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Transverse Plasma Waves and Plasma Vortices*

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Plasmas at high-electron temperatures can carry transverse waves in which self-magnetic fields and relativistic effects become important. In this paper the relativistic perturbation equations for an isotropic uniform plasma are solved as an initial-value problem, i.e., by Laplace transformation, and the propagation or dispersal of both longitudinal and transverse perturbations is calculated. In both cases transients occur which have a continuous frequency spectrum. While transverse perturbations also yield pure persistent waves (with phase velocity exceeding that of light) of all wavelengths, longitudinal perturbations of very short wavelength will not be propagated as pure waves but will die out eventually with only longer wavelengths persisting. The transverse plasma perturbations discussed in the analysis are nonvortical and the dispersal of vortices is covered by a separate discussion. The vortices do not give rise to a new mode of propagation of perturbations.

1. PLASMA PERTURBATIONS AS AN INITIAL-VALUE PROBLEM

A GOOD deal of work has been devoted to longitudinal plasma oscillations, but for high-electron temperatures, transverse plasma oscillations become important also. In this paper, a unified treatment of both types will be given. A simplification of procedure is achieved if one separates out the plasma vortices and treats them differently.

The study of plasma waves was, for a while, impeded by difficulties which arise when the wave velocity coincides with some of the electron stream velocities. These difficulties have been resolved by studying the appropriate initial-value problem, i.e., calculating how a given spatial perturbation develops in time. To anticipate a harmonic wave analysis of all perturbations is an oversimplification. (We shall see, for instance, that this anticipation automatically excludes plasma vortices.)

A full treatment of the initial-value problem for longitudinal plasma perturbations was given by Willson[1] and our general treatment will follow Willson's method closely. Instead of anticipating proportionality of all perturbations to $\exp(i\omega t-ik_1x_1-ik_2x_2-ik_3x_3)$ and identifying all the four operators $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$, and $\partial/\partial t$ with imaginary numbers $-ik_1$, $-ik_2$, $-ik_3$, and $i\omega$, one leaves $\partial/\partial t$ in the form of an operator and applies a Laplace transformation with respect to the time.

For a Laplace-transformed function, the operator $\partial/\partial t$ becomes, again, a pure number (usually denoted by s), provided the initial value of the function is zero. It follows that one can retain the mathematical procedure of the harmonic wave analysis, replace $i\omega$ by s, and adopt the interpretation of one's results according to Laplace-transform theory, as long as the initial values of the quantities concerned are zero. This is the reason why it pays to reduce vorticity to zero before one starts.

Whenever a function does not vanish initially, the rule is to write the initial value on the right side of an equation for every differentiation $\partial/\partial t$ which appears on the left (i.e., for every occurrence of s or $i\omega$ on the left). In any physical situation *some* of the variables are bound to have initial nonvanishing perturbations: otherwise perturbations of an equilibrium state could not arise. Hence, we cannot take over the wave analysis right to the end. Our final equations for, say, the electromagnetic potentials a_ν will be of the form $F(s)a_\nu=I(s)$, where the function $I(s)$ incorporates all the initial data while $F(s)$ summarizes the differential equations governing the system. The wave analysis would have given $F(s)a_\nu=0$ and hence $F(s)=F(i\omega)=0$ which is the dispersion formula, connecting the propagation constants k_1, k_2, k_3 with the frequency. But the actual behavior of the system is given by interpreting $a_\nu=I(s)/F(s)$ in the Laplace-transform sense. Our task is, not only to derive $F(s)$ for both transverse and longitudinal perturbations (i.e., to get a transverse as well as a longitudinal dispersion formula), but also to get a representation of $I(s)$ which will allow simple interpretation of the Laplace-transforms.

2. RELATIVISTIC EQUATIONS OF MOTION

Transverse plasma oscillations become interesting only when self-magnetic fields are having appreciable effects. These effects are smaller than electrostatic effects by a factor v^2/c^2, and when this is appreciable, we cannot ignore relativity effects. Thus a fully relativistic treatment becomes necessary, but the elegance of relativistic electrodynamics makes this a pleasure rather than a burden, and one need not apologize for introducing relativity even when all electrons retain small velocities.

In fact, any fully smoothed-out Maxwellian velocity distribution does contain a *few* very fast electrons, and as the upper limit of velocities plays a significant role in the dispersion formula, there is some advantage in introducing a natural extreme upper limit by means of relativity effects. A relativistically accurate longitudinal dispersion formula was obtained by Clemmow and

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[1] A. J. Willson, Ph.D. thesis, Cambridge University, 1957 (unpublished).

Willson[2] and the results deduced below are in agreement with theirs.

It is customary to begin the derivation of the plasma equations with Boltzmann's equation for a velocity distribution function, and to ignore the collision term (but to take into account macroscopic Coulomb repulsions). This treatment is tantamount to solving the equations of motion and the conservation equations of the different electron streams that are superimposed on each other in their combined fields, and we shall here adopt the latter method.

A suitable system of units is the "electron optical system."[3] In this system one can omit as multipliers the following quantities: the rest-mass of the electron, the charge of the electron (but for its sign), the velocity of light, the dielectric constant of free space, the permeability of free space, and the factors 4π of an unrationalized system. For instance, the relativistic equations of motion are

$$dU_\nu/d\tau = U_\mu(\partial_\mu a_\nu - \partial_\nu a_\mu) \quad \text{(summation convention)}, \tag{1}$$

where the U_ν are the components of four-velocity, the a_ν are the potentials, ∂_ν is short for $\partial/\partial x_\nu$, τ is proper time, and the suffixes go from 1 to 4. The fourth component is imaginary: $x_4 = it$. The components U_ν obey the "kinetic" condition $U_\nu U_\nu = -1$ while the components a_ν obey the Lorentz condition $\partial_\nu a_\nu = 0$.

The equation of conservation of mass or charge is

$$\partial_\nu N U_\nu = 0, \tag{2}$$

where N is the number density of a stream measured in its own rest-frame. The field equations are

$$\partial_\nu \partial_\nu a_\mu = \sum_{\text{streams}} N U_\mu - J_\mu{}^+, \tag{3}$$

where the summation extends over all the superimposed electron streams (not over μ) and where $J_\mu{}^+$ is the total four-current density of the ions. In future, the sign $\sum$ shall always indicate a summation over all streams, not over suffixes. Repeated suffixes are always understood to be summed over. We shall be interested in the problem of vortices, and in electrodynamics one means by vortices not those of the velocity field U_ν but those of the "generalized momentum" $P_\nu = U_\nu - a_\nu$ where U_ν is the kinetic four-momentum (in our units) and $-a_\nu$ the potential four-momentum of electrons (in our units). Since

$$d/d\tau = U_\nu \partial_\nu, \tag{4}$$

we can write the equation of motion (1) as

$$dP_\nu/d\tau + U_\mu \partial_\nu a_\mu = 0, \tag{5}$$

and convert the continuity equation (2) to the form

$$U_\nu \partial_\nu N + N \partial_\nu U_\nu = dN/d\tau + N \partial_\nu P_\nu = 0, \tag{6}$$

by using the Lorentz condition $\partial_\nu a_\nu = 0$. The field equations (3) read

$$\sum N P_\mu + (\sum N - \partial_\nu \partial_\nu) a_\mu = J_\mu{}^+. \tag{7}$$

We shall make reference to the persistence of vortices: by subtracting the identity

$$U_\mu \partial_\nu U_\mu = 0 \tag{8}$$

(see the kinetic condition $U_\mu U_\mu = -1$) from our equation of motion (5), we obtain

$$U_\mu(\partial_\mu P_\nu - \partial_\nu P_\mu) = 0, \tag{9}$$

and by an application of Stokes' theorem in four dimensions this equation can be integrated into the statement that the "circulation" of P_ν, i.e., the line-integral $\oint P_\nu dx_\nu$, is conserved when the circuit is carried along by the electron stream.[4] If there is no vorticity to begin with, none will arise, the line integral vanishes around all circuits, and P_ν is derivable from a potential: $P_\nu = \partial_\nu \zeta$.

3. SMALL PERTURBATIONS OF A UNIFORM MULTISTREAM PLASMA

In the unperturbed state U_ν, P_ν, and N are independent of space and time coordinates and the operators ∂_ν produce nothing when applied to them. The total electron current density $\sum N U_\nu$ just balances that of the ions, $J_\nu{}^+$, and there are no fields: $a_\nu = 0$, so that $P_\nu = U_\nu$.

Let the perturbations cause small modifications u_ν, p_ν, and n of the unperturbed quantities U_ν, P_ν, and N. The field is purely a perturbation. We have $p_\nu = u_\nu - a_\nu$.

The field equations (7) become

$$\sum N p_\mu + \sum n P_\mu + (\sum N - \partial_\nu \partial_\nu) a_\mu = 0, \tag{10}$$

where it is permissible to employ unperturbed values for quantities denoted by capital letters, ignoring products of perturbation terms. The equations of motion (5) and continuity (6) are perturbed to

$$dp_\nu/d\tau + U_\mu \partial_\nu a_\mu = 0, \tag{11}$$

$$dn/d\tau + N \partial_\nu p_\nu = 0, \tag{12}$$

where the proper-time derivative $d/d\tau$ may be taken along unperturbed paths (electron worldlines), with only a second order error. Hence we may put $d/d\tau = U_\nu \partial_\nu$ and again employ unperturbed values for quantities denoted by capital letters. The Lorentz condition remains $\partial_\nu a_\nu = 0$ while the kinetic condition becomes $U_\nu u_\nu = 0$.

We solve by the method described in Sec. 1 above, i.e., by writing $\partial_\nu = -ik_\nu$ and introducing initial values on the right for every occurrence of $\partial/\partial t$, that is of

[2] P. C. Clemmow and A. J. Willson, Proc. Roy. Soc. (London) **A237**, 117 (1956).

[3] See, for instance, P. A. Sturrock, *Static and Dynamic Electron Optics* (Cambridge University Press, Cambridge, 1955), Part 1, Chap. 1.2.

[4] See, for instance, O. Buneman, Proc. Cambridge Phil. Soc. **65**, 77 (1954), Part I.

$k_4(=i\partial_4=i\partial/\partial x_4=\partial/\partial t)$, on the left. Eventually we interpret by identifying k_4 with the customary s of Laplace-transform theory.

We take initial conditions such that no terms appear on the right-hand side of the equations of motion (11). We justify this restriction later (Sec. 7). Then

$$U_\mu k_\mu p_\nu + k_\nu U_\mu a_\mu = 0, \tag{13}$$

or

$$p_\nu = -k_\nu U_\mu a_\mu / U_\sigma k_\sigma, \tag{14}$$

just as in the harmonic wave analysis. (A change of dummy suffix is necessary in the denominator.) The continuity equation (12) becomes

$$U_\nu k_\nu n + N k_\nu p_\nu = U_4 n(0) + N p_4(0), \tag{15}$$

so that

$$n = (U_4 n(0) + N p_4(0) - N k_\nu p_\nu)/U_\sigma k_\sigma, \tag{16}$$

while the field equations (10) become

$$\sum N p_\mu + \sum n U_\mu + (\sum N + k_\nu k_\nu) a_\mu = k_4 a_\mu(0) + \dot{a}_\mu(0). \tag{17}$$

Here $\dot{a}_\mu(0)$ are the initial values of $\partial a_\mu/\partial t$, and in the second term the U_μ have been substituted for the unperturbed P_μ.

Finally, we substitute our explicit formulas (14) and (16) for p_ν and n into the field equations (17), changing dummy suffixes where necessary:

$$-k_\mu a_\nu \sum N U_\nu / U_\sigma k_\sigma + k_\lambda k_\lambda a_\nu \sum N U_\nu U_\mu/(U_\sigma k_\sigma)^2 + (\sum N + k_\nu k_\nu) a_\mu = I_\mu(k_4), \tag{18}$$

where

$$I_\mu(k_4) = k_4 a_\mu(0) + \dot{a}_\mu(0) - \sum U_\mu [U_4 n(0) + N p_4(0)]/U_\sigma k_\sigma. \tag{19}$$

Here are four equations for the four unknown a_μ. They lead to expressions for a_μ which can be interpreted according to the usual Laplace-transform rules.

4. ISOTROPIC DISTRIBUTIONS

The summations indicated by $\sum$ extend over all unperturbed stream velocities, i.e., over all triplets U_1, U_2, U_3 that characterize the superimposed unperturbed streams. $U_4 = i(1+U_1^2+U_2^2+U_3^2)^{\frac{1}{2}}$ is not an independent variable. In the rest-frame of the plasma, i.e., in the frame in which the ions (supposed too heavy to be perturbed) are at rest, one generally deals with isotropic distributions.

This means that, for instance, the velocity component $-U_1$ occurs with the same frequency, i.e., the same N, as $+U_1$. It also means that each component is continuously distributed, and that our sums over all streams are, strictly speaking, integrals. The symbol $\sum$ should be understood to imply this.

Let us now place the space axes such that $k_1=0$, $k_2=0$, i.e., that "propagation" takes place in the direction of the x_3 axis. Then $U_\sigma k_\sigma = U_3 k_3 + U_4 k_4$ and the following totals vanish because of the symmetry:

$$\sum N U_1/U_\sigma k_\sigma,$$
$$\sum N U_2/U_\sigma k_\sigma,$$
$$\sum N U_1 U_3/(U_\sigma k_\sigma)^2,$$
$$\sum N U_2 U_3/(U_\sigma k_\sigma)^2,$$
$$\sum N U_1 U_4/(U_\sigma k_\sigma)^2,$$
$$\sum N U_2 U_4/(U_\sigma k_\sigma)^2,$$
$$\sum N U_1 U_2/(U_\sigma k_\sigma)^2.$$

Thus the matrix of coefficients in the four equations (18) for a_μ is diagonal but for terms with suffixes (3,4).

The transverse components a_1 and a_2 are given by Eq. (18) in the desired form directly:

$$F_1(k_4)a_1 = I_1(k_4), \quad F_2(k_4)a_2 = I_2(k_4), \tag{20}$$

where

$$F_1(k_4) = (k_3^2+k_4^2)[1+\sum N U_1^2/(U_3k_3+U_4k_4)^2] + \sum N \tag{21}$$

and owing to symmetry, $F_1(k_4) = F_2(k_4)$.

For the longitudinal component one uses the Lorentz condition $\partial_\sigma a_\sigma = 0$, which becomes $k_3a_3+k_4a_4=0$ if we take a "gauge" for which $a_4(0)=0$. This can always be achieved: in fact, by adding to a_μ the four-gradient of suitable combinations of

$$\exp[ik_3(x_3-t)] \quad \text{and} \quad \exp[ik_3(x_3+t)]$$

we can always reduce $a_3(0)$ as well as $a_4(0)$ to zero. Then, incidentally, $\dot{a}_4(0)$ vanishes also, since

$$\partial_4 a_4 = -\partial_3 a_3 = ik_3 a_3.$$

Moreover, we get the longitudinal field in the form

$$E_3 = (-\partial_3 a_4 + \partial_4 a_3)/i = k_3 a_4 - k_4 a_3 = (k_3^2+k_4^2)a_4/k_3 = -(k_3^2+k_4^2)a_3/k_4. \tag{22}$$

On substitution of

$$a_3 = -k_4 E_3/(k_3^2+k_4^2)$$

and

$$a_4 = k_3 E_3/(k_3^2+k_4^2)$$

into the fourth component of Eq. (18) for the a_μ derived in Sec. 3, one obtains

$$F_0(k_4) E_3/k_3 = I_4(k_4), \tag{23}$$

where

$$F_0(k_4) = k_3^2 + \sum N k_3^2 (U_3^2+U_4^2)/(k_3U_3+k_4U_4)^2. \tag{24}$$

In the discussion of the important functions F_1 and F_0 one can employ the fact that under symmetry

$$\sum N k_3^2 U_1^2/(k_3U_3+k_4U_4)^2 + \sum N k_3 U_3/(k_3U_3+k_4U_4) = 0. \tag{25}$$

This is proved by integrating with respect to azimuth ϕ about the U_2 axis, i.e., writing $U_1 = (U_1^2+U_3^2)^{\frac{1}{2}} \sin\phi$, $U_3 = (U_1^2+U_3^2)^{\frac{1}{2}} \cos\phi$. The expression in question [left-hand side of (25)] is then equal to

$$\sum N(d/d\phi)[k_3U_1/(k_3U_3+k_4U_4)],$$

and since our summation implies an integration with respect to ϕ from 0 to 2π, this "sum" vanishes.

It is also convenient to collect the contribution from the streams with $+U_3$ and $-U_3$ and to deduce

$$\sum NU_1^2/(k_3U_3+k_4U_4)^2=\sum NU_3^2/(k_4^2U_4^2-k_3^2U_3^2) =-\sum NV_3^2/(k_4^2+k_3^2V_3^2), \quad (26)$$

where $V_3=iU_3/U_4$ is a velocity component dx_3/dt measured as displacement with respect to observer's time, not proper time. With this identity, we get F_1 (and F_2) in the form

$$\begin{aligned} F_1=F_2&=k_3^2+k_4^2+\sum N -\sum N(k_3^2+k_4^2)V_3^2/(k_4^2+k_3^2V_3^2) \\ &=k_3^2+k_4^2+\sum N(1-V_3^2) -\sum N(1-V_3^2)k_3^2V_3^2/(k_4^2+k_3^2V_3^2) \\ &=k_3^2+k_4^2+\sum N(1-V_3^2)/(1+k_3^2V_3^2/k_4^2). \quad (27) \end{aligned}$$

If k_4^2 is positive, F_1 is also positive. If k_4^2 has an imaginary part, all contributions to F_1 have imaginary parts of the same sign and hence F_1 can vanish only when k_4^2 is negative, i.e., k_4 pure imaginary. There are no transverse plasma instabilities. Also, on putting $k_4=i\omega$, we see that F_1 decreases monotonically from $\sum N$ to $-\infty$ as ω^2 rises from k_3^2 to $+\infty$. There exists just one zero of F_1 in this frequency range, i.e., one natural transverse plasma resonance. The monotonic behavior extends to frequencies as low as $k_3V_{\max}$ where $V_{\max}$ is the maximum velocity occurring in the distribution, and there can be no further resonance frequency in the extended frequency range between $k_3V_{\max}$ and k_3.

Similar arguments can be applied to F_0 which, owing to the kinetic condition, can be written as

$$F_0=k_3^2-\sum Nk_3^2(1+U_1^2+U_2^2)/(k_3U_3+k_4U_4)^2. \quad (28)$$

Here we transform the term $k_3^2\sum N/(k_3U_3+k_4U_4)^2$ while using the identity (26) already established for the remaining sums. We introduce polar coordinates about the axis of U_3 and we average over all angles, which means we average over the cosine of the polar angle between -1 and $+1$. This is easily done and leads to

$$\sum N/(k_3U_3+k_4U_4)^2 =\sum N/[k_4^2U_4^2-k_3^2(U_1^2+U_2^2+U_3^2)]. \quad (29)$$

If we let V be the speed measured as displacement with observer's (rather than proper) time, i.e.,

$$V=i(U_1^2+U_2^2+U_3^2)^{\frac{1}{2}}/U_4$$

and hence $U_4=i/(1-V^2)^{\frac{1}{2}}$, this sum becomes

$$-\sum N(1-V^2)/(k_4^2+k_3^2V^2).$$

Altogether, therefore,

$$F_0=k_3^2+\sum Nk_3^2(1-V^2)/(k_4^2+k_3^2V^2) +2\sum Nk_3^2V_3^2/(k_4^2+k_3^2V_3^2). \quad (30)$$

Again, if k_4^2 is positive or has an imaginary part, F_0 cannot vanish and hence there are no longitudinal plasma instabilities. Also, if $k_4=i\omega$, F_0 increases monotonically up to k_3^2 as ω^2 increases from $k_3^2V_{\max}^2$ to ∞, so that there is at most one longitudinal plasma resonance within this range.

Explicit forms of the dispersion laws $F_1(\omega,k_3)=0$, $F_0(\omega,\ k_3=0)$ are obtained by giving ω as a function of the phase velocity $W=\omega/k_3$, and by averaging over the polar angle in all the terms of F_1 and F_0:

$$\omega^2=W^2\sum N\left(\frac{W^2}{W^2-1}-\frac{W}{V}\tanh^{-1}\frac{V}{W}\right), \text{ (transverse)} \quad (31)$$

$$\omega^2=W^2\sum N\left(\frac{1-V^2}{W^2-V^2}+2\frac{W}{v}\tanh^{-1}\frac{V}{W}-2\right) \text{ (longitudinal).} \quad (32)$$

Since $W>V_{\max}$, one can develop in powers of V/W. Omitting terms of the order V^4/W^4, one gets

$$\omega^2/\sum N=W^2/(W^2-1)-\theta, \quad \text{(transverse)} \quad (33)$$

$$\omega^2/\sum N=1-\theta(1-3/W^2), \quad \text{(longitudinal)} \quad (34)$$

where $\theta=\sum \frac{1}{3}NV^2/\sum N$ is the temperature, in units of 6×10^9 degrees.

When comparing these formulas with nonrelativistic approximations, one must define a "plasma frequency" ω_p for reference. It is customary to use the square root of the ion density: $\omega_p^2=\sum NU_4/i$. For low temperatures ($U_4/i=1+\frac{1}{2}V^2$), this becomes $\omega_p^2=\sum N(1+3\theta/2)$ and hence

$$\frac{\omega^2-k_3^2}{\omega_p^2}=\frac{\omega^2}{\omega_p^2}(1-W^{-2})=1-(\tfrac{5}{2}-W^{-2})\theta, \quad \text{(transverse)} \quad (35)$$

$$\frac{\omega^2}{\omega_p^2}=1-(\tfrac{5}{2}-3W^{-2})\theta \quad \text{(longitudinal).} \quad (36)$$

In transverse oscillations, W always exceeds 1, so that the temperature coefficient is never large. In longitudinal oscillations, W may go as low as $V_{\max}$ which is close to $\theta^{\frac{1}{2}}$ for some distributions [$V_{\max}=(7\theta)^{\frac{1}{2}}$ for the parabolic distribution $N\propto V_{\max}^2-V^2$]. Then there is a dispersive) effect even at low temperatures, known in nonrelativistic theories.

5. WAVE VELOCITIES WITHIN THE STREAM VELOCITY RANGE

So far, we have not yet discussed the frequency range $-k_3V_{\max}<\omega<k_3V_{\max}$ corresponding to a limited range on the imaginary axis of the k_4 plane. Outside this

range F_1 and F_0 are finite and they are nonvanishing except possibly at two isolated conjugate points on the imaginary axis corresponding to plasma frequencies.

The remaining frequency range corresponds to harmonic waves whose velocity ω/k_3 coincides with that of some of the streams.[5] If such waves did exist, we should have to have $F=0$ or $I=\infty$ somewhere in this range. We shall show that for continuous velocity distributions and continuous distributions of initial perturbations this is not possible.

The proof that $F\neq 0$ rests on the fact that the imaginary part of F does not vanish in the critical frequency range. At $\omega=0$, i.e., $k_4=0$, $F_1\neq 0$ and $F_0\neq 0$ by inspection. We can therefore limit the subsequent discussion to the cases $\omega\neq 0$. Of course, the integrals over all stream velocities become meaningless when k_4 is exactly on the imaginary axis between $-ik_3V_{\max}$ and $+ik_3V_{\max}$, but we get finite answers when k_4 is just to the right or just to the left of this range (and ω just below or just above the real axis).

Now a typical "sum" in our expressions (27) and (30), such as $\sum Nk_3^2V_3^2/(k_3^2V_3^2-\omega^2)$, can, because of symmetry, be written as $\sum Nk_3V_3/(k_3V_3-\omega)$ or $\sum N+\omega\sum N/(k_3V_3-\omega)$, since N is the same for $+V_3$ as for $-V_3$. Let us now suppose that ω is just below the real axis and remember that the summation implies an integration over V_3 between $-V_{\max}$ and $+V_{\max}$. If ω/k_3 is just below a point in this interval on the real axis (see Fig. 1), then the integration amounts to describing half a clockwise circuit around the singularity and will produce a negative imaginary part in

$$\sum N/(k_3V_3-\omega).$$

This imaginary part does not vanish and will persist even as one lets ω merge into the real axis, provided the integration with respect to V_1 and V_2 for fixed $V_3=\omega/k_3$ produces a nonvanishing numerator. This it will do whenever $|\omega|<k_3V_{\max}$, for then there is bound to be at least a belt of streams whose V_3 has the value ω/k_3: the possibility of the distribution function, which depends upon $(V_1^2+V_2^2+V_3^2)^{\frac{1}{2}}$ only, having gaps between 0 and $V_{\max}$ need not be excluded.

A similar argument can be applied to the term $\sum Nk_3^2(1-V^2)/(k_3^2V^2-\omega^2)$ in that the integration over V, normally taken only from 0 to $V_{\max}$, can equally well be taken from $-V_{\max}$ to $+V_{\max}$ because of the evenness of the integrand. However, this term would not contribute an imaginary part at a gap in the distribution function. Where it does contribute, the sign of the contributions is the same as those from the other term.

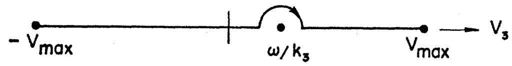

FIG. 1. Path of integration when ω approaches the real axis from below.

[5] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 and 1864 (1949).

Furthermore, the term

$$\sum N(1-V_3^2)k_3^2V_3^2/(k_3^2V_3^2-\omega^2)$$

in F_1 [Eq. (27)] contributes a nonvanishing imaginary part, by the same argument as before, in the critical interval. We conclude that neither F_0 nor F_1 can vanish when k_4 approaches the interval between $-ik_3V_{\max}$ and $+ik_3V_{\max}$ on the imaginary axis. But the sign of the (nonvanishing) imaginary parts of F_0 and F_1 depends upon the side from which k_4 approaches the interval. There is a cut in the complex plane along the interval for the purposes of representation of the analytic functions $F_0(k_4)$ and $F_1(k_4)$. Except along the cut, however, there is no ambiguity of F_0 and F_1.

One can show, incidentally, that F_0 and F_1 remain *finite* along the cut provided the density distribution has finite slope as a function of velocity and does not break off sharply at $V_{\max}$. A density N proportional to $(V_{\max}^2-V^2)dV_1dV_2dV_3$, for instance, would be adequate in order to insure that F_0 and F_1 remain finite throughout the finite part of the complex plane. Likewise the relativistic Maxwellian density proportional to $(i/U_4)\exp(iU_4/\theta)dU_1dU_2dU_3$ ($\theta=$temperature) and extending to $V_{\max}=1$, leads to bounded F_0 and F_1.[6] Relativity, of course, ensures a velocity maximum for all types of distribution law.

Considerations of boundedness are important in the study of the numerators $I_1(k_4)$ and $I_4(k_4)$ of our Laplace transforms [Eq. (19)]. Here, too, we have integrations over all velocities with the denominators $k_3U_3+k_4U_4$, proportional to $k_3V_3-\omega$, in the integrand. Provided the initial density perturbations $n(0)$, as well as $Np_4(0)$, have finite slope as functions of the velocity components and do not break off sharply at $V_{\max}$ [note that $n(0)$ need not be isotropic with respect to velocity, and that it may cover a slightly wider range than N, in which case one would modify the definition of $V_{\max}$], these integrals will remain bounded even in the critical frequency range.

To show that, say,

$$\int\int\int f(V_1V_2V_3)dV_1dV_2dV_3/(V_3-\omega/k_3)$$

is bounded, we obtain, first,

$$g(V_3)=\int\int f(V_1,V_2,V_3)dV_1dV_2$$

and write the integral in the form

$$\int_{-V_{\max}}^{+V_{\max}} g(V_3)dV_3/(V_3-\omega/k_3)$$
$$=\int[g(V_3)-g(\omega/k_3)]dV_3/(V_3-\omega/k_3)$$
$$+g(\omega/k_3)\int dV_3/(V_3-\omega/k_3). \quad (37)$$

Here the first integral is nonsingular since $g(V_3)$ has

[6] See J. L. Synge, *The Relativistic Gas* (Interscience Publishers, Inc., New York, 1957), Chap. 4, paragraph 14, for this distribution law.

finite slope at $V_3=\omega/k_3$ and the integrand remains finite, while the second term is

$$g(\omega/k_3)\{\ln(V_{\max}-\omega/k_3)-\ln(V_{\max}+\omega/k_3)\pm i\pi\}.$$

If $g(\omega/k_3)$ tends to 0 with finite slope when ω/k_3 tends to $\pm V_{\max}$, this term also remains finite. The sign of the imaginary part depends, of course, upon the direction of approach to the critical range of ω in the complex plane.

To summarize: subject to reasonable conditions of smoothness of the density distribution and the initial perturbations, the Laplace transforms of a_1, a_2, and E_3 have the following properties:

(a) They are bounded throughout the complex plane except, possibly, at an isolated pair of conjugate poles representing real logitudinal or transverse plasma waves.

(b) They are discontinuous across a cut along the part of the imaginary axis between $-ik_3V_{\max}$ and $+ik_3V_{\max}$,

(c) They tend to zero like $1/k_4$ as k_4 tends to infinity.

The latter feature is readily verified separately for $I_1(k_4)/F_1(k_4)$ and $I_4(k_4)/F_0(k_4)$, using the fact that $a_4(0)$ and $\dot{a}_4(0)$ vanish and that $|U_4|$ exceeds unity always, so that k_4 always has a nonvanishing multiplier in the combination $k_\sigma U_\sigma$ [see Eqs. (19), (20), (23), (27), and (30)].

6. INTERPRETATION OF THE LAPLACE TRANSFORMS

Since our Laplace transforms are well behaved everywhere to the right of the imaginary axis, we could use the inversion formula[7]

$$(2\pi i)^{-1}\int_{\epsilon-i\infty}^{\epsilon+i\infty}[I_1(k_4)/F_1(k_4)]\exp(k_4t)dk_4 \qquad (38)$$

for a_1 and the corresponding formula for longitudinal fields. The path of integration runs parallel and just to the right of the imaginary axis [see Fig. 2 (a)].

However, owing to the property (c) explained above, this path can, for $t>0$, be closed around a large semicircle [see Fig. 2(b)]. It can then be contracted to two circles about the poles (if any) and a contour skirting the cut from $-ik_3V_{\max}$ to $+ik_3V_{\max}$ [see Fig. 2(c)].

The circles around the poles at $k_4=\pm i\omega_0$ lead to genuine plasma waves, proportional to $\exp(\pm i\omega_0t-ik_3x_3)$. The contour around the cut represents a disturbance having a continuous spectrum with frequencies up to $k_3V_{\max}$, for we can convert the integral by changing to the variable of integration ω, and taking the right-hand limit of the Laplace transforms on the up-stroke, the left-hand limit on the down-stroke:

$$(2\pi)^{-1}\int_{-k_3V_{\max}}^{+k_3V_{\max}}\{[I_1(i\omega)/F_1(i\omega)]_{\text{right}} -[I_1(i\omega)/F_1(i\omega)]_{\text{left}}\}\exp(i\omega t)d\omega. \qquad (39)$$

[7] G. Doetsch, *Laplace Transformations* (Dover Publications, Inc., New York, 1957), Chap. 6, paragraph 5.

A disturbance having a continuous spectrum represents a pulse, not steady oscillations. After a long enough time no perturbation will be left, because of destructive interference between neighboring frequencies.

There may, of course, be some almost periodic behavior in the pulse while it lasts. The spectrum, given by the square bracket, may have pronounced maxima. If such a maximum occurs at $\omega=\omega_0$ and the logarithm of the amplitude behaves like $[\text{const}-\frac{1}{2}T_0^2(\omega-\omega_0)^2]$ in the vicinity, then the disturbance is approximately a Gaussian pulse of duration T_0.

Maxima in the spectrum can be due to maxima in I or minima in F. More precisely, let us put

$$I=L+i\pi M \text{ on the right,} \quad =L-i\pi M \text{ on the left,}$$
$$F=G+i\pi H \text{ on the right,} \quad =G-i\pi H \text{ on the left}$$

(note that G and H are real, but L and M need not be real); then the spectrum is $i(MG-LH)/(G^2+\pi^2H^2)$ and we look for maxima of $MG-LH$ and minima of $G^2+\pi^2H^2$. There is a maximum of M at frequency $k_3V_3{}^0$ when the initial perturbations predominantly occur in streams whose velocity component V_3 lies near some particular value $V_3{}^0$. This is as might be expected, since these streams will, in the first instance, carry their own perturbations along with them. But eventually the perturbations get communicated, by electromagnetic effects, to other streams and the initial perturbations will disperse.

More interesting are the minima of $G^2+\pi^2H^2$. Their nature does not depend upon the initial conditions. Minima of $G^2+\pi^2H^2$ occur when, in taking k_4 along the imaginary axis, one comes close to a zero of $F=G+i\pi H$. There are no such zeros off the imaginary axis on that sheet of the complex plane which we have employed so far. But we can, when studying the vicinity of the cut, continue the function which is valid on one side analytically across the cut on to the other side, thereby reaching a different Riemann sheet.

Suppose now that a zero of F exists on this normally concealed Riemann sheet at $k_4=i\omega_0-\gamma$, and let us approximate to F by

$$F\propto k_4-(i\omega_0-\gamma),$$

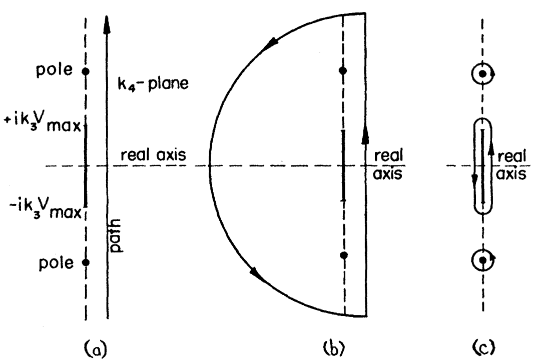

FIG. 2. Successive changes of the path of integration for Laplace-inversion.

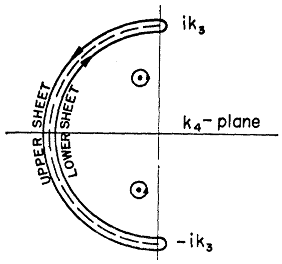

FIG. 3. Distortion of cut leading to Landau's damped exponentials.

or

$$F \propto i[\omega-\omega_0+i\gamma].$$

Then $|F|^2 \propto 1+(\omega-\omega_0)^2/\gamma^2$ and, according to our earlier remarks, the decay time will be of the order γ^{-1}.

This type of result was obtained by Landau in a study of nonrelativistic longitudinal plasma oscillations without velocity cutoff.[8] The discussion is taken up by Berz.[9] Landau's technique is, however, not readily adapted to the case of a velocity cutoff, in that his deformation of the path of integration in the inversion formula would have to be taken around the branch points at $k_4=\pm ik_3V_{\max}$.

The link between Landau's method of integration and that presented here can be established by distorting the cut from a straight line connecting $k_4=+ik_3$ with $k_4=-ik_3$ leftward into a semicircle connecting these points ($V_{\max}=1$ for a Maxwellian distribution). In pulling the right-hand branch of our path of integration up to the new semicircular edge (see Fig. 3), one moves in the lower Riemann sheet where there may be poles: their residues contribute Landau's damped exponentials. The left-hand branch is pushed leftwards in the upper Riemann sheet where there are no poles. In the nonrelativistic approximation the new cut appears, by comparison, infinitely distant and the difference between the Laplace transforms on its two edges may be ignored.

That one finds "damped oscillations" in following Landau, as zeros of F on a different Riemann sheet, does not contradict our spectrum of purely real frequencies, since both results are valid only for $t>0$. Landau's solution is $E_3=0$ for $t<0$, $E_3\propto \exp(i\omega_0-\gamma)t$ for $t>0$, *not* for all t. The Fourier spectrum of such a pulse is, indeed, proportional to $(\omega-\omega_0-i\gamma)^{-1}$.

Summarizing, we can state that transverse plasma perturbations propagate eventually as pure harmonic waves with a phase velocity exceeding that of light. But there are transients in the form of a pulse whose spectrum is continuous, covering all frequencies up to $k_3V_{\max}$.

Longitudinal plasma perturbations will be entirely of the transient variety when the wave number is large. But for low wave numbers (large-scale perturbations), a pure harmonic wave will persist, after an initial accompaniment by transients. All longitudinal transients have a spectrum which terminates with the upper frequency $k_3V_{\max}$. These results concerning relativistic longitudinal plasma waves agree with those obtained by Willson.

The critical wave number above which persistent waves do not occur is given by putting $k_4=ik_3V_{\max}$ in $F_0=0$ [Eq. (30)],

$$k_3^2=\sum N(1-V^2)/(V_{\max}{}^2-V^2) +2\sum NV_3^2/(V_{\max}{}^2-V_3^2). \quad (40)$$

For nonrelativistic velocities and the parabolic distribution law mentioned above, one obtains

$$V_{\max}{}^2k_3^2=5\sum N/2.$$

If the distribution is changed from the parabolic to the long-tailed Maxwellian distribution (extending to $V=1$), one obtains $k_3^2=\sum N$ for the critical wave number under nonrelativistic conditions. This is much smaller than before. The larger wave numbers have been drawn into the range which is damped, owing to the presence of synchronous streams. Estimates (see Landau, reference 8) show the damping to be slight. Under relativistic conditions one obtains a critical k_3 slightly above the reference plasma frequency. At most it is 2% more, viz., at $\theta=\frac{1}{5}$ approximately (see Appendix).

In a nonrelativistic analysis of the Maxwellian distribution, persistent waves do not appear: lack of relativistic cutoff allows streams to be synchronous with waves of arbitrarily high phase velocity. These streams (in cooperation with their neighbors) cause Landau damping.[5] In relativity, waves with phase velocity greater than that of light escape such damping.

7. REMOVAL AND DISPERSAL OF VORTICES

Up to now our analysis of transients has been restricted to a particular set of initial conditions, namely those which leave the equations of motion in the form given by the wave analysis in spite of the adoption of Laplace transform technique.

In Sec. 3 we took the equations of motion (13) in the form

$$U_\nu k_\nu p_\mu+k_\mu U_\nu a_\nu=0,$$

implying specific initial conditions. For general initial conditions, one must put initial values on the right for every occurrence of k_4 on the left. For $\mu=1$, 2, 3 this means that we ought to write $U_4p_1(0)$, $U_4p_2(0)$, $U_4p_3(0)$ on the right. For $\mu=4$ we should write $U_4p_4(0)+U_\nu a_\nu(0)$. Since $a_\nu=u_\nu-p_\nu$ and since $U_\nu u_\nu=0$, the right-hand side of the equation for $u_\nu=4$ can be written $-U_1p_1(0)-U_2p_2(0)-U_3p_3(0)$, and we see that there are only three arbitrary initial values in the four equations, owing to the restriction imposed on the initial velocities by the kintetic condition.

The specific choice that we made in section 3 is consistent with this requirement and amounts to taking $p_1(0)=p_2(0)=p_3(0)=0$, from which $p_4(0)=-U_\nu a_\nu(0)/U_4$ follows automatically.

In the discussion of longitudinal plasma oscillations,

[8] L. Landau, J. Phys. U.S.S.R. **10**, 25 (1946).
[9] F. Berz, Proc. Phys. Soc. (London) **B69**, 939 (1956).

moreover, we committed ourselves to a specific gauge, namely $a_3(0)=0$, $a_4(0)=0$. Thus our initial conditions yield $U_4p_4(0)=-U_1a_1(0)-U_2a_2(0)$ and we see that in the longitudinal formulas (19) and (23) the "initial" term $I_4(k_4)$ reduces to $\sum n(0)U_4^2/(k_3U_3+k_4U_4)$ by virtue of the isotropy of the velocity distribution N. For $I_1(k_4)$ we get [see Eq. (19)]

$$\sum n(0)U_1U_4/(k_3U_3+k_4U_4) \\ -a_1(0)\sum NU_1^2/(k_3U_3+k_4U_4)U_4. \quad (41)$$

If $n(0)$ is also isotropic, the *first* term disappears here, leaving the excitation of a transverse field entirely to its own initial value. As regards the velocities, we start our streams with $u_1=a_1$, $u_2=a_2$, $u_3=0$.

Since we have taken the initial generalized momenta $p_1(0)$, $p_2(0)$, $p_3(0)$ equal to zero, our distribution contains no vortices initially. This continues to be the case (as it should), for let ζ be a quantity which is initially zero and which obeys $d\zeta/d\tau=-U_\nu a_\nu$, so that its Laplace transform is $-iU_\nu a_\nu/U_\nu k_\sigma$ (remembering that $d/d\tau=U_\sigma\partial_\sigma$, transforming to $iU_\sigma k_\sigma$). Then the four-dimensional gradient of ζ, $\partial_\mu\zeta$, has the transform $-k_\mu U_\nu a_\nu/U_\sigma k_\sigma$ which is just p_μ [see Eq. (14)]. The momenta form a gradient, i.e., they have no vortices.

We see immediately that the harmonic wave analysis, in which the formula (14), $p_\mu=-k\mu U_\nu a_\nu/U_\sigma k_\sigma$, would be used without reservations, can never account for vortices in the plasma.

As regards purely transverse initial perturbations (i.e., perturbations for which $u_3=0$ initially) we may say that the absence of vortices is sufficient to justify our initial conditions. For, since we have chosen the gauge $a_3(0)=0$, we get $p_3(0)=0$ immediately while the condition of zero vorticity yields $\partial_3p_1=-ik_3p_1=0$, $\partial_3p_2=-ik_3p_2=0$ (which are not operational equations, k_3 being a real number), so that $p_1(0)=0$ and $p_2(0)=0$.

We may say, therefore, that among the transverse perturbations our analysis has just covered all the nonvortical ones. The longitudinal analysis has been restricted also, in that we have assumed $u_3(0)=0$.

Now it is easy to justify the assumption of zero initial velocities. We have treated the electron cloud in a plasma as a superposition of streams. The unperturbed cloud has a definite isotropic distribution law, given in terms of the number density of a stream N, as a function of the stream velocity components U_1, U_2, U_3. The latter remain permanent attributes of the streams in the absence of perturbations, and there is a unique way of labelling each electron according to the stream to which it belongs: its velocity components serve as labels.

When such an electron distribution is perturbed by, say, giving each electron a small impulse and also displacing it slightly, we can describe this by saying that the initial velocity components, as well as the initial density of the electrons with the label $U_1U_2U_3$, have been changed to U_1+u_1, U_2+u_2, U_3+u_3, $N(U_1,U_2,U_3)+n$, respectively. But another way of describing the perturbed initial situation is to say that each electron has changed its label: it has been removed from the stream (U_1,U_2,U_3) to the stream $(U_1+u_1,\ U_2+u_2,\ U_3+u_3)$ as well as displaced. This is certainly the way one would go about it if one used a Boltzmann distribution function. If the unperturbed velocity distribution is continuous, then it will certainly be possible to "pigeonhole" each electron in this way.

The perturbed electron cloud is then described as an assembly of streams with a density depending upon stream label and position, and this density is caused to deviate from the uniform ion density initially both by swelling one group and depleting another when imparting initial velocity changes, and by displacement in space. The groups are now labelled according to initial velocities U_1, U_2, U_3. The electrons in each group would retain the velocities indicated by their labels if the density distribution were the same as the isotropic equilibrium distribution (i.e., that of the ions). Otherwise, the electron velocity components will deviate from the original $U_1U_2U_3$ by amounts u_1, u_2, u_3 as time proceeds. But initially these deviations are zero.

In other words, the stream analysis implies an arbitrariness of labelling and this can be used to establish initial conditions such that the initial velocities vanish. In going from one description of the initial state to another, one changes, of course, the distribution $n(0)$.

From this argument we see that there is no loss of generality in assuming $u_3(0)=0$ as we did for the longitudinal perturbations. But one would be led to the natural choice $u_1(0)=0$, $u_2(0)=0$ instead of $u_1(0)=a_1(0)$, $u_2(0)=a_2(0)$ as required in our analysis.

There are several ways around this difficulty. One is to use $u_1(0)=0$ and $u_2(0)=0$ as initial conditions and modify the analysis of Sec. 3 accordingly. One readily sees that this procedure leads to squared denominators, $(U_3k_3+U_4k_4)^{-2}$, in $I_\mu(k_4)$ and without rather careful checking it is not possible to make sure that they do not indicate oscillations of increasing amplitude [the Laplace-inverse of $(k_4-i\omega_1)^{-2}$ is $t\exp(i\omega_1 t)$].

The second method is to use superposition. (This method resulted from discussions between the author and Dr. P. A. Sturrock of Stanford Microwave Laboratory.) It amounts to removing the vorticies but creating initial longitudinal velocities. We describe the initial state $u_\nu=0$ as a superposition of three states:

$u_1=a_1(0)$, $u_2=a_2(0)$, $u_3=0$, initial potentials as given;

$u_1=b_1$, $u_2=b_2$, $u_3=b_3$, initial potentials zero;

$u_1=0$, $u_2=0$, $u_3=-b_3$, initial potentials zero.

The constants b_1 and b_2 of the second state (which has no initial potentials) are then equated to the negatives of the initial potentials of the first state; b_3 is chosen such that $U_3(U_1b_1+U_2b_2)+(U_3^2+U_4^2)b_3=0$. The way in which one distributes the density perturbations among the three states is arbitrary, but we choose this in such a way that the second state is particularly simple.

The first state is nonvortical and covered by our previous analysis. The third state is also nonvortical but there is an initial longitudinal velocity which could be removed by relabelling. The second state is vortical and if we take its density perturbations zero, no fields are ever created. This occurs by virtue of our specific choice of b_3: in the absence of fields, the equations of motion are $U_\nu\partial_\nu u_\mu=0$, giving $u_\mu=U_4b_\mu/(U_3k_3+U_4k_4)$ where $b_4=-(b_1U_1+b_2U_2+b_3U_3)/U_4=b_3U_4/U_3$ in accordance with the kinetic condition and our choice of b_3. We show that the flow is "incompressible," i.e., that $\partial_\nu u_\nu=0$. Indeed, the Laplace transform of this equation, $k_3U_3+k_4u_4=b_4$, is readily checked to be true. Since $dn/d\tau=-N\partial_\nu u_\nu$, we see that no density perturbations, and hence no fields, are created as time proceeds.

The field-free solution of our equations which we have here described can be used to remove initial vortices. The expressions for u_ν are readily Laplace-inverted, leading to $u_\mu=b_\mu\exp[-ik_3(x_3-V_3t)]$. Each group of streams carries its own wave of velocity perturbations with it: the latter are (and remain) incompressible, hence no fields arise, and one has no occasion for studying the superposition of perturbations with a continuous spectrum of different time-dependences.

The removal of vortices by the superposition method is simple and instructive. But it introduces an initial transverse velocity that has to be dealt with by relabelling. This raises the question: why not remove the second solution, $u_\nu(0)=b_\nu$ by relabelling also? This procedure amounts to yet another method of dealing with initial conditions that do not conform with those of our earlier analysis. The previous method shows up in detail how the relabelling and the calculation with general initial conditions are equivalent. In fact, our perturbation equations possess a whole class of solutions which are "trivial," in the sense that they amount merely to a relabelling of the unperturbed state.

But perhaps the most elegant approach to the problem of awkward initial conditions and initial vortices is to use the initial generalized momenta P_1, P_2, P_3 as labels for the streams. This requires preliminary investigation of small magnetic fields created by anisotropies in the electron velocity distribution, and construction of potentials a_1, a_2, a_3 (with arbitrary gauge). After surveying the electron density as a function of momenta P_1, P_2, P_3 and space coordinates x_1, x_2, x_3, one subtracts off a closely fitting uniform isotropic distribution, representing the "unperturbed" state, and analyzes the balance as outlined. In unperturbed conditions the initial momenta remain unchanged and are identical with the velocities. In perturbed or nonequilibrium distributions the momenta will change from their initial values to P_1+p_1, P_2+p_2, P_3+p_3. But initially we have $p_1=p_2=p_3=0$, the condition required for our analysis of Sec. 3.

This method removes vortices simply by relabelling. If vortices have been stirred up in some of the previously unperturbed streams, we regroup the streams until in the new grouping no vortices are apparent.

This discussion suggests that in a fully relativistic treatment of the problem by a Boltzmann distribution function one should also employ generalized momenta in place of velocities. Such a procedure might prove helpful even when collisions are taken into account.

APPENDIX. CRITICAL WAVE NUMBER FOR THE RELATIVISTIC MAXWELLIAN DISTRIBUTION

The critical wave number beyond which longitudinal waves fail to be propagated undamped is given by putting $k_4=ik_3V_{\max}$, i.e., $k_4=ik_3$ in the Maxwellian distribution and solving the dispersion formula $F_0=0$. Putting $V_3=\mu V$, where μ is the cosine of the polar angle, we find, from (30),

$$k_3^2=\sum N+2\sum N\mu^2V^2/(1-\mu^2V^2)=\sum N\{2/(1-\mu^2V^2)-1\}. \quad \text{(A1)}$$

Averaging with respect to μ over its interval between -1 and $+1$ leads to

$$k_3^2=\sum N(2\gamma\coth\gamma-1),\quad \text{where}\quad \gamma=\tanh^{-1}V. \quad \text{(A2)}$$

In terms of γ, we have $U_4=i\cosh\gamma$, $(U_1^2+U_2^2+U_3^2)^{\frac{1}{2}}=\sinh\gamma$, and hence $dU_1dU_2dU_3=4\pi\sinh^2\gamma d(\sinh\gamma)$ for spherical symmetry.

In the Maxwellian distribution, N is proportional to $(i/U_4)\exp(iU_4/\theta)dU_1dU_2dU_3$, i.e., to $\exp(-\theta^{-1}\cosh\gamma)\times\sinh^2\gamma d\gamma$, and our object is to compare k_3^2 with the square of the plasma frequency, $\sum N\cosh\gamma$. Their ratio is

$$\int_{\gamma=0}^{\gamma=\infty}\exp(-\theta^{-1}\cosh\gamma)\times(2\gamma\cosh\gamma-\sinh\gamma)d(\cosh\gamma)\Big/\int_{\gamma=0}^{\gamma=\infty}\exp(-\theta^{-1}\cosh\gamma)\times\sinh\gamma\cosh\gamma d(\cosh\gamma). \quad \text{(A3)}$$

Integrations by parts allow the conversion of the integrals into combinations of

$$K_0(\theta^{-1})=\int_0^\infty\exp(-\theta^{-1}\cosh\gamma)d\gamma, \quad \text{(A4)}$$

and

$$K_1(\theta^{-1})=\int_0^\infty\exp(-\theta^{-1}\cosh\gamma)\cosh\gamma d\gamma, \quad \text{(A5)}$$

where K_0 and K_1 are Bessel functions (see Synge, reference 6). One finds the ratio to be

$$(K_1+2\theta K_0)/(K_0+2\theta K_1)$$

which is $1+\frac{1}{2}\theta$ for small θ, reaches a maximum of 1.04 at $\theta^{-1}=5$ approximately, and goes to zero for large θ.