Electromagnetic Effects in Meson-Nucleon Scattering*

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The dominant part of pion-nucleon scattering is charge independent. In view of the apparent discrepancies in the dispersion relations for meson scattering, it is desirable to investigate the effects on meson-nucleon scattering of including the electromagnetic interaction in an otherwise charge-independent theory. The calculation, which is performed in the static-nucleon, one-meson approximation of Chew and Low, is divided into two parts. In the first part, the effects of the charged-neutral meson mass difference, which is assumed to be of electromagnetic origin, are calculated. In the second part, other electromagnetic effects involving one virtual photon are calculated in the "Coulomb approximation," in which the effects of transverse photons and of graphs in which the incoming and outgoing meson lines are crossed, are ignored. A formula is given by means of which the three meson-proton differential cross sections may be analyzed in terms of the six charge-independent s- and p-wave phase shifts which would occur in the absence of the electromagnetic interaction. Aside from simple Coulomb and kinematic effects, the most important effect in π^+-proton scattering can be expressed as an alteration of the $J=\frac{3}{2}$ phase shift for that state such as to sharpen the (3,3) resonance and move it to a higher energy. The meson mass difference effect and the Coulomb effect contribute about equally to produce a phase shift alteration of about 2° at its largest.

I. INTRODUCTION

General Remarks

IT is well established that the major features of processes involving the interaction of nucleons with each other and with pi mesons are charge independent. There is, for example, evidence favoring charge independence of nuclear forces from the measured energy levels of light nuclei,[1] from proton-proton and neutron-proton scattering data,[2] and from reactions involving pi mesons.[3] To observe the charge independence one must first take into account the effect of the Coulomb force, since, although it is relatively weak, it is a charge-dependent force. It is thus widely believed that the fundamental strong meson-nucleon interaction, thought to be responsible for nuclear forces as well as for processes involving the scattering and production of pions, is exactly charge independent, and that all observed deviations from charge independence must arise from the electromagnetic interaction.

It is the purpose of this investigation to examine the effect on pion-nucleon scattering of including the electromagnetic interaction in an otherwise charge-independent meson theory. Electromagnetic effects in meson scattering have been of particular interest recently because of the apparent discrepancies, pointed out by Puppi and Stanghellini,[4] between experimental meson scattering data and the Goldberger dispersion relations.[5] The effect of the charged-neutral meson mass difference and the Coulomb effect on s-wave scattering have been calculated phenomenologically by Noyes.[6] The effect of the meson mass difference on the dispersion relations has been calculated by Agodi and Cini,[7] and the effect of the Coulomb field by Agodi, Cini, and Vitale.[8] A more recent discussion of electromagnetic effects on dispersion relations has been given by Chew and Noyes.[9] Effects of the meson mass difference computed in a manner similar in some respects to that used here have been reported by Chiu[10] and by Greenberger.[11] Except for the work of Chiu and Greenberger, recent papers are concerned with the electromagnetic effect on the form of the dispersion relations, rather than with the nature of the effect itself.

The Method

We take as our Lagrangian

$$L=L_N+L_M+L_\gamma+L_I, \qquad (1)$$

where

$$L_I=-g\bar{\psi}\gamma_5\tau_i\psi\phi_i+\bar{\psi}\delta m\psi+\tfrac{1}{2}\delta\mu_i^2\phi_i^2 -\tfrac{1}{4}\lambda_i[\phi_i\phi_i]^2-e\bar{\psi}\gamma^\mu\frac{1+\tau_3}{2}\psi A_\mu +ie\left(\frac{\partial\phi^\dagger}{\partial x^\mu}\phi-\phi^\dagger\frac{\partial\phi}{\partial x^\mu}\right)A^\mu+e^2A_\mu A^\mu\phi^\dagger\phi. \qquad (2)$$

L_N, L_M, and L_γ are the usual free Lagrangians for the

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[1] See for example J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), pp. 259–261.

[2] Reference 1, p. 94.

[3] M. Gell-Mann and K. M. Watson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 219.

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[5] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

[6] H. P. Noyes, Phys. Rev. **101**, 320 (1956).

[7] A. Agodi and M. Cini, Nuovo cimento **5**, 1256 (1957).

[8] Agodi, Cini, and Vitale, Phys. Rev. **107**, 630 (1957).

[9] G. F. Chew and H. P. Noyes, Phys. Rev. **109**, 566 (1958).

[10] Hong-Yee Chiu, Bull. Am. Phys. Soc. Ser. II, **3**, 10 (1958).

[11] D. M. Greenberger, Bull. Am. Phys. Soc. Ser. II, **3**, 11 (1958).

nucleons, mesons, and photons, respectively. In the interaction Lagrangian (2) we have taken the charge-independent pseudoscalar coupling of the pseudoscalar meson field to the nucleon field, and the gauge-invariant coupling of the mesons and nucleons to the electromagnetic field. This Lagrangian (1) is a function of the coupling constants, e and g, which appear in L_I, and of the experimental masses, μ_+, μ_0, and m, of the mesons and nucleons which appear in L_M and L_N, respectively. (For this calculation we consider the proton and neutron masses equal.) We assume that the charged-neutral meson mass difference is all electromagnetic, which means that the bare meson masses, $\mu_+-\delta\mu_+$ and $\mu_0-\delta\mu_0$, are equal when the electromagnetic charge e in L_I has its experimental value.

We now separate the electromagnetic effects in meson-nucleon scattering into two parts. We first solve the scattering problem using the Lagrangian (1), but with e set equal to zero. This Lagrangian will not lead to charge-independent scattering since the charged and neutral meson masses that appear in it are not equal. The problem is relatively simple, however, since the electromagnetic field variables are not present. This problem is treated in the static approximation in Sec. II using the effective-range approximation of Chew and Low.[12] The connection between the Lagrangian (1) and the Hamiltonian (3) of the static approximation, particularly with respect to the connection between the coupling constants, will be discussed with the numerical results of the meson mass-difference effect.

There is no reason to think that this effect will be the most important electromagnetic effect in pion-nucleon scattering, for if the charge e is turned on once more, there are many processes which can occur involving the emission and reabsorption of a virtual photon which have nothing to do with the meson self energy. All such processes should be considered, but this is too difficult to do at present. We consider in Sec. III that part of these processes which we refer to as Coulomb processes. In this Coulomb approximation we ignore the effects of transverse photons and the effects of graphs in which the incoming and outgoing meson lines are crossed. The remaining effects are of two types. First, there is the "simple Coulomb effect," which adds to the nuclear scattering amplitude the amplitude for a simple scattering of the meson by the Coulomb field of the nucleon. Effects more or less equivalent to this have been calculated by Ashkin and Smith,[13] Van Hove,[14] and Solmitz.[15] Second, there are "rescattering effects" in which the meson undergoes both Coulomb and nuclear scatterings. Calculation of these rescattering effects requires a more detailed knowledge of the meson-nucleon interaction. It is then hoped that the meson mass-difference effect together with the Coulomb effect will be the major part of the entire electromagnetic effect in meson scattering.

TABLE I. Charge representation scattering processes. The paired scatterings have the same amplitude due to charge symmetry and the charge-exchange scatterings are equal due to time-reversal invariance.

Designation	Initial state	(scatters into)	Final state
$+$	$\begin{cases}\pi^++p\\ \pi^-+n\end{cases}$		π^++p π^-+n
$-$	$\begin{cases}\pi^-+p\\ \pi^++n\end{cases}$		π^-+p π^++n
e'	$\begin{cases}\pi^-+p\\ \pi^++n\end{cases}$		π^0+n π^0+p
e	$\begin{cases}\pi^0+n\\ \pi^0+p\end{cases}$		π^-+p π^++n
0	$\begin{cases}\pi^0+n\\ \pi^0+p\end{cases}$		π^0+n π^0+p

II. THE MESON MASS DIFFERENCE EFFECT

The Hamiltonian

The effect of the charged-neutral π-meson mass difference on the scattering of mesons by nucleons is calculated using the Chew-Low-Wick[16–18] formalism. Following the notation of the paper by Chew and Low,[12] referred to hereafter as C-L, we take as our Hamiltonian

$$H=H_0+H_I, \tag{3-a}$$

where

$$H_I=\sum_k (V_k^{(0)}a_k+V_k^{(0)\dagger}a_k^\dagger), \tag{3-b}$$

$$H_0=\sum_k a_k^\dagger a_k\omega_k, \tag{3-c}$$

and

$$V_k^{(0)}=i(f_{(r)}^{(0)}/\mu)_k[\boldsymbol{\sigma}\cdot\mathbf{k}/(2\omega_k)^{\frac{1}{2}}]\tau_k v(k). \tag{4}$$

This differs from the Hamiltonian in C-L in two essential ways. First, the possibility is allowed that the unrenormalized coupling constants for charged and neutral meson emission may be different. Second, the relation between ω_k and k is different for charged and neutral mesons, the relation being $\omega_k=(\mu_k^2+k^2)^{\frac{1}{2}}$. The sum over k is to be done using states of definite charge, μ_k then being the mass of the charged or neutral meson as the case may be.

The expression for the scattering matrix follows in exactly the same way as in C-L and we have

$$\langle n|S|q\rangle=\delta_{nq}-2\pi i\delta(E_q-E_n)T_q(n), \tag{5}$$

where

$$T_q(n)=(\psi_n^{(-)},V_q^{(0)}\psi_0). \tag{6}$$

The Low equation can also be derived in exactly the

[12] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956), referred to in the text as C-L.

[13] J. Ashkin and L. Smith, Carnegie Institute of Technology Report, 1952 (unpublished).

[14] L. Van Hove, Phys. Rev. **88**, 1358 (1952).

[15] F. T. Solmitz, Phys. Rev. **94**, 1799 (1954).

[16] G. F. Chew, Phys. Rev. **95**, 1669 (1954).

[17] F. E. Low, Phys. Rev. **97**, 1392 (1955), referred to in the text as L.

[18] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

same fashion as in C-L and we have

$$T_q(p)=-\sum_n\left[\frac{T_p^\dagger(n)T_q(n)}{E_n-\omega_p-i\epsilon}+\frac{T_q^\dagger(n)T_p(n)}{E_n+\omega_p}\right]. \quad (7)$$

We define as in C-L

$$t_{qp}(z)=-\sum_n\left[\frac{T_p^\dagger(n)T_q(n)}{E_n-z}+\frac{T_q^\dagger(n)T_p(n)}{E_n+z}\right], \quad (8)$$

so that

$$T_q(p)=\lim_{z\to\omega_p+i\epsilon} t_{qp}(z). \quad (9)$$

As a result of using the modified Hamiltonian (3), the theory is not charge independent. The theory is, nevertheless, still charge symmetric since we shall assume that the unrenormalized coupling constants for the coupling of neutral mesons to protons and neutrons are equal, and that the proton and neutron masses are equal. Of course the coupling constants as well as the masses of the positive and negative mesons are assumed to be equal.

The One-Meson Approximation

In the one-meson approximation we set, in analogy with Sec. IV of C-L,

$$t_{qp}(z)=-v(q)v(p)2\pi(\omega_q\omega_p)^{-\frac{1}{2}}[pqh(z)]. \quad (10)$$

In (10), the quantity $[pqh(z)]$ is a matrix connecting the states in terms of which the scattering is to be analyzed. Since the angular momentum J is conserved, we consider states of definite angular momentum, the states $J=\frac{3}{2}$ and $J=\frac{1}{2}$ being designated by 3 and 1, respectively. In this representation, h is a diagonal matrix with respect to the angular momentum. Thus for purposes of computation one may use the angular momentum projection operators introduced in C-L.

Because of the factor pq in (10), and since isotopic spin is not conserved, it is convenient to use states of definite charge. The scattering processes are then shown in Table I.

If we use state vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix}=|0,2J\rangle,\quad \begin{bmatrix}0\\1\\0\end{bmatrix}=|-,2J\rangle,\quad \begin{bmatrix}0\\0\\1\end{bmatrix}=|+,2J\rangle, \quad (11\text{a–c})$$

we have

$$[pqh(z)]=\begin{bmatrix}p_0q_0h_{0,2J} & p_0q_+h_{e,2J} & 0\\ p_+q_0h_{e,2J} & p_+q_+h_{-,2J} & 0\\ 0 & 0 & p_+q_+h_{+,2J}\end{bmatrix}, \quad (12)$$

where $p_0=(\omega_p^2-\mu_0^2)^{\frac{1}{2}}$ and $p_+=(\omega_p^2-\mu_+^2)^{\frac{1}{2}}$. Equation (12) may be simply expressed by introducing diagonal matrices

$$[p]=\begin{bmatrix}p_0 & 0 & 0\\ 0 & p_+ & 0\\ 0 & 0 & p_+\end{bmatrix},\quad [q]=\begin{bmatrix}q_0 & 0 & 0\\ 0 & q_+ & 0\\ 0 & 0 & q_+\end{bmatrix}, \quad (13)$$

so that

$$[pqh(z)]=[p][h(z)][q], \quad (14)$$

where matrix multiplication is indicated and $[h(z)]$ is the matrix (12) without the factors of p and q.

Since isotopic spin will be nearly conserved except near threshold, it will be useful to know the connection between the matrix $[h]$ in the charge representation (14) and in the representation in which states of definite isotopic spin I and its z component I_z are used. The scattering processes are then shown in Table II.

We use state vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix}=|1,2J\rangle,\quad \begin{bmatrix}0\\1\\0\end{bmatrix}=|3,2J\rangle,\quad \begin{bmatrix}0\\0\\1\end{bmatrix}=|4,2J\rangle. \quad (15)$$

Aside from the factors of pq, the matrix of amplitudes in this isotopic representation, indicated by $[h^I]$, is related to the matrix of amplitudes in the charge representation $[h]$ by

$$[h^I]=U[h]U^{-1}, \quad (16)$$

where

$$U=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & -\sqrt{2} & 0\\ \sqrt{2} & 1 & 0\\ 0 & 0 & \sqrt{3}\end{bmatrix}. \quad (17)$$

This result may be obtained by writing $[h^I]$ in terms of projection and exchange operators as in C-L where only projection operators are needed, and then calculating the matrix elements between the appropriate charge states.

Now as in C-L the function $t_{qp}(z)$ has a pole at the origin, of residue

$$(f/\mu)_p(f/\mu)_q 2\pi v(p)v(q)(\omega_p\omega_q)^{-\frac{1}{2}} \times(\boldsymbol{\sigma}\cdot\mathbf{p}\tau_p\boldsymbol{\sigma}\cdot\mathbf{q}\tau_q-\boldsymbol{\sigma}\cdot\mathbf{q}\tau_q\boldsymbol{\sigma}\cdot\mathbf{p}\tau_p). \quad (18)$$

We now observe that the factors of p and q included in (10) are just matched by the factor qp of the residue so that each matrix of amplitudes $[h_1]$ and $[h_3]$ in the charge representation has a pole at the origin, of residue

$$[\Lambda_1]=\begin{bmatrix}-\frac{4}{3}(f/\mu)_0^2 & \frac{2}{3}\sqrt{2}(f/\mu)_+(f/\mu)_0 & 0\\ \frac{2}{3}\sqrt{2}(f/\mu)_+(f/\mu)_0 & -2(f/\mu)_+^2 & 0\\ 0 & 0 & -\frac{2}{3}(f/\mu)_+^2\end{bmatrix}, \quad (19)$$

$$[\Lambda_3]=\begin{bmatrix}\frac{2}{3}(f/\mu)_0^2 & \frac{2}{3}\sqrt{2}(f/\mu)_+(f/\mu)_0 & 0\\ \frac{2}{3}\sqrt{2}(f/\mu)_+(f/\mu)_0 & 0 & 0\\ 0 & 0 & \frac{4}{3}(f/\mu)_+^2\end{bmatrix},$$

respectively. The set $[\Lambda_1]$ refers to $J=\frac{1}{2}$ and the set

TABLE II. Isotopic-spin representation scattering processes. The exchange scatterings, 2 and 2′, occur with the same amplitude due to time-reversal invariance.

Designation	Initial state I	Initial state I_z	(scatters into)	Final state I	Final state I_z
4	$\frac{3}{2}$	$\pm\frac{3}{2}$		$\frac{3}{2}$	$\pm\frac{3}{2}$
3	$\frac{3}{2}$	$\pm\frac{1}{2}$		$\frac{3}{2}$	$\pm\frac{1}{2}$
2′	$\frac{3}{2}$	$\pm\frac{1}{2}$		$\frac{1}{2}$	$\pm\frac{1}{2}$
2	$\frac{1}{2}$	$\pm\frac{1}{2}$		$\frac{3}{2}$	$\pm\frac{1}{2}$
1	$\frac{1}{2}$	$\pm\frac{1}{2}$		$\frac{1}{2}$	$\pm\frac{1}{2}$

$[\Lambda_3]$ to $J=\frac{3}{2}$. The coefficients of f_+ and f_0 appearing here are the renormalized, unrationalized coupling constants defined by

$$f_0^{(0)}(\psi_0^\alpha,\boldsymbol{\sigma}\cdot\mathbf{q}\tau_3\psi_0^\beta)=f_0(u_\alpha,\boldsymbol{\sigma}\cdot\mathbf{q}\tau_3u_\beta), \qquad (20\text{-a})$$

$$f_+^{(0)}(\psi_0^\alpha,\boldsymbol{\sigma}\cdot\mathbf{q}\tau_\pm\psi_0^\beta)=f_+(u_\alpha,\boldsymbol{\sigma}\cdot\mathbf{q}\tau_\pm u_\beta), \qquad (20\text{-b})$$

where ψ_0^α is the state vector for a physical nucleon and u_α is the bare-nucleon spinor.

The crossing theorem of C-L leads to the relation

$$h_{i,\alpha}(z)=\sum_{\beta,j}A_{\alpha\beta}B_{ij}h_{j,\beta}(-z), \qquad (21)$$

where

$$A_{11}=-\tfrac{1}{3},\quad A_{13}=\tfrac{4}{3},\quad A_{31}=\tfrac{2}{3},\quad A_{33}=\tfrac{1}{3}, \qquad (22\text{-a})$$

$$B_{00}=B_{-+}=B_{+-}=1,\quad B_{ee'}=B_{e'e}=-1, \qquad (22\text{-b})$$

and the rest of the B's are zero. As in C-L, the A's and B's satisfy the relations

$$\sum_{\beta,j}A_{\alpha\beta}B_{ij}A_{\beta\gamma}B_{jk}=\delta_{\alpha\gamma}\delta_{ik}, \qquad (23\text{-a})$$

and

$$\sum_{\beta,j}A_{\alpha\beta}B_{ij}\Lambda_{j,\beta}=-\Lambda_{i,\alpha}. \qquad (23\text{-b})$$

From the definition of $t_{qp}(z)$ [Eq. (8)] and its relation to $T_q(p)$ [Eq. (9)], we may write

$$t_{qp}(\omega_p)=-\sum_n\left[\frac{t_{pn}^\dagger(E_n)t_{qn}(E_n)}{E_n-\omega_p-i\epsilon}+\text{crossing terms}\right]. \qquad (24)$$

That part of the sum with $E_n=0$ gives the pole (19) already discussed. In the one-meson approximation we include in addition only those terms with $E_n=\omega_n$. Then converting the sum to an integral by using $\sum_n\to\int d^3n/(2\pi)^3$ and $ndn=\omega_n d\omega_n$, we obtain from (24) the following integral equation for the matrix $[h]$:

$$[h(\omega_p)]=[\Lambda]/\omega_p+\frac{1}{\pi}\times\int_0^\infty|v(n)|^2d\omega_n\frac{[h(\omega_n)]^\dagger[n]^3[h(\omega_n)]}{\omega_n-\omega_p-i\epsilon}, \qquad (25)$$

plus crossing terms which can easily be computed using (22). Matrix multiplication is indicated on the right-hand side and the superscript dagger means the Hermitian conjugate of the matrix. We see from (25) that the matrix $[h]$ satisfies the relation

$$\mathrm{Im}[h]=\tfrac{1}{2}([h]-[h]^\dagger)=i|v|^2[h]^\dagger[p]^3[h], \qquad (26)$$

which is the analog of the unitarity condition in C-L.

The Reciprocal Matrix

Further simplification occurs and an effective-range treatment in complete analogy with that in C-L becomes possible if we now introduce a new matrix

$$[g(z)]=z^{-1}[h(z)]^{-1}, \qquad (27)$$

where this equation means that the matrix product $[g][h]$ is to equal z^{-1} times a unit matrix. We now examine the properties of $[g(z)]$. First we note that the pole of $[h(z)]$ at the origin implies

$$[g(0)]=[\Lambda]^{-1}, \qquad (28)$$

the matrix reciprocal of $[\Lambda]$. The unitarity condition (26) is quite simple. It implies

$$\mathrm{Im}[g(\omega_p)]=i(|v|^2/\omega_p)[p]^3, \qquad (29)$$

where $[p]$ is the diagonal matrix defined by (13). Thus we can write, in analogy with C-L,

$$\mathrm{Re}[g(\omega)]=[\Lambda]^{-1}-\frac{\omega}{\pi}\left\{P\int d\omega_p\frac{|v(p)|^2}{\omega_p^2(\omega_p-\omega)}[p]^3+\int d\omega_p\frac{|v(p)|^2}{\omega_p^2(\omega_p+\omega)}[H(\omega_p)]\right\}. \qquad (30)$$

Here P indicates that the principal value of the integral is to be taken and $[H(\omega_p)]$ is a matrix function which can be determined from the crossing theorem. We write the crossing theorem (21) as

$$[h(z)]=[AB(h(-z))], \qquad (31)$$

where we must remember AB does not operate on $h(-z)$ as an ordinary matrix. Then we have

$$[AB(g(z))^{-1}]=-[g(-z)]^{-1}, \qquad (32)$$

the imaginary part of which is

$$AB([g^\dagger(z)]^{-1}\,\mathrm{Im}[g(z)][g(z)]^{-1})=-[g^\dagger(-z)]^{-1}\,\mathrm{Im}[g(-z)][g(-z)]^{-1}, \qquad (33)$$

since the transformation AB does not alter the Hermiticity of the matrix. Then from (30) we see at once that

$$[H(\omega_p)]=[g(-\omega_p)]^\dagger[AB([g^\dagger(\omega_p)]^{-1}[p]^3[g(\omega_p)]^{-1})]\times[g(-\omega_p)]. \qquad (34)$$

The inclusion of this term in the equations for $[g]$, (29) and (30), makes that equation a set of nonlinear coupled integral equations for the amplitudes.

The Renormalized Coupling Constants

In the discussion of the numerical results of the meson mass-difference effect, we shall consider the effects of various possible alternatives concerning the unrenormalized coupling constants. Before presenting these results, however, we shall consider the effect of renormalizing the coupling constants. By using the method developed by Cini and Fubini[19] we explicitly show that

[19] M. Cini and S. Fubini, Nuovo cimento 3, 764 (1956).

the renormalized coupling constants are completely determined by the unrenormalized coupling constants. The relations between them involve integrals of the reduced scattering amplitudes over the meson energy. From these relations one can estimate the effect of renormalizing the coupling constants. The relations also lead to certain conditions on the amplitudes themselves.

In the charge-symmetric meson theory which we are considering, there are five renormalization constants as opposed to the two constants of the Chew-Low-Wick theory. These five are defined by

$$(\psi_0{}^\alpha,\tau_3\psi_0{}^\beta)=(1/\rho_0)\,(u_\alpha,\tau_3u_\beta), \qquad (35\text{-a})$$

$$(\psi_0{}^\alpha,\tau_\pm\psi_0{}^\beta)=(1/\rho_+)\,(u_\alpha,\tau_\pm u_\beta), \qquad (35\text{-b})$$

$$(\psi_0{}^\alpha,\sigma_p\psi_0{}^\beta)=(1/\rho_1)\,(u_\alpha,\sigma_p u_\beta), \qquad (35\text{-c})$$

$$(\psi_0{}^\alpha,\tau_3\sigma_p\psi_0{}^\beta)=\;Z_0\;(u_\alpha,\tau_3\sigma_p u_\beta), \qquad (35\text{-d})$$

$$(\psi_0{}^\alpha,\tau_\pm\sigma_p\psi_0{}^\beta)=\;Z_+\;(u_\alpha,\tau_\pm\sigma_p u_\beta). \qquad (35\text{-e})$$

In the paper by Cini and Fubini the two renormalization constants are

$$1/\rho_+=1/\rho_0=1/\rho_1, \quad\text{and}\quad Z_0=Z_+\equiv 1/\rho_2. \qquad (36)$$

These relations among the renormalization constants should be approximately true in our case. From (20a,b) we see that

$$f_0=Z_0f_0{}^{(0)}, \qquad (37\text{-a})$$

and

$$f_+=Z_+f_+{}^{(0)}. \qquad (37\text{-b})$$

From (7) it follows, as in C-L, that

$$T_p{}^\dagger(q)-T_q(p)=2\pi i\sum_n\delta(E_n-\omega_p)T_p{}^\dagger(n)T_q(n). \qquad (38)$$

Then using (6), (8), and (9) we get

$$t_{qp}(z^*)-t_{qp}(z)=2\pi i\sum_n\delta(E_n-\omega_q)\times(\psi_0,V_p{}^{(0)}\psi_n{}^{(-)})(\psi_n{}^{(-)},V_q{}^{(0)}\psi_0), \qquad (39)$$

where $z\rightarrow\omega_q+i\epsilon$ in the first term, $z\rightarrow\omega_p+i\epsilon$ in the second term, and the expression holds for $\omega_p=\omega_q$. Since $t_{qp}(z)$ is a Hermitian matrix function of z, we have $[h(z)]=[h(z^*)]^\dagger$ as in C-L. Thus (39) can be written using (4) and (10) as

$$4i[p]\,\mathrm{Im}[h(\omega_p)][q]=(f_{(r)}{}^{(0)}/\mu)_p(f_{(r)}{}^{(0)}/\mu)_q\sum_n\delta(E_n-\omega_q)\times(\psi_0,\boldsymbol{\sigma}\cdot\mathbf{p}\tau_p\psi_n{}^{(-)})(\psi_n{}^{(-)},\boldsymbol{\sigma}\cdot\mathbf{q}\tau_q\psi_0). \qquad (40)$$

Using the unrationalized coupling constants, we may now write

$$\frac{i}{\pi}\int_{\mu0}^{\infty}\mathrm{Im}[h(\omega_p)]d\omega_p=\left(\frac{f^{(0)}}{\mu}\right)_p\left(\frac{f^{(0)}}{\mu}\right)_q\{(\psi_0,\boldsymbol{\sigma}\cdot\hat{p}\tau_p\boldsymbol{\sigma}\cdot\hat{q}\tau_q\psi_0)-\sum_\alpha(\psi_0,\boldsymbol{\sigma}\cdot\hat{p}\tau_p\psi_0{}^\alpha)(\psi_0{}^\alpha,\boldsymbol{\sigma}\cdot\hat{q}\tau_q\psi_0)\}, \qquad (41)$$

where the first term on the right comes from the closure property of the $\psi_n{}^{(-)}$, and the second term is the correction to the first due to the $\delta(E_n-\omega_q)$, which does not permit the $\psi_n{}^{(-)}$ to include the real nucleon states. The caret indicates a unit vector.

We wish to write the right side of (41) in terms of bare spinors and thus reduce it to our operator notation. We need

$$(\psi_0,\boldsymbol{\sigma}\cdot\hat{p}\boldsymbol{\sigma}\cdot\hat{q}\tau_p\tau_q\psi_0)=\hat{p}\cdot\hat{q}\delta_{pq}+(i/\rho_1)\delta_{pq}\boldsymbol{\sigma}\cdot\hat{p}\times\hat{q}+(i/\rho_{p\times q})\hat{p}\cdot\hat{q}\epsilon_{ipq}\tau_i-Z_{p\times q}\boldsymbol{\sigma}\cdot\hat{p}\times\hat{q}\epsilon_{ipq}\tau_i, \qquad (42\text{-a})$$

and

$$\sum_\alpha(\psi_0,\tau_p\boldsymbol{\sigma}\cdot\hat{p}\psi_0{}^\alpha)(\psi_0{}^\alpha,\boldsymbol{\sigma}\cdot\hat{q}\tau_q\psi_0)=Z_pZ_q(\delta_{pq}+i\epsilon_{ipq}\tau_i)(\hat{p}\cdot\hat{q}+i\boldsymbol{\sigma}\cdot\hat{p}\times\hat{q}). \qquad (42\text{-b})$$

In these expressions the subscript $p\times q$ indicates the meson type i of ϵ_{ipq}. The matrix elements of (41) may now be computed, and if we use the notation

$$H_{i,\alpha}=\frac{3}{\pi}i\left(\frac{\mu}{f^{(0)}}\right)_p\left(\frac{\mu}{f^{(0)}}\right)_q\int_{\mu0}^{\infty}\mathrm{Im}h_{i,\alpha}(\omega)d\omega, \qquad (43)$$

where p and q are the two meson types indicated by the subscript designation i (see Table I), then (41) may be written

$$\begin{aligned}
&H_{0,1}=1+2/\rho_1-3Z_0{}^2,\\
&H_{0,3}=1-1/\rho_1,\\
&H_{e,1}=H_{e',1}=\sqrt{2}(-1/\rho_+-2Z_++3Z_0Z_+),\\
&H_{e,3}=H_{e',3}=\sqrt{2}(-1/\rho_++Z_+),\\
&H_{-,1}=1+2/\rho_1+1/\rho_0+2Z_0-6Z_+{}^2,\\
&H_{-,3}=1-1/\rho_1+1/\rho_0-Z_0,\\
&H_{+,1}=1+2/\rho_1-1/\rho_0-2Z_0,\\
&H_{+,3}=1-1/\rho_1-1/\rho_0+Z_0.
\end{aligned} \qquad (44)$$

Since (36) is approximately true, it is convenient to introduce small quantities defined by

$$r_0=1/\rho_0-1/\rho_1, \qquad (45\text{-a})$$

$$r_+=1/\rho_+-1/\rho_1, \qquad (45\text{-b})$$

$$z_+=Z_+-Z_0. \qquad (45\text{-c})$$

We now consider as our five renormalization constants, Z_0, $1/\rho_1$, z_+, r_0, and r_+, and we discard terms quadratic in the last three since they are small compared to the first two. It is clear at this stage that the quantity of interest, namely z_+, can be obtained in terms of the $H_{i,\alpha}$'s. It is also clear that relations exist among the $H_{i,\alpha}$'s that do not involve the renormalization constants since there are eight equations and only five constants. We shall now write (44) in a different form which shows more directly the deviation from charge independence.

Since isotopic spin is nearly conserved, if we change our $H_{i,\alpha}$'s to the isotopic representation using (16), the following relations which are exactly satisfied in C-L will be approximately satisfied here:

$$H_{4,3}{}^I-H_{3,3}{}^I=0, \tag{46-a}$$

$$H_{4,1}{}^I-H_{3,1}{}^I=0, \tag{46-b}$$

$$H_{4,1}{}^I-H_{1,3}{}^I=0, \tag{46-c}$$

$$H_{2,1}{}^I=0, \tag{46-d}$$

$$H_{2,3}{}^I=0, \tag{46-e}$$

where the second subscript is $2J$ and the first is the isotopic designation (see Table II). We now write Eqs. (46) exactly, in terms of the charge representation $H_{i,\alpha}$'s. Using (44) and (45), we get in the same order as (46a-e)

$$H_{+,3}-\tfrac{2}{3}H_{0,3}-\tfrac{2}{3}\sqrt{2}H_{e,3}-\tfrac{1}{3}H_{-,3} =-\tfrac{4}{3}r_0+\tfrac{4}{3}r_+-\tfrac{4}{3}z_+, \tag{47-a}$$

$$H_{+,1}-\tfrac{2}{3}H_{0,1}-\tfrac{2}{3}\sqrt{2}H_{e,1}-\tfrac{1}{3}H_{-,1} =-\tfrac{4}{3}r_0+\tfrac{4}{3}r_++(8/3)z_+, \tag{47-b}$$

$$H_{+,1}-\tfrac{1}{3}H_{0,3}+\tfrac{2}{3}\sqrt{2}H_{e,3}-\tfrac{2}{3}H_{-,3} =-(5/3)r_0-\tfrac{4}{3}r_++\tfrac{4}{3}z_+, \tag{47-c}$$

$$\tfrac{1}{3}\sqrt{2}H_{0,1}-\tfrac{1}{3}H_{e,1}-\tfrac{1}{3}\sqrt{2}H_{-,1} =-\tfrac{1}{3}\sqrt{2}r_0+\tfrac{1}{3}\sqrt{2}r_++\tfrac{2}{3}\sqrt{2}z_++3\sqrt{2}Z_0z_+, \tag{47-d}$$

$$\tfrac{1}{3}\sqrt{2}H_{0,3}-\tfrac{1}{3}H_{e,3}-\tfrac{1}{3}\sqrt{2}H_{-,3} =-\tfrac{1}{3}\sqrt{2}r_0+\tfrac{1}{3}\sqrt{2}r_+-\tfrac{1}{3}\sqrt{2}z_+. \tag{47-e}$$

Including two more equations

$$2H_{+,1}+H_{+,3}+3H_{-,3}=6(1-Z_0), \tag{47-f}$$

$$H_{+,3}+H_{-,3}+H_{0,1}=3(1-Z_0{}^2), \tag{47-g}$$

makes the seven equations (47) equivalent to the eight equations (44) with the renormalization constant ρ_1 eliminated. Various relations follow at once from (47). Particularly simple is the relation obtained by combining (47-a) and (47-e). This gives

$$H_{+,3}-2H_{0,3}+H_{-,3}=0. \tag{48}$$

This means that if the unrenormalized coupling constants $(f^{(0)}/\mu)_+$ and $(f^{(0)}/\mu)_0$ are equal, the following relation is exactly true even though the charged and neutral meson masses are not equal:

$$\mathrm{Im}\int_{\mu_0}^{\infty}[h_{+,3}(\omega)-2h_{0,3}(\omega)+h_{-,3}(\omega)]d\omega=0.$$

This relation could not have been anticipated from considerations of the charge symmetry of the theory alone.

We now return to our original purpose in this section, finding the effect of renormalizing the coupling constants. A relatively simple expression for z_+ is obtained from (47-b) and (47-d) as follows:

$$H_{+,1}-2H_{0,1}+H_{-,1}=-12Z_0z_+. \tag{49}$$

This may be considered an equation for z_+ since the constant Z_0 has been calculated in the Chew-Low-Wick theory by Miyazawa[20] as

$$f/f^{(0)}=Z_0=0.65. \tag{50}$$

We then have

$$z_+/Z_0=-[3i/(12\pi Z_0{}^2)]\,\mathrm{Im}\int_{\mu_0}^{\infty}[(\mu/f^{(0)})_+{}^2(h_{+,1}-h_{-,1}) -2(\mu/f^{(0)})_0{}^2h_{0,1}]d\omega. \tag{51}$$

This quantity (51) is a measure of the meson mass-difference effect on the renormalization of the coupling constants since, from (37) and (45),

$$\frac{(f/\mu)_+}{(f/\mu)_0}=\frac{(f^{(0)}/\mu)_+}{(f^{(0)}/\mu)_0}\left(1+\frac{z_+}{Z_0}\right). \tag{52}$$

From (51) it appears likely that z_+/Z_0 is quite small. We shall show in fact that it is probably less than $\tfrac{1}{3}\%$. The smallness of (51) is easily seen. First, all of the amplitudes involved are "small" experimentally up to energies well beyond resonance. Second, the coefficient is a small number,

$$3/(12\pi Z_0{}^2)=0.19. \tag{53}$$

Numerical Results for the Meson Mass Difference Effect

We wish to determine how the cross sections for meson-nucleon scattering are affected by the charged-neutral meson mass difference. We examine the mass-difference effect on the reduced scattering amplitudes, $[h]$, given by (27), (29), and (30). We use the effective-range approximation of C-L which consists of setting

$$\mathrm{Re}[g(\omega)]=[\Lambda]^{-1}-\omega[r(\omega)], \tag{54}$$

and then ignoring the ω dependence of $[r(\omega)]$. We write

$$[h(\omega_p)]=\frac{\omega_p{}^{-1}}{[\Lambda]^{-1}-\omega_p[r]-i(|v|^2/\omega_p)[p]^3}. \tag{55}$$

The charged-neutral meson mass difference will then appear in $[\Lambda]$, which depends on the renormalized coupling constants, in $[r]$, and in $[p]$, since the momenta of charged and neutral mesons are different at a given energy.

We shall first show that the effect on $[r]$ is small. From (30) we see that $[r]$ is the sum of two terms. The first term is a principal-value integral which we explicitly evaluated as a function of ω for $[p]\to p_0=(\omega_p{}^2-\mu_0{}^2)^{\frac{1}{2}}$ and $[p]\to p_+=(\omega_p{}^2-\mu_+{}^2)^{\frac{1}{2}}$, and for two values of the cutoff energy. The effect of changing the cutoff energy in this term, which must be done if the cutoff function v is a function of the momentum, is to produce only a 0.1% difference in the terms for charged and neutral mesons. The effect of the factor of $[p]^3$ in

[20] H. Miyazawa, Phys. Rev. 101, 1564 (1956).

this integral is somewhat larger. For $\omega<1.9\mu$ the integral is larger for neutral mesons and for $\omega>1.9\mu$ the integral is larger for charged mesons. For $1.5\mu<\omega<2.5\mu$ the effect is always less than $\frac{1}{2}\%$. The second term in $[r]$ is an integral involving $H(\omega_p)$ given by (34). This integral is too complicated to evaluate, but the factor of p^3 and the nonvanishing denominator should make the high-energy part of the integral, where we expect little deviation from charge independence, the most important part. Thus we expect the meson mass-difference effect in $[r]$ to be small and we shall ignore this effect.

Next we shall show that the meson mass-difference effect on $[\Lambda]$ is probably small. There are two possible causes for a difference between the renormalized coupling constants for neutral and charged mesons. First, the bare coupling constants will depend on the Lagrangian that is postulated. We are using (2) as our interaction Lagrangian. The relation between the unrenormalized pseudoscalar coupling constant $g^{(0)}$ of (2) and the coupling constant $(f^{(0)}/\mu)$ of the interaction Hamiltonian (3) and (4) is

$$(f^{(0)}/\mu)=g^{(0)}/2m, \tag{56}$$

where m is the nucleon mass. The coupling constants $g^{(0)}$ must be equal for charged and neutral mesons since all deviations from isotopic spin independence are assumed to be of electromagnetic origin. Thus the quantity $(f^{(0)}/\mu)$ is taken to be the same for neutral and charged mesons and the difference is all due to the effect of renormalization. There are electromagnetic effects other than the meson mass difference which produce a renormalized coupling constant difference. However, these effects are zero in the Coulomb approximation considered in the next section (see Appendix A).

To calculate the meson mass difference effect on the renormalization, we use (51) for the quantity z_+/Z_0. The "small" amplitudes, $h_{\alpha,1}$ of (51), are not known with sufficient accuracy for experimental results to be of any use in evaluating (51). Thus we use the effective-range formula (55). The value of $[r]$ is also not known for the $J=\frac{1}{2}$ amplitudes, and thus in order to get an estimate of the possible size of (51) we take $[r]=0$. Since the fractional coupling-constant difference will turn out to be small compared to the mass difference, we may ignore the effect of the $[\Lambda]$ term in the effective-range formula (55), for the amplitudes needed in (51), and consider only the effect of the $[p]^3$ term. Under these conditions the integral of (51) may be explicitly evaluated, and a numerical integration shows that

$$z_+/Z_0=-0.002, \tag{57}$$

verifying the statement at the end of the section entitled "The Renormalized Coupling Constants." This is sufficiently small that the effect due to (57) will only change the cross sections by 1% or 2%. Because it is not large, and because of the uncertainty in the method of computing the coupling constant difference, we shall not include this effect but shall indicate after the next paragraph how it may be calculated.

Finally, we consider the effect of the $[p]^3$ term in the denominator of (55). Since only the charged-neutral meson mass difference has significance we choose $\bar{\mu}$, the bare mass (as far as electromagnetic effects are concerned), in such a way as to give the effect in the simplest form. This choice is

$$\bar{\mu}=\mu_+-\tfrac{2}{3}\delta\mu=\mu_0+\tfrac{1}{3}\delta\mu. \tag{58}$$

We now set $[h]=[\bar{h}]+\delta[h]$, where $[\bar{h}]$ is the reduced scattering amplitude obtained when the meson masses are $\bar{\mu}$. From (55) we see, for $\delta[h]\ll[\bar{h}]$, that

$$\delta[h]\ (\text{mass difference})=i[\bar{h}]\delta([p]^3)[\bar{h}], \tag{59}$$

where

$$\delta([p]^3)=3\bar{p}\bar{\mu}\delta\mu\begin{bmatrix}\frac{1}{3} & 0 & 0\\ 0 & -\frac{2}{3} & 0\\ 0 & 0 & -\frac{2}{3}\end{bmatrix}. \tag{60}$$

Here $\bar{p}=(\omega_p{}^2-\bar{\mu}^2)^{\frac{1}{2}}$ and we have set $|v|^2=1$. For purposes of simplification and because it should give the largest effect, we consider only the $I=\frac{3}{2}$, $J=\frac{3}{2}$ part of the amplitude $[\bar{h}]$ in (59). Then we have

$$[\bar{h}^I{}_{J=\frac{3}{2}}]=\begin{bmatrix}0 & 0 & 0\\ 0 & h_{3,3} & 0\\ 0 & 0 & h_{3,3}\end{bmatrix}, \tag{61}$$

and from (16)

$$[\bar{h}_{J=\frac{3}{2}}]=\tfrac{1}{3}\begin{bmatrix}2 & \sqrt{2} & 0\\ \sqrt{2} & 1 & 0\\ 0 & 0 & 3\end{bmatrix}h_{3,3}. \tag{62}$$

The other amplitudes can easily be included in the $[\bar{h}]$ of (59), but with only the (3,3) state we get the simple result

$$\delta[h_{J=\frac{3}{2}}]=-2ih_{3,3}{}^2\bar{p}\bar{\mu}\delta\mu\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{63}$$

with only the π^+-proton scattering amplitude in the $J=\frac{3}{2}$ state altered. Then, since

$$h_{3,3}=e^{i\delta_{33}}\sin\delta_{33}(1/\bar{p}^3), \tag{64}$$

we have for this amplitude, corrected by the meson mass-difference effect,

$$\begin{aligned}h_{+,3}&=e^{i\delta_{33}}\sin\delta_{33}(1/\bar{p}^3)-2i\bar{\mu}\delta\mu e^{2i\delta_{33}}\sin^2\delta_{33}(1/\bar{p}^5)\\ &=e^{i[\delta_{33}+\Delta_{+,3}(m)]}\sin[\delta_{33}+\Delta_{+,3}(m)]\\ &\qquad\times(1+2\bar{\mu}\delta\mu/\bar{p}^2)/\bar{p}^3,\end{aligned} \tag{65}$$

where the magnitude of the amplitude is increased and the phase shift is changed by an amount

$$\Delta_{+,3}(m)=-2(\bar{\mu}\delta\mu/\bar{p}^2)\sin\delta_{33}\cos\delta_{33}. \tag{66}$$

The effect of this term is to sharpen the 3,3 resonance in the π^+-proton state and to increase the magnitude of the amplitude.

If the coupling-constant difference were known with sufficient accuracy to warrant its inclusion, its effect could be calculated in the same approximation as that used above. In fact, if we use

$$(\bar{f}/\bar{\mu})=(f/\mu)_{+}+\tfrac{2}{3}\delta(f/\mu)=(f/\mu)_{0}-\tfrac{1}{3}\delta(f/\mu), \quad (67)$$

we get

$\delta[h_{J=\frac{3}{2}}]$ (coupling constant difference)

$$=\omega_p[\bar{h}][\bar{\Lambda}]^{-1}\delta[\Lambda][\bar{\Lambda}]^{-1}[\bar{h}]$$
$$=-\omega_p h_{3,3}{}^2(\bar{\mu}/\bar{f})^3\delta(f/\mu)\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad (68)$$

so that the only effect is to increase the (3,3) phase shift for π^+-proton scattering by an amount

$$\Delta_{+,3}(f)=-(\omega_p/\bar{p}^3)(\bar{\mu}/\bar{f})^3\delta(f/\mu)\sin^2\delta_{33}. \quad (69)$$

After discussing the Coulomb effect we shall write down expressions for the cross sections for elastic π^+ and π^- scattering on protons and for charge-exchange scattering. In these expressions we shall include two other simple kinematical effects of the π mass difference. First, the p-wave scattering amplitudes [see (10)–(12)] contain a factor pq which will be larger for charge-exchange scattering than for elastic scattering at a given energy. Second, the cross sections themselves contain a factor q/p where q and p are the momenta of the outgoing and incoming mesons, respectively. This factor, which comes from the incident flux and the density of final states, will increase the charge-exchange scattering.

III. THE COULOMB EFFECT

The Integral Equation

We proceed in the manner of Low,[17] referred to hereafter as L. Using our interaction Lagrangian (2), we may write down the result for the S matrix for scattering of a meson of momentum q and type i to momentum q' and type j, the nucleon going from momentum p to p', analogous to equation (1.11) of L:

$$\begin{aligned}\langle p'q_j'|S|pq_i\rangle &=(-i)^2\int dxdy(4\omega_q\omega_{q'})^{-\frac{1}{2}}e^{-iqx+iq'y}\\ &\quad\times\langle p'|P\{\mathcal{O}_j(y)\mathcal{O}_i(x)\}|p\rangle\\ &\quad-i\lambda\int dx(4\omega_q\omega_{q'})^{-\frac{1}{2}}e^{-i(q-q')x}\\ &\quad\times\langle p'|P\{\delta_{ij}\phi_k(x)\phi_k(x)+2\phi_i(x)\phi_j(x)\}|p\rangle\\ &\quad-ie^2\int dx(4\omega_q\omega_{q'})^{-\frac{1}{2}}e^{-i(q-q')x}\\ &\quad\times\langle p'|P\{A^\mu(x)A_\mu(x)\epsilon_{ik3}\epsilon_{jk3}\}|p\rangle\\ &\quad+ie\int dx(4\omega_q\omega_{q'})^{-\frac{1}{2}}e^{-i(q-q')x}\\ &\qquad\times\langle p'|P\{(q^\mu+q^{\mu\prime})A_\mu(x)i\epsilon_{ij3}\}|p\rangle, \quad (70)\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}_i(x)&=(-\square^2-\mu^2)\phi_i(x)\\ &=g\bar{\psi}\gamma_5\tau_i\psi-\delta\mu^2\phi_i+\lambda\phi_i\phi_k\phi_k-e^2\phi_jA^\mu A_\mu\epsilon_{ik3}\epsilon_{jk3}\\ &\quad+(2eA^\mu\phi_{j,\mu}+eA^\mu{}_{,\mu}\phi_j)\epsilon_{ij3}. \quad (71)\end{aligned}$$

These equations are the same as in L, except for the extra terms which arise in an obvious way from the extra terms in our Lagrangian.

Proceeding as in Sec. III of L, we may derive the integral equation to first order in e^2 in the one-meson approximation:

$$\begin{aligned}&\langle p'q_j'|\mathcal{O}_i(0)|p\rangle\\ &=-\frac{1}{(2\omega_{q'})^{\frac{1}{2}}}\left[\frac{(\psi_{p'},\mathcal{O}_j\psi_{p'+q'})(\psi_{p'+q'},\mathcal{O}_i\psi_p)}{E(p'+q')-E(p')-\omega_{q'}}\right.\\ &\quad\left.+\frac{(\psi_{p'},\mathcal{O}_i\psi_{p-q'})(\psi_{p-q'},\mathcal{O}_j\psi_p)}{E(p-q')-E(p)+\omega_{q'}}\right]\\ &\quad-\frac{1}{(2\omega_{q'})^{\frac{1}{2}}}\int\frac{d^3q''\,d^3p''}{(2\pi)^3}\\ &\quad\times\left[\frac{\langle p''q_k''|\mathcal{O}_j|p'\rangle^*\langle p''q_k''|\mathcal{O}_i|p\rangle\delta(p''+q''-p'-q')}{p_0''+\omega_{q''}-p_0'+\omega_{q'}-i\alpha}\right.\\ &\quad\left.+\frac{\langle p''q_k''|\mathcal{O}_i|p'\rangle^*\langle p''q_k''|\mathcal{O}_j|p\rangle\delta(p''+q''-p+q')}{p_0''+\omega_{q''}-p_0+\omega_{q'}}\right]\\ &\quad+e(2\omega_{q'})^{-\frac{1}{2}}\langle p'|(2iq_\mu'+\partial_\mu)A^\mu|p\rangle\epsilon_{ij3}. \quad (72)\end{aligned}$$

The term arising from the e^2 term of (71) has been omitted since it involves the two electromagnetic field operators at the same space-time point and is a meson mass renormalization cancelled by the counter term. (72) is exactly the same in form as in L except for the last term, which corresponds to the incoming and outgoing mesons interacting at the same space-time point with a photon.

We now make the usual static-nucleon approximation and also the approximation in the last term of (72), which we call X_{qp}, that the photon which interacts with the scattering meson also interacts with the static charge distribution of the nucleon to produce Coulomb scattering of the meson. Then in the notation of C-L, (72) becomes

$$\begin{aligned}T_p(q)&=-\left[\frac{T_q{}^\dagger(0)T_p(0)}{-\omega_p}+\frac{T_p{}^\dagger(0)T_q(0)}{\omega_q}\right]\\ &\quad-\int\frac{d^3n}{(2\pi)^3}\left[\frac{T_q{}^\dagger(n)T_p(n)}{\omega_n-\omega_q-i\epsilon}\right.\\ &\quad\left.+\frac{T_p{}^\dagger(n)T_q(n)}{\omega_n+\omega_q}\right]+X_{qp}, \quad (73)\end{aligned}$$

TABLE III. Table showing the proton electrostatic potential $V(r)$. $a^{-1}=(2/3)\times 0.49[10^{-13}\text{ cm}]^2$.

$a^{\frac12}r$	$(a^{\frac12}r)^{-1}$	$\dfrac{(1/e)V(r)}{a^{\frac12}}$	$(a^{\frac12}r)^{-1}-\dfrac{(1/e)V(r)}{a^{\frac12}}$
0	∞	1.127	∞
0.4	2.5	1.071	1.429
0.8	1.25	0.9274	0.3226
1.2	0.8333	0.7586	0.0747
1.6	0.6250	0.6103	0.0147
2.0	0.5000	0.4977	0.0023
2.4	0.4167	0.4164	0.0003

where p and q are initial and final meson momenta and

$$X_{qp}=-ie\epsilon_{ij3}(\omega_p+\omega_q)(4\omega_p\omega_q)^{-\frac12}\times\int d^3x\, e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}V(x), \quad (74)$$

$V(x)$ being the electrostatic potential due to the nucleon charge distribution. We now take the derivative, with respect to the electromagnetic charge e, of Eq. (73). The left-hand side is then the quantity which we desire. The first term on the right involves the derivative of the single-nucleon expectation of the operator $\mathcal{O}_i$. This term is zero in the static-nucleon, Coulomb approximation which we are making, when the meson mass effects are properly eliminated, as we shall show in Appendix A. The second term on the right will make the equation a linear integral equation for the quantity on the left. The last term, finally, is the inhomogeneous term of this integral equation.

We solve this integral equation in an approximation in which we ignore the crossing terms both in d/de of (73) and in the e-independent part of (73). This may be a reasonable approximation since the crossing term seems to be less important than the noncrossing term in the charge-independent theory, particularly for the (3,3) state. In solving the approximated integral equation we use the simplified notation

$$T_p(q)=(q|p)h_q{}^q \text{ (for } e\text{-independent part)}, \quad (75\text{-a})$$

$$X_{qp}=[q|p], \quad (75\text{-b})$$

$$e\frac{d}{de}T_p(q)=\{q^-|p\}. \quad (75\text{-c})$$

For the h's, which are essentially the same as those of the previous section, the subscript refers to the meson energy and the superscript to the isotopic type. In this notation our approximate integral equation may be written

$$\{q^-|p\}=[q|p]-\sum_l\frac{\{q|l^-\}(l|p)h_l{}^l+(q|l)\{l^-|p\}h_l{}^{l*}}{\omega_l-\omega_q-i\epsilon}. \quad (76)$$

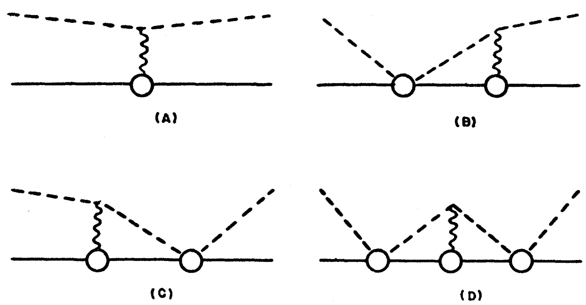

FIG. 1. Graphs representing the four terms of (77) for the Coulomb scattering of a meson by a nucleon. The solid line represents the nucleon, the dashed line the meson, and the wavy line the photon.

The solution to this equation is

$$\{q^-|p\}=[q|p]-\sum_l\frac{(q|l)[l|p]\omega_q h_q{}^q+[q|l](l|p)\omega_q h_q{}^p}{\omega_l(\omega_l-\omega_q-i\epsilon)}+\sum_{kl}\frac{(q|k)[k|l](l|p)\omega_q{}^2h_q{}^p h_q{}^q}{\omega_k\omega_l(\omega_l-\omega_q-i\epsilon)(\omega_k-\omega_q-i\epsilon)}. \quad (77)$$

The four terms in this solution correspond to the four graphs shown in Fig. 1. Equation (77) can be shown to be the solution to (76), to our approximation, by direct substitution and by use of the e-independent Low equation with the crossing terms omitted, which we write as

$$(q|p)h_q{}^q=\frac{(qp)}{\omega_q}-\sum_l\frac{(q|l)(l|p)|h_l{}^l|^2}{\omega_l-\omega_q-i\epsilon}, \quad (78)$$

(qp) being the no-meson term. Thus (77) gives us the Coulomb corrections to meson scattering.

The Simple Coulomb Scattering Term

The first term of (77) involves adding to the nuclear scattering amplitude the Born approximation to the amplitude for the Coulomb scattering of the meson by the charge distribution of the nucleon. From (74) we see that for this, as well as for the rescattering corrections, we need to know $V(x)$, the electrostatic nucleon potential. From the Hofstadter[21] experiments on electron-proton scattering we know that the proton has a root-mean-square charge radius of 0.7×10^{-13} cm, and a Gaussian charge distribution gives a good fit. The proton potential is then essentially $1/r$ times the normal probability integral which is tabulated. This potential is shown in Table III. The simple Coulomb scattering term thus includes the scattering of a point proton plus corrections to this due to the fact that the proton potential differs from $1/r$ near $r=0$. By calculating the s-wave part of this correction we showed it to be negligible. Thus, for the simple Coulomb term we keep only the point nucleon part, which has already been

[21] E. E. Chambers and R. Hofstadter, Phys. Rev. **103**, 1454 (1956).

calculated.[13–14] This term occurs with opposite sign in π^+-proton and π^--proton scattering and is absent in charge-exchange scattering.

The Rescattering Effect I

The term in (77) involving the sum on l, which we call R_1, has two terms, one of which represents the nuclear rescattering of a Coulomb-scattered meson, and the other of which represents the Coulomb rescattering of a nuclear-scattered meson. Using

$$(q|l)h_q{}^q = T_l(q) = (4\omega_l\omega_q)^{-\frac{1}{2}}\mathbf{l}\cdot\mathbf{F}(q), \tag{79}$$

and omitting the isotopic spin dependence, which will be put back in later, this term may be written

$$R_1 = -2e\int\frac{d^3l}{(2\pi)^3}\int d^3x\,\frac{[\mathbf{l}\cdot\mathbf{F}(q)]\exp[i(\mathbf{l}-\mathbf{p})\cdot\mathbf{x}]V(x)}{l^2-q^2-i\epsilon}\times\frac{(\omega_l+\omega_q)(\omega_l+\omega_p)\omega_q}{4(\omega_p\omega_q)^{\frac{1}{2}}\omega_l^2}, \tag{80}$$

which is to be evaluated for $\omega_q=\omega_p$. We now approximate the factors of (80) involving ω by a function which will enable us to perform the integration more easily. We set

$$(\omega_q+\omega_l)^2/(4\omega_l^2)\sim(7/12)+(5/12)(\omega_q{}^2/\omega_l{}^2). \tag{81}$$

This approximation is exact for $\omega_q=\omega_l$, as it must be in order that an infinite part of (80), associated with the long-range Coulomb potential, may be properly identified. For other values of the variables the approximation is quite good and should give the integral at least within 10%. Using (81) and replacing the vector $\mathbf{l}$ of the dot product in (80) by a gradient, the d^3l integration may be performed giving

$$R_1 = -[2e/(2\pi)^3]\int d^3x\,V(x)e^{-i\mathbf{p}\cdot\mathbf{x}}[-i\mathbf{F}(q)\cdot\nabla_x]\times(2\pi^2/r)[e^{iqr}-(5/12)e^{-\mu r}]. \tag{82}$$

The first term in the square bracket comes from the poles at $l=\pm(q+i\epsilon)$ and the second from the poles at $l=\pm i\mu$. Taking the derivative and doing the angular integration of d^3x, we get

$$R_1 = -2e[\hat{p}\cdot\mathbf{F}(q)]\int_0^\infty dr\,V(r)[(\cos qr)/qr-(\sin qr)/(qr)^2]\times\{[-qr\sin qr-\cos qr]+(5/12)[(1+\mu r)e^{-\mu r}]+i[qr\cos qr-\sin qr]\}. \tag{83}$$

The imaginary part of the integral is logarithmically divergent since only the p-wave part of the Coulomb scattered amplitude, which is infinite in the Born approximation, is rescattered. This infinite amplitude is exactly out of phase with the p-wave part of the nuclear amplitude, however, and so corresponds to a higher-order correction to the cross section in e^2 than the real part. In the exact theory with small Coulomb phase shifts, this term would be dropped. This term will be identified later in a discussion of s- and p-wave interference effects. The remaining integral has been evaluated, with the result

$$R_1 = -2e^2\hat{p}\cdot\mathbf{F}(q)A_1(q), \tag{84}$$

where $A_1(q)$ is a function which varies more or less linearly from a value of 0.79 at $q=0$ to 0.45 at $q=1.5\mu$. The function $A_1(q)$ is shown in Table IV.

Table IV. Table showing the function $A_1(q)$.

q/μ	$A_1(q)$
0	0.79
0.4	0.68
0.8	0.57
1.2	0.49
1.6	0.43
2.0	0.37

We have omitted the isotopic dependence of X_{qp} which is $-i\epsilon_{ij3}$. This operator has the eigenvalues $+1$, -1, and 0 for positive, negative, and neutral mesons, respectively. Thus, the R_1 term is in negative phase with the nuclear amplitude for π^+-proton scattering, and in positive phase for elastic π^--proton scattering. In charge-exchange scattering the relative phase is the same as with negative mesons, but the effect is only half as large since only one of the diagrams $(B-C)$ of Fig. 1 enters.

The Rescattering Effect II

The remaining rescattering effect is given by the term in (77) involving the sum of k and l. This term represents a nuclear scattering of the meson followed by a Coulomb scattering and then another nuclear scattering. Using expressions like (79) with $\mathbf{k}\cdot\mathbf{F}(q)$ for the first nuclear scattering and $\mathbf{l}\cdot\mathbf{G}(p)$ for the second, this term may be written

$$R_2 = \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\times\frac{[\mathbf{k}\cdot\mathbf{F}(q)]\int d^3x\exp[i(\mathbf{k}-\mathbf{l})\cdot\mathbf{x}]V(x)[\mathbf{l}\cdot\mathbf{G}(p)]}{(l^2-q^2-i\epsilon)(k^2-q^2-i\epsilon)}\times\frac{\omega_q{}^2(\omega_l+\omega_q)(\omega_k+\omega_q)(\omega_k+\omega_l)v(k)v(l)}{8\omega_k{}^2\omega_l{}^2\omega_q}. \tag{85}$$

The cutoff factors $v(k)$ and $v(l)$ are included here since

R_2 depends on the cutoff somewhat more strongly than R_1. We use the form $v(k)=\Gamma^2/(\Gamma^2+k^2)$ where $\Gamma=7\mu$. Then, to facilitate the integration, we approximate the factors of (85) depending on ω and v by the simpler form

$$\frac{1}{2}\left[\frac{\omega_q^2}{\omega_k^2}\left(\frac{\Gamma^2}{\Gamma^2+l^2}\right)+\frac{\omega_q^2}{\omega_l^2}\left(\frac{\Gamma^2}{\Gamma^2+k^2}\right)\right]. \tag{86}$$

This approximation should give the integral at least to 15%. The integrations may now be done, giving

$$R_2=\frac{e}{12\pi}[\mathbf{F}(q)\cdot\mathbf{G}(p)]\times\int_0^\infty dr\,V(r)\left[-\frac{(e^{iqr}-e^{-\Gamma r})}{r}+iqe^{iqr}+\Gamma e^{-\Gamma r}\right]\times\left[-\frac{(e^{iqr}-e^{-\mu r})}{r}+iqe^{iqr}+\mu e^{-\mu r}\right]\times\left(\frac{\Gamma^2}{q^2+\Gamma^2}\right). \tag{87}$$

The integral has been evaluated, giving

$$R_2=e^2\mathbf{F}(q)\cdot[(1+\tau_3)/2](-i\epsilon_{ij3})\mathbf{G}(p)A_2(q), \tag{88}$$

where we have included the isotopic dependence. $\mathrm{Re}A_2(q)$ is flat near resonance with the value $1.1\mu^2/12\pi$, but drops to $0.78\mu^2/12\pi$ at $q=0$. The imaginary part is much smaller at low energies, but becomes greater than the real part at resonance and above. The calculated values of $A_2(q)$ are shown in Table V.

From (77) it may be seen that, although the crossing theorem is not obeyed by that equation, the unitarity condition is still satisfied. This gives an exact relation between A_1 and A_2 of (84) and (88):

$$12\pi\,\mathrm{Im}A_2=i(2q^2A_1). \tag{89}$$

Because of the approximations made, the calculated A_1 and A_2 only satisfy this relation to about 15% accuracy. Since A_1 is more accurately known than A_2, we shall use (89) for $\mathrm{Im}A_2$ rather than the calculated values.

As with the meson mass-difference effect, we shall only include the (3,3) state in the e-independent part of $T_q(p)$ in computing R_1 and R_2. Then the isotopic spin dependence of R_2 given by (88) is quite simple. For positive mesons on protons we get the full effect with the plus sign. For negative mesons on protons there is a 2:1 chance that the meson will charge-exchange scatter in the first nuclear scattering. Since there is no R_2 effect in this case, the effect is reduced to a factor of $\frac{1}{3}$ for negative scattering or charge-exchange scattering, and the sign is opposite to that for positive mesons. A factor of 3 is obtained in R_2 from the angular momentum dependence of $\mathbf{F}\cdot\mathbf{G}$.

Since the unitarity condition is maintained by (77), the effect of R_1 and R_2 on π^+-proton scattering can be expressed as a change in the scattering phase shift. The (3,3) amplitude for this state, corrected by R_1 and R_2, is

$$\begin{aligned}T_{+,3}&=-(4\pi q^2/2\omega_q)h_{3,3}+2e^2q4\pi A_1h_{3,3}+3e^2q^2(4\pi)^2A_2h_{3,3}^2\\&=-(4\pi/2q\omega_q)e^{i[\delta_{33}+\Delta_{+,3}(R)]}\sin[\delta_{33}+\Delta_{+,3}(R)].\end{aligned} \tag{90}$$

Using (64) and (89), we get

$$\Delta_{+,3}(R)=-(2\omega_q/q)e^2[2A_1\cos\delta_{33}\sin\delta_{33}+(12\pi/q^2)\,\mathrm{Re}A_2\sin^2\delta_{33}], \tag{91}$$

sharpening the resonance and moving it to a higher energy. For π^--proton and charge-exchange scattering the effect is not expressible as a change in the phase shifts since the corrected $h_{i,j}$ is not diagonal in either the isotopic or charge representation. Rather than diagonalize the scattering matrix, we shall write the corrections to the $J=\frac{3}{2}$ amplitudes (for π^- elastic and charge-exchange scattering) in the charge representation. These amplitudes, corrected by R_1 and R_2, are

$$\begin{aligned}T_{-,3}=&-(4\pi q^2/2\omega_q)(\tfrac{1}{3}h_{3,3})-2e^2q4\pi A_1(\tfrac{1}{3}h_{3,3})\\&-e^2q^2(4\pi)^2A_2(\tfrac{1}{3}h_{3,3}^2),\end{aligned} \tag{92}$$

$$\begin{aligned}T_{e,3}=&-(4\pi q^2/2\omega_q)(\tfrac{1}{3}\sqrt{2}h_{3,3})-e^2q4\pi A_1(\tfrac{1}{3}\sqrt{2}h_{3,3})\\&-e^2q^2(4\pi)^2A_2(\tfrac{1}{3}\sqrt{2}h_{3,3}^2).\end{aligned} \tag{93}$$

The s-p-Wave Interference Effect

In scatterings which involve the Coulomb field, the scattering amplitude is the sum of the Coulomb amplitude and a nuclear amplitude, with each of the partial waves of the nuclear amplitude multiplied by $e^{2i\eta_l}$, η_l being the Coulomb phase shift for the same orbital angular momentum as the nuclear partial wave.[22] Since the η_l are small, we write this factor as $1+2i\eta_l$. The term containing the product of $2i\eta_1$ with the p-wave nuclear amplitude appeared explicitly as the imaginary part of the integral R_1 of (83). This infinite integral can be shown to correspond to the Born-approximation p-wave Coulomb phase shift by considering a cut-off Coulomb potential and letting the cutoff radius become infinite. The only effect of these terms to lowest order in the η_l will appear in the interference between partial waves of different angular momenta. Including only s and p waves, it is easily seen that the extra cross-section contribution is proportional to

$$\eta_1-\eta_0=\pm e^2/v, \tag{94}$$

for $\pi^\pm$-proton scattering. For charge-exchange scattering, the sign is as with π^--proton scattering, but with

TABLE V. Table showing the function $A_2(q)$.

q/μ	$12\pi A_2(q)$
0	$0.78\mu^2$
0.74	$(1.1+0.5i)\mu^2$
1.65	$(1.1+1.9i)\mu^2$
2.0	$(1.1+2.6i)\mu^2$

[22] L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949); see Eq. (20.24).

half the magnitude since the Coulomb field is active only during half the scattering. In Eq. (94), v is the relative velocity. This effect only alters the coefficient B of the $\cos\theta$ term of the cross section. If we include both s-wave phase shifts, δ_1 and δ_3, but only the δ_{33} p-wave phase shift, the result is

$$\Delta B_+ = -8p^{-2}(e^2/v)\sin\delta_3 \sin\delta_{33} \sin(\delta_{33}-\delta_3), \tag{95-a}$$

$$\Delta B_- = +8p^{-2}(e^2/v)[(2/9)\sin\delta_1 \sin\delta_{33}\sin(\delta_{33}-\delta_1) + (1/9)\sin\delta_3 \sin\delta_{33}\sin(\delta_{33}-\delta_3)], \tag{95-b}$$

$$\Delta B_e = +4p^{-2}(e^2/v)(2/9)[-\sin\delta_1 \sin\delta_{33}\sin(\delta_{33}-\delta_1) + \sin\delta_3\sin\delta_{33}\sin(\delta_{33}-\delta_3)]. \tag{95-c}$$

IV. CONCLUSIONS

We may now write expressions for the differential cross sections in the center-of-mass system for the three processes (+), (−), and (e), in terms of the six charge-independent phase shifts, δ_1, δ_3, δ_{33}, δ_{31}, δ_{13}, δ_{11}, and the mass-difference and Coulomb effects previously described:

$$\begin{aligned} d\sigma(+)/d\Omega = &|\{-S_3-[2T_{+,3}+T_{3,1}]\cos\theta\}\omega_p/2\pi \\ &-(\omega_p e^2)/[p_+^2\sin^2(\theta/2)]|^2 \\ &+(\omega_p/2\pi)^2|[T_{+,3}-T_{3,1}]|^2\sin^2\theta \\ &+\Delta B_+\cos\theta, \end{aligned} \tag{96-a}$$

$$\begin{aligned} d\sigma(-)/d\Omega = &|\{-[\tfrac{1}{3}S_3+\tfrac{2}{3}S_1]-[2(T_{-,3}+\tfrac{2}{3}T_{1,3}) \\ &+(\tfrac{1}{3}T_{3,1}+\tfrac{2}{3}T_{1,1})]\cos\theta\}\omega_p/2\pi \\ &+(\omega_p e^2)/[p_+^2\sin^2(\theta/2)]|^2 \\ &+(\omega_p/2\pi)^2|[(T_{-,3}+\tfrac{2}{3}T_{1,3}) \\ &-(\tfrac{1}{3}T_{3,1}+\tfrac{2}{3}T_{1,1})]|^2\sin^2\theta+\Delta B_-\cos\theta, \end{aligned} \tag{96-b}$$

$$\begin{aligned} d\sigma(e)/d\Omega = &(p_0/p_+)|\{-[\tfrac{1}{3}\sqrt{2}S_3-\tfrac{1}{3}\sqrt{2}S_1] \\ &-(p_0/p_+)[2(T_{e,3}-\tfrac{1}{3}\sqrt{2}T_{1,3}) \\ &+(\tfrac{1}{3}\sqrt{2}T_{3,1}-\tfrac{1}{3}\sqrt{2}T_{1,1})]\cos\theta\}\omega_p/2\pi|^2 \\ &+(p_0/p_+)(\omega_p/2\pi)^2|(p_0/p_+)[(T_{e,3}-\tfrac{1}{3}\sqrt{2}T_{1,3}) \\ &-(\tfrac{1}{3}\sqrt{2}T_{3,1}-\tfrac{1}{3}\sqrt{2}T_{1,1})]|^2\sin^2\theta \\ &+\Delta B_e\cos\theta. \end{aligned} \tag{96-c}$$

In (96)

$$S_i = -2\pi(p_+\omega_p)^{-1}e^{i\delta_i}\sin\delta_i, \tag{97}$$

TABLE VI. The meson mass-difference and Coulomb effects on the $J=\frac{3}{2}$, π^+-proton scattering phase shift. After the s-p-wave interference effect, which will be discussed below, and the simple Coulomb and kinematic effects [all of which are also included in (96)], have been taken into account, the indicated amount must still be subtracted from the (+, 3) phase shift to get the charge-independent part. The electromagnetic effect is to sharpen the resonance and move it to a higher energy.

p/μ	$\Delta_{+,3}(m)$	$\Delta_{+,3}(R)$	$\Delta_{+,3}$	Lab energy
0.8	−0.74°	−0.26°	−1.0°	54 Mev
1.2	−1.03°	−0.67°	−1.7°	110 Mev
1.6	−0.12°	−0.53°	−0.7°	185 Mev
2.0	+0.33°	+0.01°	+0.34°	270 Mev

δ_i being the s-wave phase shift for $I=i/2$; and

$$T_{3,1} = -2\pi(p_+\omega_p)^{-1}e^{i\delta_{31}}\sin\delta_{31}, \tag{98}$$

δ_{31} being the p-wave phase shift for $I=\frac{3}{2}$, $J=\frac{1}{2}$. Equation (98) holds also for $T_{1,3}$ and $T_{1,1}$ with the corresponding p-wave phase shifts, δ_{13} and δ_{11}. No attempt has been made to furnish kinematic corrections to these "small" p-wave amplitudes or to the s-wave amplitudes. In (96-a) we have

$$T_{+,3} = -2\pi(p_+\omega_p)^{-1}e^{i(\delta_{33}+\Delta_{+,3})}\sin(\delta_{33}+\Delta_{+,3}), \tag{99}$$

where δ_{33} is the charge-independent (3,3) p-wave phase shift and

$$\Delta_{+,3} = \Delta_{+,3}(m)+\Delta_{+,3}(R), \tag{100}$$

given by (66) and (91), respectively. In (96-b) we have

$$\begin{aligned} T_{-,3} = &-\tfrac{2}{3}\pi(p_+\omega_p)^{-1} \\ &\times[1-2(\mu\delta\mu/p_+^2)+4(\omega_p/p_+)e^2A_1]e^{i\delta_{33}}\sin\delta_{33} \\ &-[(4\pi)^2/(3p_+^4)]e^2A_2e^{2i\delta_{33}}\sin^2\delta_{33}, \end{aligned} \tag{101}$$

and in (96-c) we have

$$\begin{aligned} T_{e,3} = &-\tfrac{2}{3}\sqrt{2}\pi(p_+\omega_p)^{-1} \\ &\times[1-2(\mu\delta\mu/p_+^2)+2(\omega_p/p_+)e^2A_1]e^{i\delta_{33}}\sin\delta_{33} \\ &-[\sqrt{2}(4\pi)^2/(3p_+^4)]e^2A_2e^{2i\delta_{33}}\sin^2\delta_{33}. \end{aligned} \tag{102}$$

In these equations A_1 is given in Table IV, ReA_2 in Table V, and ImA_2 by Eq. (89). The three quantities ΔB are given by (95a-c). p_+ is the momentum of either the nucleon or mass-μ_+ meson in the center-of-mass system, computed relativistically. In (96-c) the approximation $p_0/p_+=1+\mu\delta\mu/p_+^2$ is good except at the lowest energies. θ is the angle of scattering of the meson. To the extent to which we have included all electromagnetic effects, an analysis of the pion-nucleon scattering data according to (96) should give the six charge-independent phase shifts for mesons of mass $\bar{\mu}$ [see Eq. (58)] which would occur in the absence of all electromagnetic effects.

To indicate how large these effects are, we look at π^+-proton scattering, for which the effects are most simply expressed. First, we show in Table VI the magnitude of the (3,3) phase-shift alteration brought about by the meson mass-difference and the Coulomb effects.

With our choice of $\bar{\mu}$ there is no effect analogous to $\Delta_{+,3}(m)$ for π^- elastic and charge-exchange scattering. The effect analogous to $\Delta_{+,3}(R)$ cannot be expressed as a phase-shift alteration, but is given in (96) by (101) and (102). To indicate how large the effect of ΔB_+ is on the s-p-wave interference we consider the π^+-proton scattering at the energy at which $\delta_{33}-\delta_3=\frac{1}{2}\pi$. At this energy the $\cos\theta$ term is zero except for the correction, $\Delta B_+\cos\theta$, and this correction is equivalent to an increase of $\delta_{33}-\delta_3$ by an amount $2e^2\omega_p/p$ or about 1.0°. Thus, the ΔB_+ electromagnetic effect is to reduce the energy at which the $\cos\theta$ term goes to zero. The analogous effect for π^- elastic and charge-exchange scattering is given in (96) by (95b-c). The significance of all these electromagnetic effects in meson-nucleon scattering can be correctly determined only by using (96) to analyze the scattering data at all energies through the (3,3) resonance.

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APPENDIX A

Using a method similar to that which gives (70), we have in the Heisenberg representation

$$\frac{d}{de}\langle p|\,\mathcal{O}_i(x)\,|p'\rangle = -\frac{1}{e}\int dy\,dz\,\langle p|P\{\mathcal{O}_i(x)j^\mu(y)j^\nu(z)\}|p'\rangle \times\langle(A_\mu(y)A_\nu(z))_+\rangle-\frac{1}{e}\int dy\langle p|P\{J_i{}^\mu(x)j^\nu(y)\}|p'\rangle \times\langle(A_\mu(x)A_\nu(y))_+\rangle+\cdots, \quad \text{(A-1)}$$

where, to lowest order in e,

$$J_{i\mu}=e(2\partial_\mu+ik_\mu)\phi_j\epsilon_{ij3}, \quad \text{(A-2)}$$

k_μ being the four-momentum of the photon associated with the contraction $\langle(A_\mu(x)A_\nu(y))_+\rangle$ in (A-1), and where $j_\mu=-\partial L_I/\partial A^\mu$ is the usual current operator. The terms omitted in (A-1) are proportional to the derivative with respect to e of the various mass renormalization counter terms, and are cancelled by parts of the two terms of (A-1) which are written down.

The Coulomb approximation is equivalent to taking the two current operators to have the same time coordinate. Thus, in the first term on the right of (A-1), the ordering of the operators with the $\mathcal{O}_i$ between the two j_μ's never occurs. We now eliminate the time-ordering operator and rewrite this term using a sum over intermediate states:

$$\langle p|\,\mathcal{O}(x)j(y)j(z)\,|p'\rangle =\textstyle\sum_n\langle p|\,\mathcal{O}(x)\,|n\rangle\langle n|\,j(y)j(z)\,|p'\rangle. \quad \text{(A-3)}$$

Using the one-meson approximation, we now obtain an integral equation for the desired matrix element:

$$\left[e\frac{d}{de}\langle p|\,\mathcal{O}_i(\mathbf{x})\,|p'\rangle\right] =-\int dy\langle p|P\{J_i{}^\mu(\mathbf{x})j^\nu(y)\}|p'\rangle\langle(A_\mu(\mathbf{x})A_\nu(y))_+\rangle -\int d^3y\sum_{qp''}\left\{\langle p|\,\mathcal{O}_i(\mathbf{x})\,|p''q_j\rangle\frac{e^{-i\mathbf{q}\cdot\mathbf{y}}}{(2\omega_q)^{\frac{1}{2}}} \times\left[e\frac{d}{de}\langle p''|\,\mathcal{O}_j(\mathbf{y})\,|p'\rangle\right]\frac{1}{p_0''+\omega_q-p_0-i\epsilon} +\left[e\frac{d}{de}\langle p|\,\mathcal{O}_j(\mathbf{y})\,|p''\rangle\right]\frac{e^{i\mathbf{q}\cdot\mathbf{y}}}{(2\omega_q)^{\frac{1}{2}}} \times\langle p''q_j|\,\mathcal{O}_i(\mathbf{x})\,|p'\rangle\frac{1}{p_0''+\omega_q-p_0'-i\epsilon}\right\}. \quad \text{(A-4)}$$

The inhomogeneous term of (A-4) is cancelled in part by the meson mass renormalization terms. Part of the inhomogeneous term, as well as terms which have already been dropped, will produce an electromagnetic renormalization of the meson-nucleon coupling constant in addition to the renormalization brought about by the charged-neutral meson mass difference. If this effect could be calculated, it could be included in $\Delta_{+,3}(f)$ [see Eq. (69)].

If we consistently use the Coulomb approximation, however, this type of coupling-constant renormalization does not occur. From (A-2) we see that the inhomogeneous term of (A-4) has two terms, one containing the meson momentum $\partial_\mu\phi$ and the other containing the photon momentum k_μ. After a change of variables, the first of these may be written as

$$2\int dz\left\langle p\left|P\left\{\left(\frac{\partial}{\partial x^\mu}+\frac{\partial}{\partial z^\mu}\right)[\phi_i(x)j_\nu(x-z)]\right\}\right|p'\right\rangle \times\langle(A^\mu(x)A^\nu(x-z))_+\rangle, \quad \text{(A-5)}$$

to be evaluated at $x_0=0$. In the Coulomb approximation the term $\partial/\partial x^\mu$ gives a factor (p_0-p_0') since only $j_\mu=j_0$ is considered. This is zero in the static-nucleon approximation. The term $\partial/\partial z^\mu$ has one part involving $(\partial j_\nu/\partial z^\mu)g^{\mu\nu}$, which is zero, and another part, coming from the time-ordering symbol, involving a δ function of z_0 multiplied by the commutator of ϕ with j_0. The integration on z_0 makes this the commutator for equal times which is proportional to the δ function of $\mathbf{z}$. This part of (A-5) thus contains the contraction of the two electromagnetic field operators at the same space-time point. This cannot be a real effect and must correspond to part of the meson mass renormalization to be cancelled. By similar arguments the part of the inhomogeneous term of (A-4) proportional to the photon momentum k_μ is also seen to be zero.

Thus to our approximation the solution to (A-4) is zero.