial^{A\rho}S^{\sigma}-C_{A\nu}{}^{\rho}\partial^{A\mu}S^{\sigma})_{,\rho\sigma}.$$

Thus, we can write

$$U_{\nu}{}^{[\mu\rho]}=L_{\nu}{}^{[\mu\rho]}+\tfrac{2}{3}(C_{A\nu}{}^{\rho}\partial^{A\mu}S^{\sigma}-C_{A\nu}{}^{\mu}\partial^{A\rho}S^{\sigma})_{,\sigma}. \quad (9)$$

The antisymmetric part of S^{σ} can be treated in an even simpler manner. In this case we have directly that

$$(C_{A\nu}{}^{\mu}\partial^{A\rho}S^{\sigma})_{,\rho\sigma}\equiv 0, \quad (10)$$

and so we can take

$$U_{\nu}{}^{[\mu\rho]}=L_{\nu}{}^{[\mu\rho]}. \quad (11)$$

We can combine Eqs. (9) and (11) to obtain finally

$$U_{\nu}{}^{[\mu\rho]}=L_{\nu}{}^{[\mu\rho]}+\tfrac{2}{3}(C_{A\nu}{}^{\rho}\Gamma^{A(\mu\sigma)}-C_{A\nu}{}^{\mu}\Gamma^{A(\rho\sigma)})_{,\sigma}. \quad (12)$$

In this form, the expressions for the superpotentials are general enough to be applicable to most problems of interest.