Theoretical Calculations of the Scattering of π^- Mesons by Complex Nuclei*

W. F. Baker, H. Byfield, and J. Rainwater
Columbia University, New York, New York

(Received August 4, 1958)

The angular distributions of 80-Mev negative pions elastically scattered from Li, C, and Al have been analyzed on the basis of optical models by using I.B.M.-650 computers. It was not possible to fit the data with the usual V_1+iV_2 type model having either a square or diffuse edge and V_1 and iV_2 proportional to the nuclear density. A Kisslinger-type model, which takes better account of the p-wave nature of the basic π-nucleon scattering process, was also used with a modification to avoid a physically unreasonable behavior of the effective interaction. Using $r_0=1.08$ and a diffuseness parameter $a=0.25$ in a Saxon-type density distribution, good agreement with experimental data was obtained.

I. INTRODUCTION

THE preceding paper[1] reports on experimental results for the scattering of 80-Mev π^- mesons on Li, C, Al, and Cu with much better energy resolution than in previous similar measurements.[2-4] In the case of carbon the elastic scattering was clearly separated from inelastic scattering corresponding to nuclear excitation. The scattering from Li (treated as Li^7) is also favorable for separation into mainly elastic and inelastic components. The contribution of excitation of the 0.48-Mev level should be small[1] compared to the elastic scattering. Al and Cu still have unknown contributions from inelastic scattering to <5-Mev nuclear levels; however, these contributions are smaller than in previous measurements. We have thus carried out an extensive series of calculations based on optical models to see if agreement with experiment can be obtained for reasonable choices of the optical model parameters. The calculations were carried out by using I.B.M.-650 Computers.[5] The calculations were based on an exact phase shift analysis including the effect of a Coulomb field.

$$f(\theta)=f_c+f_{nc} \tag{1}$$

is the scattering amplitude where

$$f_c=\frac{\alpha}{2k_0\sin^2(\theta/2)}\exp[2i\eta_0-i\alpha\ln\sin^2(\theta/2)], \tag{2}$$

and

$$f_{nc}=\frac{1}{k_0}\sum_{l=0}^{\infty}(2l+1)\exp[i(2\eta_l+\delta_l)]\sin\delta_l P_l(\cos\theta), \tag{3}$$

where $p_0=\hbar k_0$ is the momentum at $r\to\infty$, $\alpha=Z_1Z_2e^2/\hbar v$, $\exp[2i\eta_l]=\Gamma(1+l+i\alpha)/\Gamma(1+l-i\alpha)$. The radial parts, ϕ_l, of $r\psi$ have behavior at large r

$$k_0\phi_l\to\sin[k_0r-\tfrac{1}{2}l\pi-\alpha\ln(2k_0r)+\eta_l] \tag{4}$$

for a pure Coulomb field and

$$k_0\phi_l\to\sin[k_0r-\tfrac{1}{2}l\pi-\alpha\ln(2k_0r)+\eta_l+\delta_l] \tag{5}$$

for the final radial differential equation which includes interactions.

The δ_l were found by a method of numerical integration of the radial differential equations for the model case and for the pure Coulomb case to $r\gg$nuclear edge. The details of the method of determining the δ_l in terms of these solutions have been described in Nevis Cyclotron Laboratory Report No. 71.

II. OPTICAL MODEL THEORY

The usual optical model[6,7] for the elastic scattering of particles by nuclei assumes that the nuclear interaction can be approximated by a complex nuclear potential $V_n=V_1+iV_2$, where the real and imaginary parts are respectively related to the particle-nucleon forward scattering amplitude and total cross sections. The value of V_n is usually taken proportional to the nuclear matter density, although for nucleon-nucleus scattering a somewhat larger effective nuclear size is used to account for the range of the nuclear forces. A favored choice for the nuclear density function $\rho(r)=F(r)\rho_0$ is

$$F(r)=\left[1+\exp\left(\frac{r-R}{a}\right)\right]^{-1}, \tag{6}$$

where $R=r_0A^{\frac{1}{3}}\times10^{-13}$ cm gives the half-density radius. $t=4.40a$ is the edge thickness to go from $0.9\rho_0$ to $0.1\rho_0$. (a is understood to be in units of 10^{-13} cm throughout.) This shape has been successfully used in interpreting the Stanford electron scattering experiments[8] which determine the nuclear charge distribution giving $r_0\approx1.08$ and $a=0.5$. For nucleon-nucleus elastic scattering[9-13] the

* This research is supported by the Office of Naval Research and the U. S. Atomic Energy Commission.

[1] See Baker, Rainwater, and Williams, preceding paper [Phys. Rev. **112**, 1763 (1958)].

[2] Pevsner, Rainwater, Williams, and Lindenbaum, Phys. Rev. **100**, 1419 (1955).

[3] Williams, Rainwater, and Pevsner, Phys. Rev. **101**, 412 (1956).

[4] Williams, Baker, and Rainwater, Phys. Rev. **104**, 1695 (1956).

[5] We wish to thank International Business Machines Watson Laboratory and Columbia University Hudson Laboratories for the use of their computers.

[6] K. M. Watson, Phys. Rev. **89**, 575 (1953).

[7] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

[8] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

[9] R. E. LeLevier and D. S. Saxon, Phys. Rev. **87**, 40 (1952).

[10] D. M. Chase and F. Rohrlich, Phys. Rev. **94**, 87 (1954).

[11] R. W. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).

[12] Melkanoff, Moszkowski, Nodvik, and Saxon, Phys. Rev. **101**, 507 (1956).

[13] Glassgold, Cheston, Stein, Schuldt, and Erickson, Phys. Rev. **106**, 1207 (1957).

best value of a is about the same, but the best r_0 is larger due to the range of nuclear forces.

The above theory takes account of all l values in the elementary particle-nucleon scattering process only with respect to the forward propagation of the wave front in nuclear matter. For scattering through angle θ, the nuclear interactions are treated as if the elementary scattering processes were isotropic. This is obviously wrong since for a very weak interaction an impulse (or Born) approximation should be accurate and this would modify the above predicted $f(\theta)$ by the factor $f_1(\theta)$ which is the elementary particle-nucleon (coherent average) angular distribution shape normalized such that $f_1(0)=1$. For stronger interactions it becomes incorrect to multiply the above optical model $f(\theta)$ by $f_1(\theta)$ since multiple processes become important. Kisslinger[14] has developed a theory for pion-nucleus scattering which tries to take account of the predominantly p-wave nature of the elementary pion-nucleon coherent scattering. This leads to a term in the optical model interaction proportional to $\nabla\cdot(\rho\nabla\psi)$. It contains a kinetic energy like term $\rho\nabla^2\psi$, which combines with the usual $\nabla^2\psi$ term in the wave equation to give a revised weighting of $\nabla^2\psi$ relative to the other terms, and a term $\nabla\rho\cdot\nabla\psi$ which is proportional to the rate of change of the density, and thus gives a large emphasis to the edge region in the analysis. For reasons described below, we believe that the straight Kisslinger theory should not be used. It gives a denominator term $(1+CF)$ in the interaction term U. In the center $CF\approx-1$ to -1.4, so the denominator goes to zero or changes sign to cause U to become unreasonably large and/or change sign inside the nucleus. We essentially employ an arbitrary convergence factor to prevent this while otherwise maintaining the feature of a $\nabla\cdot(\rho\nabla\psi)$ term. We have been able to obtain good fits to the experimental results using this modified theory, but not when U is taken proportional to ρ. An alternative simpler "derivation" is given which we believe involves the same approximation as used by Kisslinger, who uses the notation of Watson and Francis.[7]

Consider the wave equation

$$\nabla^2\psi+k^2\psi=U\psi, \tag{7}$$

where $U(r)$ is spherically symmetric and is different from zero only in the nucleus region. Let

$$\psi=\psi_{\text{inc}}+\psi_{\text{scatt}},$$

$$\psi_{\text{inc}}=e^{i\mathbf{k}_0\cdot\mathbf{r}},$$

$$G=\frac{e^{ikr}}{r}e^{i\mathbf{k}_f\cdot\mathbf{r}_1} \quad \text{as} \quad r\to\infty,$$

$$\nabla^2G+k^2G=-4\pi\delta(\mathbf{r}-\mathbf{r}_1),$$

$$\phi_f=e^{i\mathbf{k}_f\cdot\mathbf{r}}.$$

[14] L. S. Kisslinger, Phys. Rev. 98, 761 (1955).

Then, for large r, an exact solution has the form

$$\psi_{\text{scatt}}(\mathbf{r})=\frac{f(\theta)}{r}e^{ikr}=-\frac{1}{4\pi}\int G(U\psi)d\tau_1, \tag{8}$$

$$f(\theta)=-\frac{1}{4\pi}\langle\phi_f|U|\psi\rangle.$$

The problem is to find a suitable form for the operator U.

Suppose that only nuclear volume element $d\tau_1$ were present. Then one must take $|\psi\rangle\approx|\psi_{\text{inc}}\rangle$ in Eq. (8) for the Born approximation to apply

$$\frac{df(\theta)}{d\tau_1}\to-\frac{1}{4\pi}\langle\phi_f|U|\psi_{\text{inc}}\rangle. \tag{9}$$

This should correspond to the coherent scattering amplitude due to $\rho(r_1)N/Ad\tau_1$ neutrons and $\rho(r_1)Z/Ad\tau_1$ protons in $d\tau_1$, of the form

$$df(\theta)/d\tau_1=\rho(r_1)[s'+s\mathbf{k}_f\cdot\mathbf{k}_0], \tag{10}$$

where s' and s are related, respectively, to the $l=0$ and $l=1$ coherent scattering amplitudes per average nucleon. $\mathbf{k}_i\cdot\mathbf{k}_f$ gives the $l=1$ angular dependence, $\cos\theta$. The operator U can be written using $\mathbf{k}\cdot\rho\mathbf{k}$ in place of $\rho\mathbf{k}_f\cdot\mathbf{k}_0$ since $\langle\phi_f|$ and $|\psi_{\text{inc}}\rangle$ are eigenstates of operator $\mathbf{k}$, so operator U in Eq. (9) becomes

$$U=-4\pi[s'\rho(\mathbf{r}_1)+s\mathbf{k}\cdot\rho\mathbf{k}]. \tag{11}$$

For energies well below that of the $T=\frac{3}{2}$, $p_{\frac{3}{2}}$ resonance, the p-wave scattering amplitude does tend to vary as k^2 and the s wave to be independent of k, so Eq. (11) has the right energy dependence if s' and s are considered roughly independent of energy. The rough k^2 dependence does not continue much above 80 Mev. At the $p_{\frac{3}{2}}$ resonance the real part of the coherent scattering amplitude $\to 0$ in fact. We assume that Eq. (11) gives the form for U in terms of the elementary π^--nucleon scattering when we return to the real nucleus problem. In this case we use Eq. (8) and identify $U|\psi\rangle$ with the right side of Eq. (7) to establish the basic wave equation. The operators $\mathbf{k}$ have the form $-i\nabla$ in a space representation so we obtain an equation of the form

$$U\psi=C'k_0^2F\psi-C\nabla\cdot(F\nabla\psi). \tag{12}$$

Including center-of-mass transformation effects and differences in the elastic scattering kinematic and phase space factors for the elastic scattering by free nucleons and nucleons bound in a relatively heavy atom, we obtain (for π^- mesons only)

$$C=-\frac{4\pi A\rho_0}{k_0k_{\text{c.m.}}}\left[\frac{Z}{A}f_p(0)+\frac{N}{A}f_n(0)\right]_{l=1}, \tag{13a}$$

$$C'=-\frac{4\pi A\rho_0}{k_0k_{\text{c.m.}}}\left[\frac{Z}{A}f_p(0)+\frac{N}{A}f_n(0)\right]_{l=0}, \tag{13b}$$

where hk_0 is the meson-laboratory system momentum and $hk_{\text{c.m.}}$ is the relative motion momentum when the meson is incident on a single nucleon at rest. The factor $(k_0/k_{\text{c.m.}})$ is the amplitude solid angle transformation factor from the center-of-mass system to the laboratory system.

$$[f_p(0)]_{l=0}=(w_3+2w_1)/3k_{\text{c.m.}},$$
$$[f_p(0)]_{l=1}=(w_{31}+2w_{11}+2w_{33}+4w_{13})/3k_{\text{c.m.}},$$
$$[f_n(0)]_{l=0}=w_3/k_{\text{c.m.}},$$
$$[f_n(0)]_{l=1}=(w_{31}+2w_{33})/k_{\text{c.m.}}.$$

The subscripts refer to the isotopic spin and j of the $l=0$ or $l=1$ scattering as customary, and

$$w=\frac{1}{2i}[e^{2i\delta}-1]=e^{i\delta}\sin\delta.$$

To obtain the rest of Eq. (7) we consider the Klein-Gordon equation including the Coulomb potential V_c formed from $(E-V_c)^2=c^2p^2+\mu^2c^4$:

$$\nabla^2\psi+k_0^2\psi=[(2E-V_c)V_c/c^2p_0^2]k_0^2\psi, \tag{14}$$

and add the nuclear $U\psi$ to right side. Let $r\psi=\sum\phi_l P_l(\cos\theta)$ to give the radial differential equation

$$\frac{d^2\phi_l}{dr^2}+\left[k_0^2+\frac{l(l+1)}{r^2}\right]\phi_l=\frac{k_0^2[(C+C')F+(2E-V_c)V_c/(cp_0)^2]\phi-CF'(\phi'-\phi/r)}{(1+CF)}, \tag{15}$$

where the primes on F' and ϕ' refer to differentiation with respect to r.

If one uses the real and imaginary parts of C and C' predicted by the single nucleon scattering, this is probably reasonably correct for the real part, but overestimates the imaginary part. Brueckner[15] has suggested that the imaginary part of C' be increased to account for all absorption effects. The absorption is decreased by Pauli-principle effects but increased by quasi-deuteron absorption effects. Reasonable choices of nuclear size and shape parameters lead to the real part of C more negative than -1, and thus a zero denominator results for some intermediate $F<1$ and the expression has reverse the sign of the numerator for $F=1$ (at the center). If we note what happens for $CF=-1$ where $\nabla^2\psi\to\infty$, we note that this comes from

$$\nabla^2\psi+k_0^2\psi=-CF\nabla^2\psi+\text{other terms},$$

where the $\nabla^2\psi$ on the left is from c^2p^2 and the $-CF\nabla^2\psi$ on the right is the p-wave scattering term involving $\nabla^2\psi$. Thus $\nabla^2\psi$ varies as $(1+CF)^{-1}$ as $CF\to-1$ if we take the expression literally. We know, however, that the k^2 dependence of the p-wave pion-nucleon scattering does not continue very much above 80-Mev kinetic energy, so this infinity and reversal of sign of $\nabla^2\psi$ as CF goes through -1 should not be taken seriously. With this idea in mind we looked for the simplest way of modifying Eq. (15) to have the right side increase with increasing F but to maintain the sign associated with small F. We chose to replace $(1+CF)^{-1}$ on the right side by the first two terms $(1-CF)$ in its Taylor series expansion about $F=0$. This gives our "modified Kisslinger model" which we have mainly employed.

As a result of the above modifications, and because of various other theoretical questions, we decided that we should treat C and C' as adjustable parameters in the fitting of the theoretical to the experimental curves.

[15] We wish to thank Professor K. A. Brueckner for discussions of this matter.

The point emphasized is the *form* of Eq. (15) after modification by $(1+CF)^{-1}\to(1-CF)$ without insisting that the exact calculated values of C and C' above be used. An example of another theoretical consideration leading to this view is the effect of the attractive potential well in considerably increasing the effective meson momentum in the collisions, and the effect of the motion of the nucleons. In principle one would like to use some operator for p-waves which behaves as $\mathbf{k}\cdot\mathbf{k}$ at low k, but follows more realistically the k dependence of the p-wave scattering above 80-Mev kinetic energy. A simple power series in k^2 brings questions of commutation with ρ. When $k\to i\nabla$ in operator form there will appear higher than second derivatives of ψ in the wave equation. This would greatly complicate numerical solution of the problem. Our final choice represents one of many possible semi-empirical compromise procedures.

III. CALCULATIONS AND COMPARISON WITH EXPERIMENT

The calculations for carbon were carried out for about 100 different selections of parameters. While most attention was focussed on the "modified Kisslinger model" described above, we also tried the usual type optical model obtained by adding a $V_n=V_1+iV_2$ to V_c in Eq. (14). The family of shape parameters determined by r_0 and a in Eq. (6) was used. An approximate square well shape was obtained by choosing $a=0.01$ which was $\sim\frac{1}{6}$ of the grid spacing chosen for the numerical integrations.

Figures 1 through 3 show theoretical angular distributions for 80-Mev π^- scattering from carbon using this V_1+iV_2 type model; the bars on the experimental points show the statistical accuracy. With an "approximate square well" as r_0 is increased with the same V_1+iV_2, the computed angular distribution becomes higher at small angles, has a larger slope at small angles, and the minimum moves in toward smaller

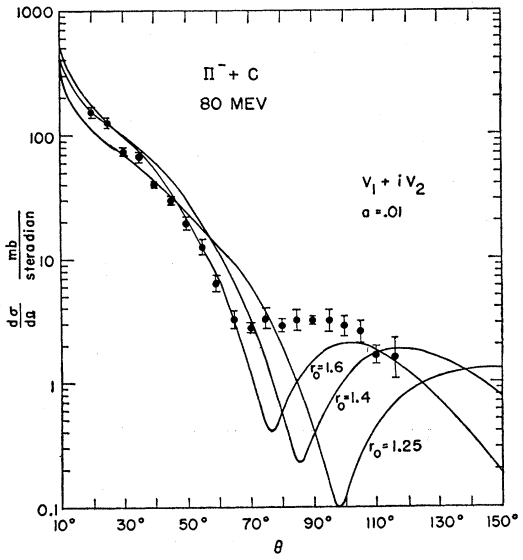

FIG. 1. Carbon. V_1+iV_2 type model with approximate square well; $a=0.01$. Variation with nuclear radius. (r_0 and a are in units of 10^{-13} cm.)

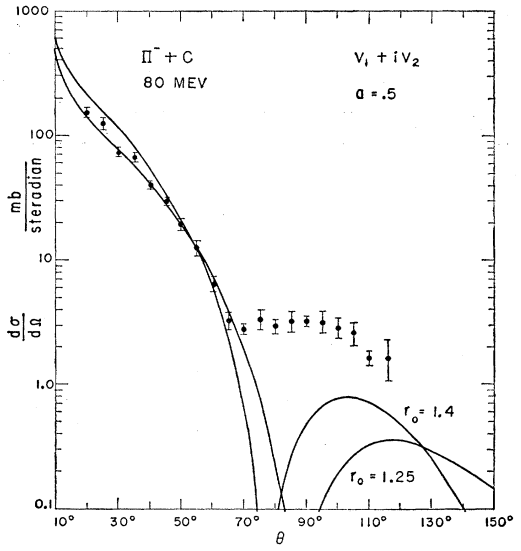

FIG. 3. Carbon. V_1+iV_2 type model with diffuseness parameter $a=0.5$. Variation with nuclear radius for $V=-30-22i$ Mev.

angles. A diffuse edge, obtained by increasing a, has an effect similar to increasing r_0 slightly except that the larger angle cross sections are lowered even farther below the experimental data. Using values of V_1 and V_2 to fit the small-angle points and various choices of r_0 and a, we were unable to get an over-all fit with carbon using this type of model either with or without a diffuse edge after trying twenty choices of parameters.

Applying this same model to aluminum, somewhat better agreement with experiment is obtained (Fig. 4) for angles less than 55°. If edge effects are important in pion scattering, then this model might be expected to be better for the heavier nuclei where the edge region is a smaller fraction of the total nuclear volume.

Because of the large mass of the nucleus compared to that of the pion, the computed cross sections in the pion-nucleus system are compared directly with the cross sections measured in the laboratory system. The angular resolution of the experiments varied from $\pm 3°$ at small angles to $\pm 5\frac{1}{2}°$ at large; this would tend to smooth out slightly the structure of the calculated curves shown if it were folded into the calculated curves.

Using the modified Kisslinger model, attempts were first made to fit the carbon data. Carbon was

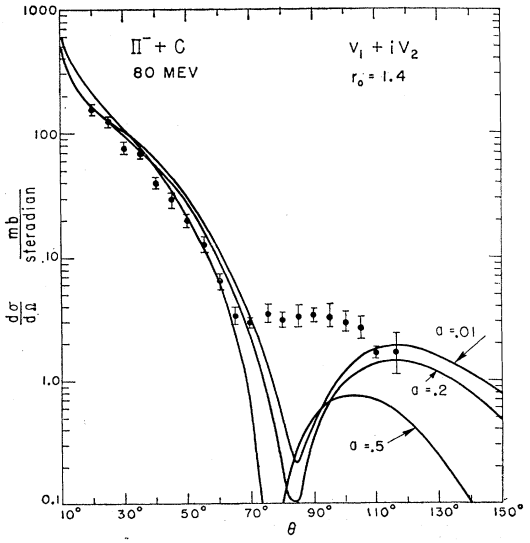

FIG. 2. Carbon. V_1+iV_2 type model with various edge diffusenesses. $V=-30-22i$ Mev.

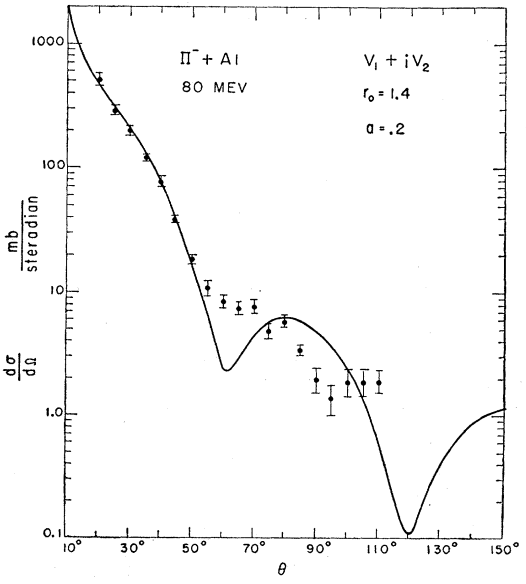

FIG. 4. Aluminum. V_1+iV_2 type model with $r_0=1.4$, $a=0.2$, and $V=-25-22i$ Mev.

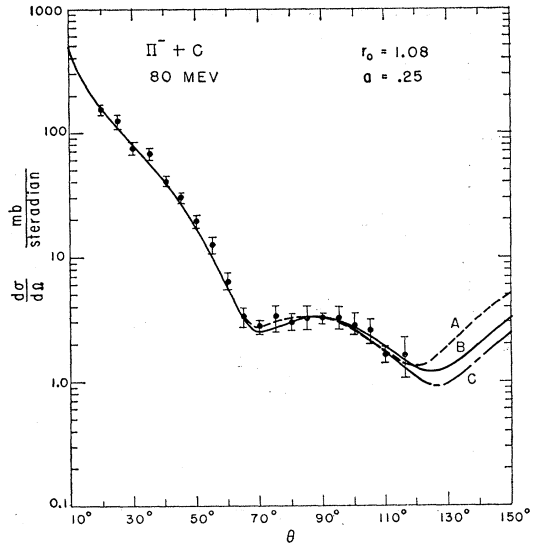

FIG. 5. Carbon. Modified Kisslinger model with $r_0=1.08$ and $a=0.25$. Fits for (A) $C=-1.2$, $C'=+0.45-0.15i$, (B) $C=-1.1-0.1i$, $C'=+0.35-0.15i$, and (C) $C=-1.1-0.2i$, $C'=+0.35-0.25i$.

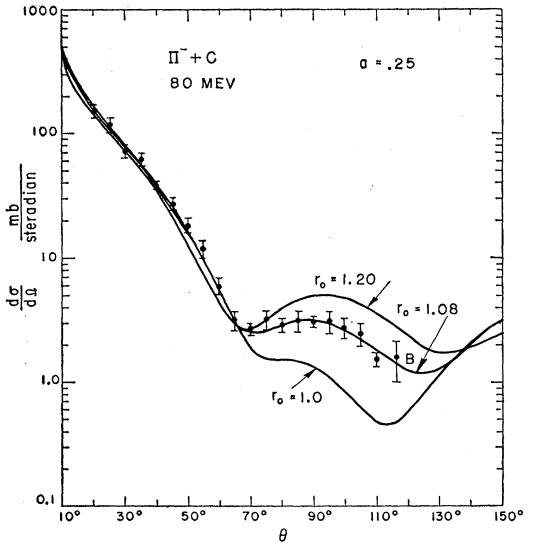

FIG. 7. Carbon. Modified Kisslinger model with various nuclear radii and $a=0.25$. C and C' varied inversely as nuclear volume to fit small angles. The fit is for the set of parameters (B) of Fig. 5. *Correction noted in proof.*—In the region $\theta>70°$ the upper curve is for $r_0=1.0$ and the lower for $r_0=1.20$.

emphasized since its first excited level is at 4.43 Mev which is higher than the first levels of the other elements measured; and, therefore, it is more certain that purely elastic scattering was measured.

With r_0 between 1.2 and 1.4 as predicted by the analysis of nucleon scattering,[9–13] and $a=0.5$ in agreement with nucleon and electron scattering results, it was not possible to fit the data. With a smaller radius, $r_0=1.08$, as suggested by the electron results, a fit to carbon was obtained for several sets of values of C and C' (Fig. 5); however, a smaller value of a was required.

An increase in a produces a lowering of the large angle plateau (Fig. 6). This tendency can be counteracted somewhat by varying C and C', but not sufficiently for values of a as large as 0.5. The plateau can also be raised by decreasing r_0 (Fig. 7), thus allowing larger values of a for $r_0<1.08$.

When the parameters which give good agreement with the carbon data are applied to lithium and aluminum (Figs. 8 and 9), the angular distributions show the

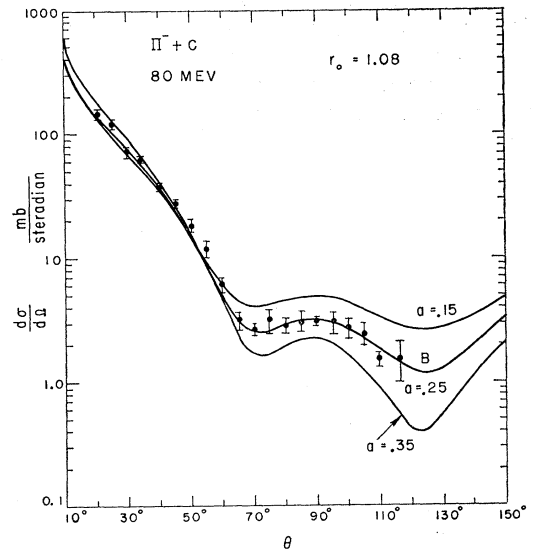

FIG. 6. Carbon. modified Kisslinger model with various values of the edge diffuseness parameter. $r_0=1.08$, $C=-1.1-0.1i$, and $C'=+0.35-0.15i$.

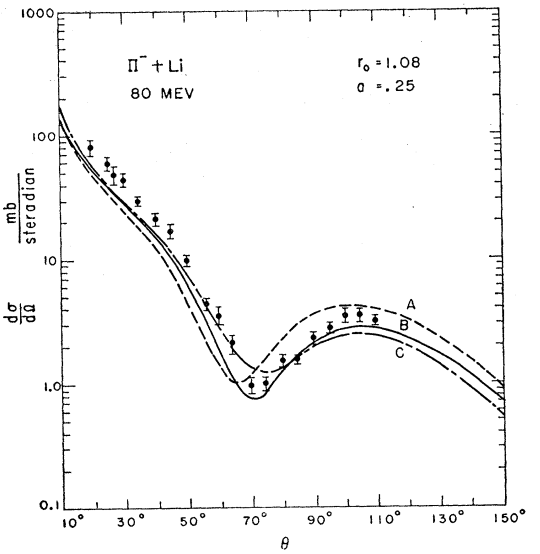

FIG. 8. Lithium. Modified Kisslinger model with the parameters of the carbon fits (see Fig. 5). Some of the small-angle points are from Williams *et al.*[3]

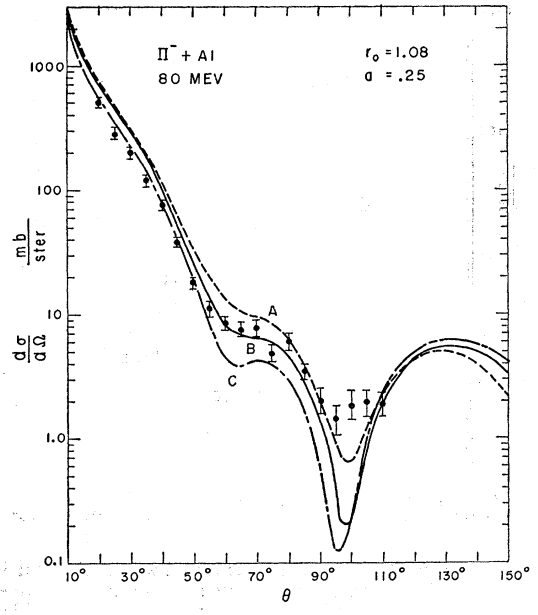

FIG. 9. Aluminum. Modified Kisslinger model with the parameters of the carbon fits (see Fig. 5).

same general shape as the experimental data. Of these, that set of parameters which is in best agreement with the lithium data is also in best agreement with the aluminum data. Because of the differences between nuclei including different relative numbers of neutrons and protons, one would not necessarily expect the exact same set of parameters to fit all nuclei. Particularly in the case of aluminum, also, the many low-lying levels make it questionable that only elastic scattering events are included in the experimental points.

Using the phase shifts of Orear[16] in Eq. (13) for carbon with $r_0=1.08$ and $a=0.25$ yields $C=-1.43-0.31i$ and $C'=+0.075-0.083i$. Thus for the best fits C is somewhat less negative than, and C' several times larger than the predicted quantities.

[16] J. Orear, Nuovo cimento 4, 856 (1956).

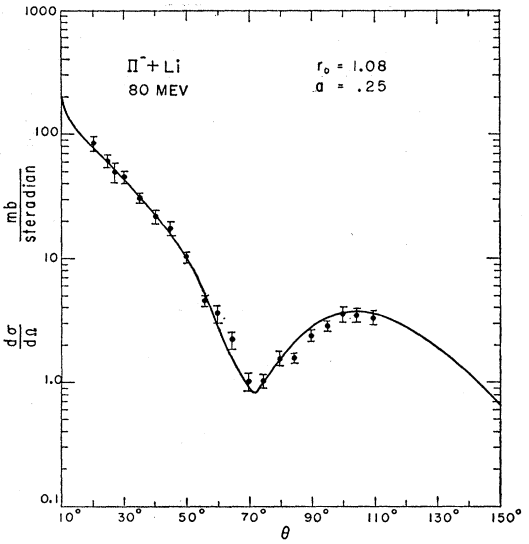

FIG. 10. Lithium. A fit with the modified Kisslinger model. $r_0=1.08$, $a=0.25$, $C=-1.3-0.15i$, and $C'=+0.3-0.15i$.

With C slightly larger than, and the other parameters the same as for a carbon fit, a good match to the lithium data is possible (Fig. 10).

IV. CONCLUSIONS

For the elements investigated it was not possible to fit the data with the usual type of optical model, which assumes an interaction proportional to the nuclear density, either with or without a diffuse nuclear surface. Using a Kisslinger model which includes an interaction term of the form $\nabla\cdot(\rho\nabla\psi)$ and modified as indicated in the text, it is possible to obtain agreement with the experimental results for reasonable values of the parameters involved. The data for various elements can be fit for very similar values of these parameters. Emphasizing carbon, good fits to the data were obtained for a smaller radius more in agreement with electromagnetic methods of measurement than with nucleon scattering results.