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Kinematics of Growing Waves*

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This paper is concerned with the problem of distinguishing between amplifying and evanescent waves. These have, in the past, been distinguished by considerations of energy transfer or of the initial and boundary conditions with which a wave must be associated. Both procedures are open to criticism.

The problem is here interpreted kinematically: we investigate the classes of wave functions which a given propagating system may support, without inquiring into the way these disturbances may be set up, and postponing inquiry into the boundary conditions necessary. In this way, we may distinguish between amplifying and evanescent waves by determining whether a wave function which may be analyzed into "real-frequency" waves may also be analyzed into "real-wave-number" waves. This question may be answered by means of a certain diagram, which may be constructed from knowledge of the dispersion relation.

Interchange of the roles of time and space leads to the statement and solution of a further problem. If a propagating system is unstable, the instability may be such that a disturbance grows, but is propagated away from the point of origin: this is termed "convective instability." On the other hand, the instability may be such that the disturbance grows in amplitude and in extent, but always embraces the original point of origin: this is termed "nonconvective instability." The statement that the system supports amplifying waves is synonymous with the statement that the system exhibits convective instability. A system which exhibits nonconvective instability may not be used as an amplifier, but may be used as an oscillator. It is possible to distinguish between convective and nonconvective instability by a further diagram which also may be constructed from knowledge of the dispersion relation.

Our theory enables us to make the following assertions. If ω is real for all real k, then any complex k, for real ω, denotes an evanescent wave. Conversely, if k is real for all real ω, then any complex ω, for real k, denotes nonconvective instability.

The theory is illustrated by certain simple examples and by discussion of the result of weak coupling between certain types of waves.

1. INTRODUCTION

In many branches of physics, one is interested in the propagation of waves through a complex medium. In some instances, such as the propagation of radio waves through the ionosphere or the production of electromagnetic power in electron tubes, these waves represent the physical phenomenon of interest. In other cases, notably in problems concerned with stability,[1,2] the waves which appear in the mathematical analysis are not accorded individual but rather "collective" significance in the sense that it is supposed that any real disturbance may be represented by a combination of such waves. One of the problems which, it has seemed, could not be resolved by this procedure, which is known as "substitution analysis,[3] is that of distinguishing between amplifying and evanescent waves.

The principal difficulty which one faces in approaching this problem is that the terms "amplifying" and "evanescent" are never defined: they are normally used as if their meanings were intuitively obvious whereas they have in fact been vague. It would seem that certain controversies concerning growing waves which have arisen are due primarily to a lack of understanding of the nature of these concepts. What attention these terms have received has been directed towards the statement of rules for recognizing these wave types rather than towards their definition. Our primary objective will be to elucidate the meaning of these terms, and our secondary objective that of deriving criteria for recognizing when a wave is amplifying and when it is evanescent.

We shall see, in the course of this communication, that there is a second, hitherto unrecognized, problem concerning instabilities of propagating media which also admit classification into two physically distinct types, the nature of which will be discussed later.

The first problem, that of distinguishing between amplifying and evanescent waves, may be expressed as follows. We suppose that the problem of wave propagation has been reduced to one-dimensional form; the "transverse boundary conditions" have therefore been taken into account. The variables in our problem, which we shall represent collectively as ϕ, may therefore be expressed as functions of one spatial coordinate z and time t, that is as $\phi(z,t)$. If the medium is homogeneous in z and t, one proceeds to look for solutions of the relevant equations which may be expressed in the form

$$\phi(z,t)=e^{i(kz-\omega t)}. \tag{1.1}$$

If the system is periodic in z or in t, the right-hand side of (1.1) must include a term of this periodicity, for instance,

$$\phi(z,t)=f(z)e^{i(kz-\omega t)}, \tag{1.2}$$

wherein $f(z)$ would be a function of the same periodicity

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[1] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932), Sec. 231.

[2] M. D. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A223**, 348 (1954).

[3] Substitution analysis has also been critized in its application to the problem of plasma oscillations in an electron gas of nonzero temperature [L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946); R. Q. Twiss, Phys. Rev. **88**, 1392 (1952)], but the difficulty is not fundamental—it may be traced to the fact that the plasma-oscillation modes do not represent a complete set but should be supplemented by field-free modes.

as the medium itself.[4] One normally finds that solutions of type (1.1) may exist only if ω and k are appropriately related: the equation which determines permissible relations,

$$D(\omega,k)=0, \tag{1.3}$$

is known as the "dispersion relation."

We shall restrict our attention to the case that the dispersion relation yields a finite, or denumerably infinite, number of solutions: these may be expressed either as

$$\omega=\Omega_\alpha(k), \tag{1.4}$$

or as

$$k=K_\alpha(\omega), \tag{1.5}$$

wherein α will enumerate the "modes" of the system.

It may be noted that there is some arbitrariness in analyzing the behavior of a propagating system into modes. Equation (1.3) represents a relation between two complex quantities, or between the four real quantities ω_r, ω_i, k_r, k_i, where

$$\omega=\omega_r+i\omega_i, \tag{1.6}$$

$$k=k_r+ik_i. \tag{1.7}$$

It therefore represents a set of two-surfaces in a four-dimensional space. These surfaces may have points, or curves, of contact, which may lead to some indeterminacy in relating the functions $\Omega_\alpha(k)$ which appear in (1.4) with the functions $K_\alpha(\omega)$ which appear in (1.5). However, where such indeterminacy occurs, one will normally be considering two or more modes at a time [for instance, a function $\Omega_\alpha(k)$ and its complex conjugate function] so that this indeterminacy is not likely to cause trouble in practice.[5]

Suppose, now, that Eq. (1.3) is solved in the form (1.5), since, in the given problem, we are concerned with the propagation of waves of a definite (real) frequency. If the resulting functions $K_\alpha(\omega)$ are real, there is no difficulty in interpreting our results: the wave propagates without attenuation, with a known phase velocity and with a group velocity[6,7] which is given by the derivative of $K_\alpha(\omega)$,

$$v_{\mathrm{gr}}=[dK(\omega)/d\omega]^{-1}, \tag{1.8}$$

or, equivalently, by

$$v_{\mathrm{gr}}=d\Omega(k)/dk. \tag{1.9}$$

However, now suppose that the function $K(\omega)$ is complex for the value of ω of interest. Two possible interpretations are known: either the wave is evanescent (the wave in a cut-off wave guide is of this category), or the wave is amplifying (one of the waves in a traveling-wave tube is of this category).

In some analyses which have followed the lines indicated, the distinction between amplifying and evanescent waves has been ignored completely. In some cases, this gap in the mathematical analysis was filled by experimental evidence[8,9]; in other cases, it was not.[10–12]

Among those who seek to distinguish between amplifying and evanescent waves, one may distinguish two schools, which may be labeled the "energetics" school, and the "boundary-condition" school. The criteria adopted by these schools will be reviewed in the next section. The point of view of the energetics school is, briefly, that one should distinguish between amplifying and evanescent waves by determining whether or not one may extract energy from the wave.[13–15] The point of view of the boundary-condition school is that one should distinguish between amplifying and evanescent waves by determining what wave form will be set up by prescribed initial and boundary conditions.[16–21] The first school is open to the following criticism: in setting up the dispersion relation, all conservation laws have implicitly been taken into account; one would therefore expect that whatever information is to be derived from the law of conservation of energy is already contained in the dispersion relation. The second school is open to this criticism: one may seek to classify a wave as "amplifying" or "evanescent" by making statements which are independent of any particular choice of boundary conditions or initial conditions, for instance, that waves of the first type may, with appropriate boundary and initial conditions, lead to the design of an amplifier, whereas waves of the second type cannot, with any choice of boundary and initial conditions, lead to the design of an amplifier; one would therefore expect —or at least hope— that the distinction between these two waves might be arrived at by some procedure other than the explicit analysis of a specified variety of initial and boundary conditions.

The rationale of the present treatment of this problem is that it should be possible to distinguish between amplifying and evanescent waves without introducing arguments of energetics or of Laplace-transform

[4] L. Brillouin, *Wave Propagation in Periodic Structures* (McGraw-Hill Book Company, Inc., New York, 1946), p. 140.

[5] Generalization of the theory here presented will nevertheless involve further analysis of the topology and interpretation of these "mode surfaces."

[6] Lord Rayleigh, *Theory of Sound* (MacMillan and Company, Ltd., London, 1894), Vol. I, p. 301 ff. and appendix.

[7] See reference 1, p. 357 ff.

[8] A. V. Haeff, Proc. Inst. Radio Engrs. **37**, 4 (1949).

[9] R. G. E. Hutter, *Advances in Electronics*, edited by L. Marton (Academic Press, Inc., New York, 1954), Vol. 6, p. 372.

[10] J. R. Pierce, J. Appl. Phys. **19**, 231 (1948).

[11] J. A. Roberts, Phys. Rev. **76**, 340 (1949).

[12] J. Feinstein and H. K. Sen, Phys. Rev. **83**, 405 (1951).

[13] J. R. Pierce, Bell System Tech. J. **33**, 1343 (1954).

[14] V. A. Bailey, Phys. Rev. **78**, 428 (1950); **83**, 439 (1951); **106**, 1356 (1957).

[15] L. J. Chu and H. A. Haus, Massachusetts Institute of Technology Internal Report, 1957 (unpublished).

[16] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).

[17] R. Q. Twiss, Phys. Rev. **88**, 1392 (1952).

[18] R. Q. Twiss, Proc. Phys. Soc. (London) **B64**, 654 (1951); Phys. Rev. **84**, 448 (1951).

[19] J. R. Pierce, Bell System Tech. J. **30**, 626 (1951).

[20] R. W. Gould, IRE Trans. on Electron Devices **2**, 37 (1955).

[21] J. R. Pierce and L. R. Walker, Phys. Rev. **104**, 306 (1956).

analysis. It is believed that the distinction may be looked for in what is here called "wave kinematics," by which we mean the study of the space-time distribution of wave amplitude of *free* waves or combinations of free waves. We shall consider, in particular the existence and nature of wave packets in media which support growing waves. The term "growing waves" is here used as a neutral term for a wave described by complex $K(\omega)$ for real ω, when it is yet to be decided whether this wave is amplifying or evanescent.

Our view of what constitutes wave kinematics is such as to lead to the following consequence: all statements which may be classed as "wave kinematics," which one may make with reference to a particular medium and a particular wave type, may be derived from the dispersion relation characterizing that wave. The aim of this paper will therefore be seen to be that of deriving a criterion for distinguishing between amplifying and evanescent waves which requires knowledge only of the dispersion relation, requiring no explicit knowledge either of energy transfer or of the influence of initial and boundary conditions.

In Sec. 2, we shall review the criteria which have previously been proposed or used for distinguishing between amplifying and evanescent waves. In Sec. 3, we shall give a kinematic interpretation of the distinction between amplifying and evanescent waves, and so derive a criterion for the distinction of these wave types which involves only the dispersion relation of the system. We shall find, in Sec. 4, that our kinematic interpretation of the distinction between amplifying and evanescent waves leads naturally to a corresponding distinction between two types of instability of a propagating medium, which we term "convective instability" and "nonconvective instability." It will be seen that a medium which exhibits convective instability may be used as an amplifier, whereas a medium which exhibits nonconvective instability is self-oscillatory, and so may not be used as an amplifier. We shall derive a criterion, involving only the dispersion relation, for distinguishing between these two types of instability. If, as seems likely, energetic considerations offer no simple way of distinguishing between the two types of instabilities which we shall be led to introduce, this fact constitutes a further objection to the energetic approach to the problem of growing waves.

Section 5 will be devoted to a discussion of certain mechanical models which are simple enough for one to understand intuitively the dynamical nature of their characteristic wave patterns. Section 6 will discuss certain complications of the theory which may on occasion arise, and certain more interesting examples.

2. EARLIER AND DIFFERENT APPROACHES TO THIS PROBLEM

It was stated, in the Introduction, that earlier approaches to the problem of distinguishing between amplifying and evanescent waves have turned upon arguments of energetics or of boundary conditions; certain general objections to these approaches were registered. In this section, we shall review these, and certain other, methods in a little more detail.

In attempting to classify waves by arguments of energy transfer, we should first notice that such a classification is impossible if one considers only the total energy transfer of a given pure wave, characterized by a real value of ω and complex value of k. The reason for this is that, if the system is conservative (and it is only if the system is conservative that one expects arguments of energy transfer to be fruitful), the mean energy flow (or "power") in the direction of the wave must be zero. This follows from the following simple argument: the energy density E and the energy flow S are related by the conservation equation

$$\partial E/\partial t+\partial S/\partial z=0. \tag{2.1}$$

For a wave which is sinusoidal in time, (2.1) implies that $\bar{S}(z)$, the mean energy flow, satisfies

$$d\bar{S}/dz=0, \tag{2.2}$$

so that $\bar{S}$ is constant. However, E and S are quadratic functions of the field variables ϕ, so that

$$\bar{S}\propto e^{-2k_i z}. \tag{2.3}$$

Since we are assuming k_i to be nonzero, it follows from (2.2) and (2.3) that

$$\bar{S}\equiv 0. \tag{2.4}$$

It is an immediate consequence of the argument of the preceding paragraph that if we are to distinguish between amplifying and evanescent waves by energy considerations, it will be necessary to divide the total energy flow into components which are to be assigned to the various "carriers" which constitute the propagating system. The necessity of such an analysis represents a further objection to the energetics method.

Let us now suppose that the propagating medium is subdivided into carriers which we enumerate by a suffix i. The total energy density and the total energy flow may thereby be analyzed as follows:

$$E=\textstyle\sum_i E_i, \quad S=\sum_i S_i. \tag{2.5}$$

Note that we cannot assume, in advance, that each energy component is essentially positive; it is, indeed, a characteristic of conservative amplifying systems that one of the carriers admits of negative-energy states.

Assuming that the energy and power have been subdivided as in (2.5), how is one to distinguish between amplifying and evanescent waves? In many problems, one is concerned with the interaction between streams of charged particles and electromagnetic fields, and one is looking for mechanisms for converting particle energy into electromagnetic-field energy. Some writers[14] have therefore adopted as the criterion for a genuine amplifying wave that the Poynting vector should be parallel to the direction in which the wave-amplitude is growing, since this seems to imply a progressive transfer of energy from particles to the field. However, this argument is not valid, as we may see from the schematic diagram shown in Fig. 1. This diagram indicates that it may be necessary to inject, at some boundary to the right of the growing wave, energy into the electron beam which is transferred, as shown, to the electromagnetic field and so appears as a growing Poynting flux parallel to the direction of growth. If energy must be injected at the high-amplitude end of the propagating system, the wave must be classed as evanescent rather than amplifying. Note also that it is not sufficient to adopt, as the criterion for amplification, that the energy transfer from the beam to the field should increase in the direction in which the wave grows.

A more persuasive criterion, based upon energetics, is obtained only by supplementing the analysis of energy and energy flow represented by (2.5) by an assumption about the "direction of flow."[13,15] If, in some sense which should be clarified but usually is not, the disturbance is propagating in the direction of growth, and if the Poynting vector is parallel to the direction of growth, then one would believe that the wave is amplifying rather than evanescent. However, there remains the problem of clarifying what is meant by the "direction of flow" and of giving a rule for determining this direction in any particular system. One might at first be tempted to identify this direction with the direction of the "group velocity," but one immediately faces the difficulty of fitting the idea of group velocity into the framework of growing waves.[22,23] The simple rule represented by (1.8) and (1.9) clearly cannot be valid since the expressions on the right-hand sides of these equations are complex.

In physical systems, one can often form a definite idea as to what the direction of flow must be, even if one cannot be precise as to what exactly is "flowing." For instance, if there are a number of interacting electron streams, all of which are moving in the same direction, one has the intuitive idea that disturbances represented by growing waves are, in some sense, being

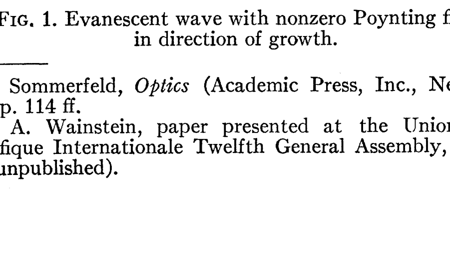

FIG. 1. Evanescent wave with nonzero Poynting flux in direction of growth.

[22] A. Sommerfeld, *Optics* (Academic Press, Inc., New York, 1954), p. 114 ff.
[23] L. A. Wainstein, paper presented at the Union Radio-Scientifique Internationale Twelfth General Assembly, Boulder, 1957 (unpublished).

propagated in the direction of the beam velocities. This assumption, which seems plausible but really calls for clarification and analysis, is usually an explicit or implicit ingredient of criteria employed by the energetics school.

The above argument may be given a particularly persuasive form by interpreting growing waves as being due to a "weak coupling" between the constituent carriers of the propagating system. Assuming that the individual carriers, when uncoupled, propagate unattenuated waves, and that the group velocities of these waves is in the same direction, it is reasonable to assume that, if the carriers are only weakly coupled, waves in the resulting system are propagated in the original direction, even if the waves so obtained are no longer unattenuated. It is seen that this argument, which underlies the "coupled mode" theory of electron tubes,[13] should lead to a reliable distinction between amplifying and evanescent waves, but only at the expense of analyzing the energy flow into constituents, and of appealing to the weak-coupling hypothesis.

The case in favor of distinguishing between amplifying and evanescent waves by calculation which takes the boundary and initial conditions explicitly into account—for instance, by Laplace-transform analysis—has been forcefully presented by Twiss.[18] There can be no objection to carrying through the mathematical analysis which duplicates precisely the physical process which occurs when an electron tube is switched on, and so arriving at the steady state which would be set up, except that this may prove a discouragingly laborious calculation, and except that one may have the suspicion that the distinction which we seek may be arrived at more simply. This means that we cannot take exception to the results of the transient calculation carried through by Laplace transformation in the t coordinate. If the medium were infinite in extent, so that there were no spatial boundary conditions to be considered, such a calculation might show certain waves to be evanescent, since their amplitude decayed away from the point of origin, or it might show the system to be amplifying, since the disturbance is propagated away from the origin but grows in amplitude, or it may prove the system to be unstable in the sense that the wave-amplitude in the neighborhood of the origin grows indefinitely.

However, since the preceding calculation leads more naturally to an analysis in real wave numbers than in real frequencies, it is sometimes proposed that a Laplace-transform calculation in the z variable should be carried out. It may seem that the validity of this procedure stands or falls with the validity of the Laplace-transform calculation in time, but this is not so. There is an important physical distinction between the two cases, in consequence of which Laplace-transform calculation in the z variable may lead to erroneous conclusions.

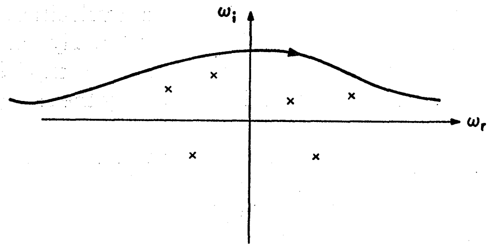

FIG. 2. Contour of integration for Laplace-transform analysis in time.

In applying Laplace-transform theory in time, one assumes that the system is undisturbed up to time $t=0$, when a disturbance is initiated by "forcing terms." Laplace-transform calculation, based on the equations of motion, will then lead to an expression of the following type for the wave amplitude[18]:

$$\phi_i(z,t)=e^{ikz}\int d\omega\, e^{-i\omega t}\frac{F_{ij}(k,\omega)\psi_j(k,\omega)}{D(k,\omega)}. \qquad (2.6)$$

For simplicity, we consider a disturbance which is sinusoidal in the space coordinate z. In the above equation, $\psi_j(k,\omega)$ characterizes the forcing term: the consideration of impulse-like forcing terms would enable us to interpret $\psi_j(k,\omega)$ alternatively as determined by the initial values of the potentials ϕ_i. The coefficients $F_{ij}(k,\omega)$ are characteristic of the propagating medium: the denominator $D(k,\omega)$ is identical with the function appearing in (1.3), and appears in Laplace-transform theory as the determinant of the matrix F_{ij}:

$$D(k,\omega)=|F_{ij}(k,\omega)|. \qquad (2.7)$$

The integral (2.6) may be analyzed into contributions arising from the poles of the integrand, that is, from the zeros of the dispersion function $D(k,\omega)$. However, in order to obtain a definite answer, it is necessary to specify the contour of integration. In the case we are considering, this is determined uniquely by the requirement that $\phi_i(z,t)\equiv 0$ for $t<0$. The contour must be that shown in Fig. 2, that is, it must be a curve which passes above all zeros of the function $D(k,\omega)$. This curve may be displaced in the direction $\omega_i\to+\infty$, which ensures that $\phi_i(z,t)\equiv 0$ if $t<0$. For positive values of t, the contour may be closed by a return path for which $\omega_i=-\infty$, which shows that every zero of $D(k,\omega)$ then contributes to the wave.

Now suppose that we attempt to repeat these arguments, interchanging the roles of z and t. We may imagine that we are considering the modulation of a beam incident from $z=-\infty$ at the plane $z=0$, the modulation being periodic in time. The resulting wave function may again be written in a form similar to (2.6),

$$\phi_i(z,t)=e^{-i\omega t}\int dk\, e^{ikz}\frac{F_{ij}(k,\omega)\psi_j(k,\omega)}{D(k,\omega)}, \qquad (2.8)$$

where the function $\psi_i(k,\omega)$ now characterizes the external force applied at the plane $z=0$. The problem still remains of choosing the appropriate contour of integration. If it could be asserted that $\phi_i(z,t)\equiv 0$ for $z<0$, then the contour would be analogous to that appropriate to the integral (2.6), that is, the integral shown in Fig. 3. (This contour now runs *below* the zeros of $D(k,\omega)$ since (1.2) gives different signs to the space and time exponents.) However, the important point to which we wish to draw attention is this: in the previous case, in which forces were applied at time $t=0$ to a system which was undisturbed for $t<0$, our understanding of causality led to the unmistakable conclusion that $\phi_i(z,t)\equiv 0$ for $t<0$; however, it is impossible to deduce from the principles of causality that, in the case we are considering, the wave functions $\phi_i(z,t)$ should vanish for $z<0$. There is no reason why unattenuated waves (for instance, electromagnetic waves traveling with approximately the velocity of light) should not propagate "upstream," in the negative z direction. Indeed, it may be that in some recalcitrant cases it is impossible for the wave function even to remain finite for negative values of z, since an amplifying wave propagates against the direction of the electron stream. There is also the more likely possibility that an evanescent wave will extend upstream from the modulating plane $z=0$, a sort of "bow wave." (A careful analysis shows that such space-charge bow waves exist in klystron amplifiers.) If we now suppose that such bow waves exist, it is clear that the contour of integration should not be that shown in Fig. 3: it should instead be that shown in Fig. 4, the contour running below all zeros of $D(k,\omega)$ except those representing evanescent waves which fall off in the negative z direction, that is, with negative values of k_i. Note, however, that there may be also genuine amplifying waves which will also have negative values of k_i: it is clearly necessary that the contour should run below the zeros of $D(k,\omega)$ representative of these modes, as shown in Fig. 4. We now see that, in carrying out Laplace-transform theory in z, the following basic difficulty has arisen: the contour may not be assumed to run below all poles of the dispersion function $D(k,\omega)$; moreover, in order to know

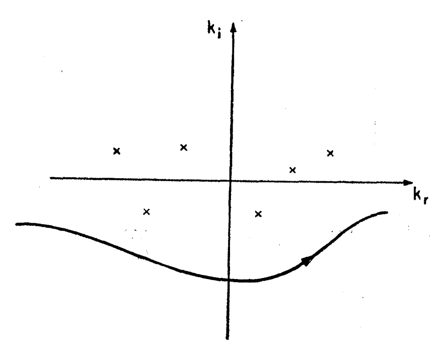

FIG. 3. Incorrect contour of integration for Laplace-transform analysis in space.

just what contour is appropriate, it is necessary to have some means of classifying waves which grow exponentially in the $+z$ direction into "amplifying" and "evanescent" waves. Hence the application of Laplace-transform theory in the z coordinate by no means leads to a classification of growing waves into amplifying and evanescent types: on the contrary, such a classification must be obtained by other methods if Laplace-transform theory in the z coordinate is to be employed at all.

It may be noted that the above difficulty is not alleviated by interpreting the integral (2.8) as giving the wave function due to prescribed initial values at the plane $z=0$ (which is, of course, the conventional interpretation of the Laplace-transform integral). The reason is that it may be physically impossible to set up the prescribed initial values of $\phi_i(z,t)$ at $z=0$ by operations at the plane $z=0$. For instance, if there is an evanescent wave with a negative value of k_i, this wave can be excited only by an appropriate forcing term at some positive value of z.

An unusual approach to the problem of resolving the distinction between amplifying and evanescent waves has been proposed by Piddington.[24] Piddington shows that it is possible to misinterpret waves which are known to be evanescent as amplifying, but deduces from this fact the unwarranted conclusion that all waves for which ω is real and k complex must, in fact, be evanescent. However, Piddington's contributions are valuable in emphasizing the relationship between amplification and instability: Piddington clearly interprets amplification as a kind of "convective instability," a point of view which our investigation leads us to endorse.

A new approach to the problem we are discussing has recently been proposed by Buneman.[25] The procedure here is to investigate whether or not it is possible to draw energy from a given growing wave by means of an appropriate probe. Since this theory is quite new, it is impossible to give a critical account of it here. It must suffice to observe that the theory, as at present proposed, makes it necessary to divide the system under consideration into independent components, and to study the interaction of these components with a "virtual probe." The results of this theory agree with our own insofar as they demonstrate that amplification is associated with instability.

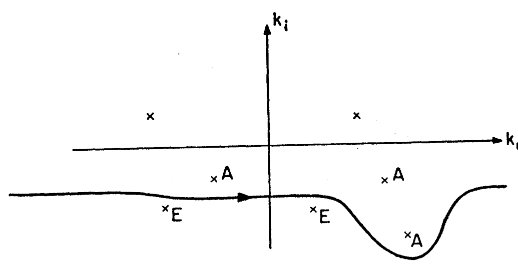

FIG. 4. Correct contour of integration for Laplace-transform analysis in space.

[24] J. H. Piddington, Phys. Rev. **101**, 9 (1956); **101**, 14 (1956).
[25] O. Buneman (to be published).

3. KINEMATIC FORMULATION OF THE PROBLEM

The principal thesis of this communication is the assertion that the distinction between amplifying and evanescent waves may be interpreted within the framework of wave kinematics. This term is intended to denote the space-time "geometry" of wave functions, and to exclude all specifically dynamical concepts such as energy, momentum, etc. In this section, we derive a kinematic classification of growing waves into two types, and adopt this classification as defining the terms "amplifying" and "evanescent." This kinematic classification will lead us to a criterion for distinguishing between these wave types which requires knowledge only of the dispersion relation.

We consider the relation (1.5), characterizing a particular mode of a propagating system. We suppose that, for some value ω_0 of ω, the quantity k, which is given by $K(\omega_0)$ if we drop the suffix α, is complex. We wish to find out whether this wave is amplifying or evanescent without looking into the energetics of the system and without explicit discussion of the boundary conditions necessary to excite this wave. This means that we must consider a "free" wave function which is not necessarily limited to any finite region of space and time. If this wave is monochromatic, there is no way of telling whether it is amplifying or evanescent.

It is proposed that we consider a wave function which is quasi-monochromatic. That is, we consider a wave which is expressible in the form

$$\phi(z,t)=\int_{-\infty}^{\infty}d\omega\, f(\omega)e^{i[K(\omega)z-\omega t]}, \tag{3.1}$$

where the integration is here assumed to run over all real values of ω, and it is assumed that $f(\omega)$ has a sharp peak at $\omega=\omega_0$ and is negligibly small elsewhere. For definiteness, we shall consider the following functional representation of $f(\omega)$:

$$f(\omega)=\exp\left[-\left(\frac{\omega-\omega_0}{\Delta\omega}\right)^{2N}\right], \tag{3.2}$$

where N takes some positive integral value and $\Delta\omega$ may be made arbitrarily small.

We first observe that the wave function represented by (3.1) is a "time-like packet" in the sense that, for any value of z, the wave function is bounded in extent. This follows at once from Riemann's Lemma.[26] Provided that $K_i(\omega)$ is bounded, the function $f(\omega)e^{iK(\omega)z}$ is bounded for all real values of ω. Hence, by Riemann's Lemma,

$$\phi(z,t)\rightarrow 0 \quad \text{as} \quad t\rightarrow\pm\infty. \tag{3.3}$$

[26] E. C. Titchmarsh, *Fourier Integrals* (Clarendon Press, Oxford, 1937), p. 11 ff.

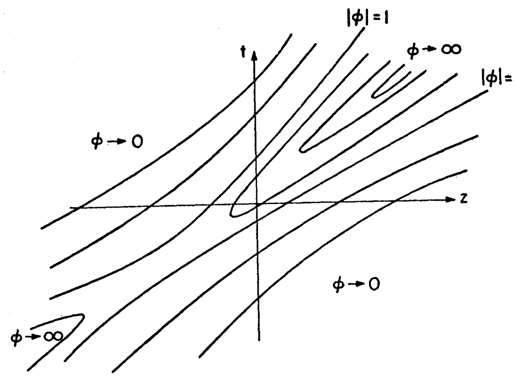

FIG. 5. Time-like packet also a space-like packet.

It is in this sense that we describe $\phi(z,t)$ as a "time-like packet."

The important question which now arises is the following: Is the wave function (3.1) also a space-like packet? If it is, then a "contour diagram" of the amplitude of the wave function over the z-t plane must be as shown in Fig. 5; on the other hand, the contour diagram might be as shown in Fig. 6, in which case the wave function could not be interpreted as a space-like packet. It is clear, from Fig. 5, that a wave function which may be interpreted as both a time-like packet and a space-like packet represents a disturbance which is being propagated in the medium: we might say that such a wave function is "launchable" since if it were generated, at some time, in a localized region of the z axis, the disturbance would be "launched" from this region since the disturbance in the neighborhood of this region ultimately vanishes. We may conceive that such a disturbance could also be generated by an appropriate time-varying signal applied at the plane $z=0$. Wave functions of the type shown in Fig. 6, on the other hand, do not represent propagated disturbances of the medium, since they are at no time localized in space. In order to realize a wave function of the type shown in Fig. 6, one would expect that it would be necessary to apply appropriate signals at *two* points, with large positive and negative z coordinates.

It is clear that wave functions of the type shown in Fig. 6 are those which we expect to arise in systems which support only evanescent waves, whereas we expect that wave functions of the type shown in Fig. 5 will arise in systems which support amplifying waves. We propose to adopt this classification as our kinematic formulation of the distinction between amplifying and evanescent waves. We therefore assert that *if a quasi-monochromatic spatially growing wave which is, by construction, a time-like packet is also a space-like packet, then the wave under consideration is amplifying; if, on the other hand, the time-like packet is not a space-like packet, then the wave is evanescent.*

It follows from the above definition that in order to determine whether or not a growing wave is amplifying, we should determine whether the function defined by (3.1) is such that, for arbitrary t,

$$\phi(z,t) \rightarrow 0 \quad \text{as} \quad z \rightarrow \pm\infty. \tag{3.4}$$

If this condition is satisfied, then the wave function is a space-like packet and the wave is amplifying; if this condition is not satisfied, the wave function is not a space-like packet, and the wave is evanescent.

If $\phi(z,t)$ is to be a space-like packet, it is necessary that, for arbitrary t, ϕ should be expressible as

$$\phi(z,t) = \int_{-\infty}^{\infty} dk\, g(k,t) e^{ikz}, \tag{3.5}$$

where the integration is to run over real values of k. By Riemann's Lemma, in order that $\phi(z,t)$ should be a space-like packet, it is sufficient that it should be expressible in the form (3.5), in which the integration runs over all real values of k, and, for arbitrary t, $g(k,t)$ is a bounded function of k.

The problem is now to decide how investigation of the dispersion relation will enable us to see whether or not the wave function defined by (3.1) may be expressed in the form (3.5). The information which we seek is given by a certain diagram. We construct, in the ω_r-ω_i plane, the locus Γ which is traced by the function (1.4), appropriate to the mode under consideration, as k takes all real values between $-\infty$ and ∞. This curve will not pass through the point $\omega=\omega_0$ since, by hypothesis, this value of ω corresponds to a complex value of k. We shall suppose that $K(\omega)$ is real for real ω outside the range ω_1 to ω_2. The path of integration of the integral (3.1) therefore corresponds to integration over real values of k, except for the contribution to the integral between the limits ω_1 and ω_2. However, we remember that, by appropriate choice of $f(\omega)$—for instance, by making $\Delta\omega$ in (3.2) arbitrarily small—we may make the contribution to the integral (3.1) outside a small neighborhood of ω_0 negligibly small. If the functions occurring in (3.1) are analytic, we are entitled to displace the path of integration of (3.1) so that the integration runs from $\omega=-\infty$ to $\omega=\infty$ by some path other than the real axis. The question which interests us is this: Is it possible to displace the contour of integration in such a way that the integral (3.1) then represents

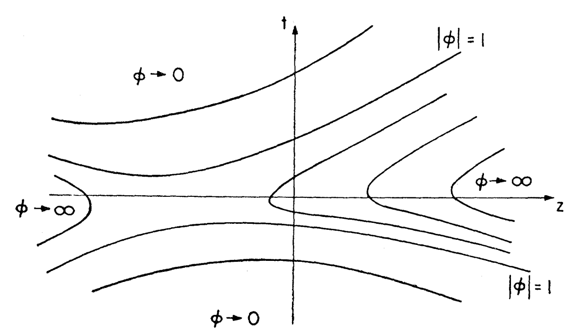

FIG. 6. Time-like packet not a space-like packet.

integration over real values of k? This will be possible if and only if the curve Γ "bridges" the gap between ω_1 and ω_2. If Γ has the form shown in Fig. 7, it is clearly possible to re-express the integral (3.1) as

$$\phi(z,t)=\int_{-\infty}^{\infty} dk\,\frac{d\Omega}{dk} f(\Omega(k))e^{i[kz-\Omega(k)t]}, \qquad (3.6)$$

where k runs over all real values.

If $\phi(z,t)$ is to represent a space-like packet, it is certainly necessary that Γ should so bridge the gap between ω_1 and ω_2 that the integral (3.1) may be re-expressed as (3.6). Hence we may immediately assert that if we find the contour Γ to be of the form shown in Fig. 8, then the mode under consideration represents an evanescent wave over the band ω_1 to ω_2. In fact, the situation shown in Fig. 8 necessarily involves a complication which was referred to in the Introduction. In analyzing media which support growing waves, we shall normally consider modes in groups of at least two: for instance, the waves characterized by $K(\omega)$ and by $K^*(\omega)$, where K^* denotes the complex conjugate of K. We expect, of course, that all the modes of one group will be of the same type—either evanescent or amplifying, although some of the amplifying modes might prove to be uninteresting since they represent negative amplification.

Let us now assume that the contour Γ is of the form shown in Fig. 7, so that the integral (3.1) may be expressed in the form (3.6). We shall consider two questions: With what range of the curve Γ is the dominant contribution to the integral (3.6) associated, and what further conditions must be satisfied in order that the integral (3.6) should represent a space-like packet? The first question is easily answered if we consider the particular functional form (3.2). By making $\Delta\omega$ arbitrarily small, we may insure that the contribution to the integral arising from that part of the

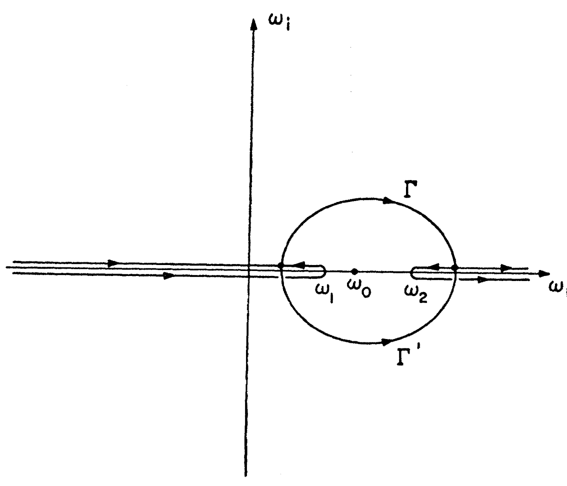

FIG. 7. Frequency band corresponding to growing waves bridged by locus of frequencies corresponding to real wave numbers.

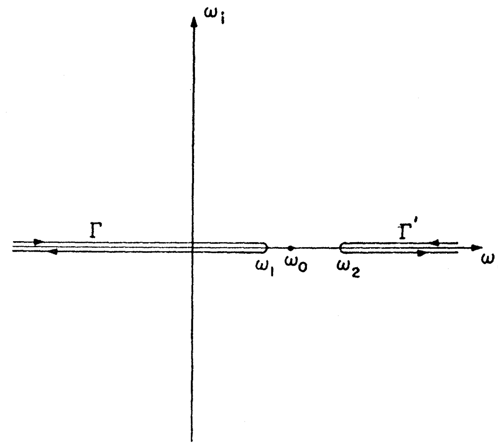

FIG. 8. Frequency band corresponding to growing waves not bridged by locus of frequencies corresponding to real wave numbers.

contour for which

$$|\omega_i|<\alpha|\omega_r-\omega_0| \qquad (3.7)$$

is made negligibly small, where α is the smallest real root of the equation

$$1-\binom{2N}{2}\alpha^2+\binom{2N}{4}\alpha^4-\cdots+(-)^{2N}\alpha^{2N}=0. \qquad (3.8)$$

If $N=1$, $\alpha=1$; if $N=2$, $\alpha=[(3-2\sqrt{2})]^{\frac{1}{2}}$. It is clear that, whatever form the function $f(\omega)$ may take, the dominant contribution must come from that portion of the curve Γ for which $\omega_i\neq 0$.

We may now turn to the second question. It follows from Riemann's Lemma that the integral (3.6) will represent a space-like packet, for which (3.4) is satisfied, provided that $(d\Omega/dk)f(\Omega(k))e^{\Omega_i(k)t}$ is bounded for all real k. However, the consideration of the preceding paragraph indicates that it is, indeed, sufficient that this function should be finite over the range of values of k necessary to bridge the gap in the real ω axis, excluding the points of intersection with the axis. If we restrict our attention to functions $f(\omega)$ which do not happen to have poles on the curve Γ, this condition is satisfied provided that Γ remains in the finite part of the plane, and provided that the complex function $d\Omega/dk$ is bounded over the relevant part of the curve Γ and has no singularities between Γ and the real axis.

4. CONVECTIVE AND NONCONVECTIVE INSTABILITY

We now pass on to the consideration of a new problem. By considering wave functions which were defined in such a way as to be time-like packets, and then enquiring into conditions necessary for these functions to be also space-like packets, we have arrived at a classification between amplifying and evanescent waves, and a criterion for distinguishing them. What

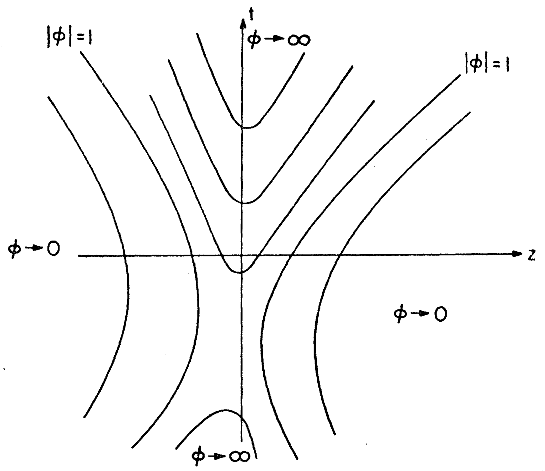

FIG. 9. Space-like packet not a time-like packet.

analogous information may we obtain by interchanging the roles of space and time?

We now begin by considering modes as characterized by the form (1.4) of the dispersion relation. We shall consider that, for a particular value k_0 of k, $\Omega(k)$ is complex. Clearly, we are now considering the problem of *instability* of the given medium for, if $\Omega_i>0$, the component of the disturbance characterized by k_0 will grow in time. We may therefore anticipate that the analysis given in Sec. 3 will enable us to define a classification of unstable modes into two types, and that we shall be able to obtain a criterion for distinguishing between these types.

We now consider, in place of the expression (3.1), the integral

$$\phi(z,t)=\int dk\, g(k)e^{i[kz-\Omega(k)t]}, \tag{4.1}$$

in which we assume that the function $g(k)$ is sharply peaked at $k=k_0$. Hence, if $\Omega_i(k)$ is bounded, (4.1) represents a space-like packet.

We may again distinguish, for the functions defined by (4.1), two possibilities: the function $\phi(z,t)$, which is by hypothesis a space-like packet, may or may not also be a time-like packet. In order to see the significance of this distinction, we may draw diagrams for the two possibilities. If the wave function represents a time-like packet, we obtain once more contours of the form shown in Fig. 5. If, on the other hand, the space-like packet is not a time-like packet, we obtain the set of contours shown in Fig. 9. These two diagrams represent two types of instability: it is suggested that that shown in Fig. 5 be termed "convective instability," and that shown in Fig. 9 be termed "nonconvective instability." Hence we arrive at the following definition: *if a wave function growing in time, which is composed of a narrow spectrum of wave numbers, and which is therefore a space-like packet, is also a time-like packet, then the instability represented by this wave is convective; if, on the other hand, the space-like packet is not a time-like packet, then the instability is nonconvective.*

The classification of instability which we have so easily arrived at is instructive and illuminating. The theory of dynamical systems of a finite number of degrees of freedom leads one to expect that any system which is shown to be unstable cannot persist in a quiescent state if random disturbances, no matter how small, must be supposed to be present. This simple view suggests that a propagating system characterized by a dispersion relation which admits complex values of ω for real values of k must be supposed to disrupt some time after an arbitrarily small disturbance is admitted. However, electron tubes such as the traveling-wave tube and the two-stream amplifier are represented by dispersion relations with this property, but it is known that the tube is not unstable in the sense indicated. This paradox, to which Twiss has drawn attention,[18] is resolved by our classification of instability. If a propagating system exhibits convective instability, a finite length of the system may persist in a quiescent state, even in the presence of small random disturbances, since these disturbances, although amplified, are carried away from the region in which they originate. Such systems may be used as amplifiers, and the traveling-wave tube[27] and two-stream amplifier[8] are of this type. If, on the other hand, a propagating system exhibits nonconvective instability, an arbitrary perturbation of the system will give rise to a disturbance which grows in amplitude at the point at which the perturbation originated; we also expect that the disturbance will spread until it extends over an arbitrarily large region of the system. If an electron tube were to exhibit nonconvective instability, it could not be used as an amplifier; it would be said to be "unstable" or "self-oscillatory."

We may distinguish between convective and nonconvective instability by constructing diagrams analo-

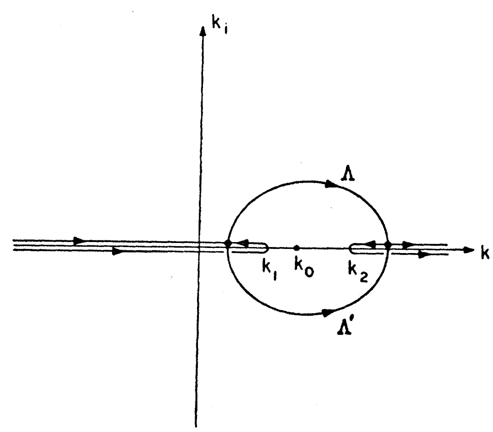

FIG. 10. Band of wave numbers corresponding to instability bridged by locus of wave numbers corresponding to real frequencies.

[27] J. R. Pierce, *Traveling-Wave Tubes* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950).

gous to those shown in Figs. 7 and 8. We construct, in the k_r-k_i plane, the curve Λ which is the locus of points $k=K(\omega)$, where K is the function (or functions) appropriate to the mode under investigation, and ω takes all real values. We are assuming that, for $k=k_0$, ω is complex: hence the curve Λ may not pass through the point k_0. The two types of curve which we may expect to obtain are shown in Figs. 10 and 11. If the curve Λ bridges the interval k_1 to k_2, for which ω is complex, the space-like packet (4.1) may be rewritten in the form

$$\phi(z,t)=\int d\omega \frac{dK}{d\omega} g(K(\omega))e^{i[K(\omega)z-\omega t]}, \qquad (4.2)$$

in which the integration is over real values of ω, so that the packet is time-like; the instability is therefore convective. If, on the other hand, the curves Λ, Λ' representing a pair of modes have the configuration shown in Fig. 11, the integral (4.1) may not be re-

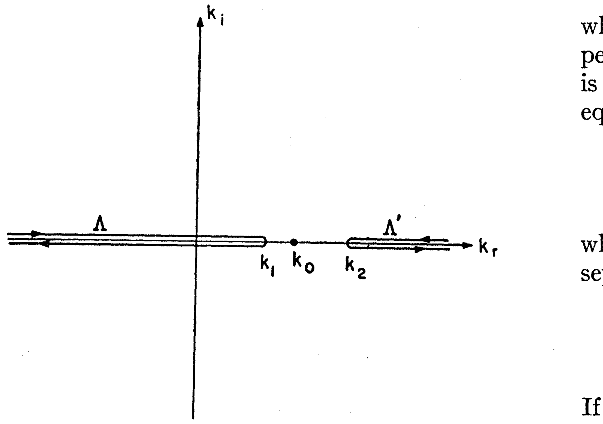

FIG. 11. Band of wave numbers corresponding to instability not bridged by locus of wave numbers corresponding to real frequencies.

expressed as an integral for a time-like packet, so that the instability is nonconvective.

5. DISCUSSION OF MECHANICAL MODELS

In order to compare the kinematic classification of wave types which has been set out in the preceding two sections with one's more intuitive dynamical ideas about growing waves, it is convenient to discuss simple examples. In this section we shall consider a mechanical device which is flexible enough to display nongrowing and growing waves of all types. We shall find that, for this model, it is easy to decide on mechanical grounds whether a spatially growing wave is amplifying or evanescent, but not at all easy to decide whether a wave growing in time represents convective or nonconvective instability. Hence, even in this simple case, the kinematic theory confirms what one knows dynamically and also adds something which one did not know dynamically.

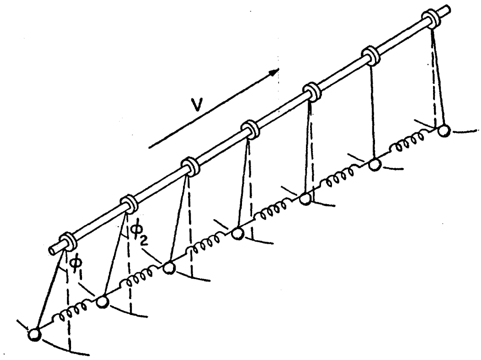

FIG. 12. Mechanical model of simple propagating system.

We consider a linear array of oscillators, such as pendula vibrating in planes transverse to the z axis. The equation of motion of each oscillator is then of the form

$$d^2\phi/dt^2=-\lambda\phi, \qquad (5.1)$$

where ϕ might measure the angular displacement of the pendulum. If we now suppose that each pendulum bob is connected to its neighbors by elastic strings, the equation of motion of the nth pendulum is of the form

$$\frac{d^2\phi_n}{dt^2}=-\lambda\phi_n-\tilde{\mu}(\phi_{n+1}-2\phi_n+\varphi_{n-1}), \qquad (5.2)$$

which becomes, in the limiting case of infinitely short separation between oscillators,

$$\frac{d^2\phi}{dt^2}=-\lambda\phi+\mu\frac{\partial^2\varphi}{\partial z^2}. \qquad (5.3)$$

If we now suppose that the whole assembly, shown schematically in Fig. 12, is translated in the z direction with velocity v, the total derivative in (5.3) may be re-expressed as

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)^2\phi=-\lambda\phi+\mu\frac{\partial^2\phi}{\partial z^2}. \qquad (5.4)$$

We see at once that the dispersion relation derivable from (5.4) is

$$(\omega-vk)^2=\lambda+\mu k^2. \qquad (5.5)$$

If, as we have implicitly assumed, the pendula are hanging in their stable positions and the connecting links are under tension, $\lambda>0$ and $\mu>0$ so that (5.5) may be rewritten as

$$(\omega-vk)^2=\omega_0^2+c^2k^2. \qquad (5.6)$$

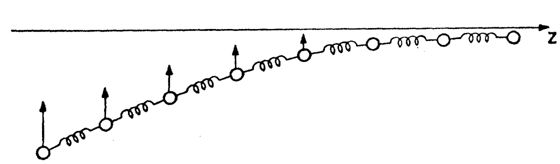

FIG. 13. Wave pattern for low-frequency evanescent wave.

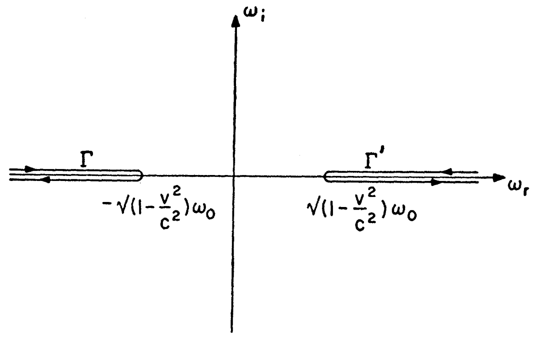

FIG. 14. Diagram characterizing waves known to be evanescent.

(This equation is identical with the long-wave approximation to the dispersion relation of a drifting electron plasma of nonzero temperature.[28]) We know that if spatially growing waves appear in such energetically passive systems, these are evanescent waves (see Fig. 13). Equation (5.6) leads to complex values of k, for real ω, only if $c>|v|$ and $|\omega|<(1-v^2/c^2)^{\frac{1}{2}}\omega_0$. Within this range,

$$k_r=-\frac{v\omega}{c^2-v^2},\quad k_i=\pm\frac{[(c^2-v^2)\omega_0{}^2-c^2\omega^2]^{\frac{1}{2}}}{c^2-v^2}. \qquad (5.7)$$

In order to compare our conjecture with our kinematic criterion, we construct the curves Γ from the formula

$$\omega=vk\pm(\omega_0{}^2+c^2k^2)^{\frac{1}{2}}. \qquad (5.8)$$

This diagram is found to be as shown in Fig. 14, which is of the type shown in Fig. 8 and so denotes evanescent waves, as we expect. Since ω is real for all real k, the system is of course stable so that no investigation of instability is required. We may, indeed, notice that *the fact that the dispersion relation admits of no complex values of* ω *for real values of* k *is sufficient reason for one to class any spatially growing waves which may occur as evanescent.*

We now rearrange the model in such a way that it is clear, from a dynamical point of view, that it will act as an amplifier. We first assume that the coupling between pendula is removed and that the pendula are initially in their positions of unstable equilibrium: thus $\lambda<0$ and $\mu=0$. If ω is real, k is complex so that the system will again support spatially growing waves.

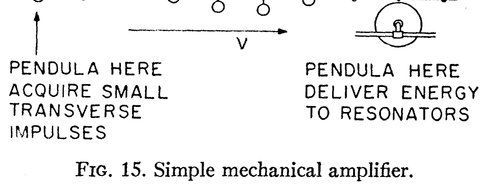

FIG. 15. Simple mechanical amplifier.

[28] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).

This device is an amplifier in the sense that more energy may be extracted from the wave than is used to modulate the wave. The initial modulation may be accomplished by arbitrarily small disturbances given to undisturbed pendula as they pass a given plane whereas one may, sufficiently far down the line, extract a finite amount of power from the system by arranging for the bobs to strike massive resonators such as large pendula suspended in their stable positions. (See Fig. 15.)

We now apply the kinematic criterion to this system, relaxing the condition that the connecting strings should be removed. The dispersion relation may now be written as

$$(\omega-vk)^2=-\nu^2+c^2k^2. \qquad (5.9)$$

We must distinguish between the cases $|v|>c$ and $|v|<c$; the former is relevant to our above model of an amplifier. Equation (5.9) leads to the following formulas for $K(\omega)$:

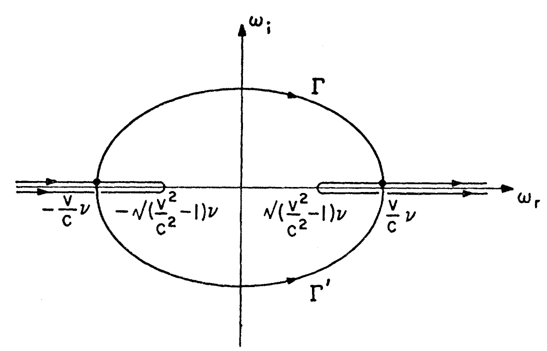

FIG. 16. Diagram characterizing waves known to be amplifying.

$$\begin{aligned} k_r&=\frac{v\omega}{v^2-c^2},\quad k_i=\frac{[(v^2-c^2)\nu^2-c^2\omega^2]^{\frac{1}{2}}}{v^2-c^2} \\ &\qquad\text{if}\quad |\omega|<(v^2/c^2-1)^{\frac{1}{2}}\nu, \\ k_r&=\frac{v\omega\pm[c^2\omega^2-(v^2-c^2)\nu^2]^{\frac{1}{2}}}{v^2-c^2},\quad k_i=0 \\ &\qquad\text{if}\quad |\omega|>(v^2/c^2-1)^{\frac{1}{2}}\nu. \end{aligned} \qquad (5.10)$$

Hence the system admits growing waves only in the band $|\omega|<(v^2/c^2-1)^{\frac{1}{2}}\nu$. In order to determine the nature of these growing waves, we derive the functions $\Omega(k)$,

$$\begin{aligned} &\omega_r=vk,\quad \omega_i=\pm(\nu^2-c^2k^2)^{\frac{1}{2}},\quad \text{if}\quad |k|<\nu/c, \\ &\omega_r=vk\pm(c^2k^2-\nu^2)^{\frac{1}{2}},\quad \omega_i=0,\quad \text{if}\quad |k|>\nu/c, \end{aligned} \qquad (5.11)$$

and so construct the curves Γ which are as shown in Fig. 16. These curves are of the type shown in Fig. 7, and so confirm that the growing waves are amplifying. If $|v|<c$ there are no spatially growing waves.

It is clear from dynamical considerations that the above model is unstable, and it is also clear that this instability is convective if $c=0$, $v\neq 0$, and nonconvective if $c\neq 0$, $v=0$. We now consider the question of stability kinematically, first for the case $|v|>c$. Formulas (5.11) show that this instability is restricted to the range $|k|<\nu/c$. The nature of this instability may be determined by constructing the curves Λ by means of formulas (5.10), which are as shown in Fig. 17. Our expectation that the instability is convective for the case $c=0$ is confirmed, since this diagram is of the type shown in Fig. 10; however our kinematic arguments have given us more information (for $c\neq 0$) than could be deduced dynamically.

We now consider the instability exhibited by the system when $|v|<c$. The band of instability is as before, but the curves Λ are now determined by the formulas

$$k_r=\frac{-v\omega\pm[(c^2-v^2)\nu^2+c^2\omega^2]^{\frac{1}{2}}}{c^2-v^2},\quad k_i=0. \tag{5.12}$$

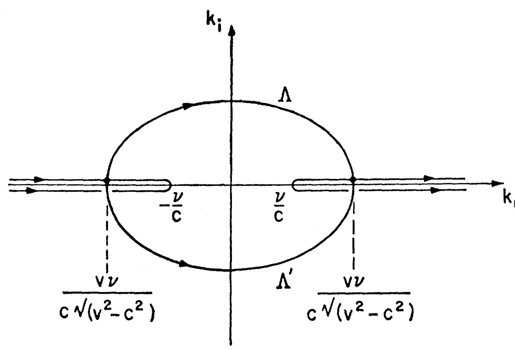

FIG. 17. Diagram characterizing convective instability in mechanical model.

These curves are as shown in Fig. 18. Since they are of the type shown in Fig. 11, we conclude that the instability is now nonconvective, in agreement with our earlier conjecture concerning the particular case $v=0$, $c\neq 0$. This result may be regarded as a special case of the following general assertion: *if the dispersion relation admits only real values of k for real values of ω, then any instability must be nonconvective.*

6. DISCUSSION

In Sec. 3, we considered a mode for which the function $K(\omega)$ is real outside the finite band between frequencies ω_1 and ω_2; a similar simplification was made in Sec. 4. This assumption is appropriate to the discussion of systems composed of lossless elements, except that one of the frequencies ω_1 and ω_2 might be supposed to be infinite, a point which we shall return to in a later paragraph. However, many physical systems involve lossy elements for which we must assume that it is the exception, rather than the rule, for $K(\omega)$ to be real. It is interesting to note briefly modifications which must be made in our theory in this case.

It is easy to see what effect a small amount of loss or dissipation will have upon the curves Γ determined by

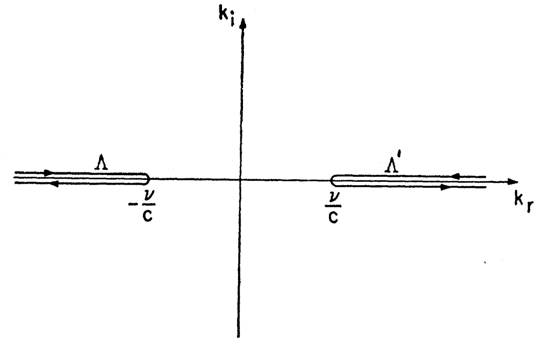

FIG. 18. Diagram characterizing nonconvective instability in mechanical model.

the functions $\Omega(k)$ in the ω_r-ω_i plane. We see from (1.1) that the function $\Omega(k)$ will generally have a small imaginary part which is negative in sign. This will have the effect of displacing the curves Γ slightly in the negative ω_i direction. Hence the typical diagrams, shown in figures 7 and 8, for amplifying and evanescent waves will be replaced by those of Figs. 19 and 20, respectively. In looking for amplifying waves, it is clear that we should look for ranges of values of ω, which correspond to complex k, and which are "bridged" by a curve Γ which enters the upper half-plane, for this is essential if disturbances are to grow in time. It follows that a system will cease to be amplifying if the losses are so great that the curves Γ are depressed entirely below the ω_r axis. These minor facts apart, the analysis of Sec. 3 is unaffected.

The analysis of stability given in Sec. 4 will also be affected slightly by dissipation terms. However, we find that the sense of the displacement of the curves Λ depends upon the value of the derivative $d\omega/dk$.

Let us now ignore dissipation, but suppose that one of the frequencies ω_1, ω_2—say ω_2—is infinite. The analysis of Sec. 3 would now be affected in a significant way if Ω_i/Ω_r tends to a nonzero limit as $k\rightarrow\infty$. Suppose, for instance, that $\Omega_i/\Omega_r\rightarrow 0.5$. We now find that wave functions of the type (3.1) may be re-expressed as integrals over real values of k if, in (3.2), $N=1$. This

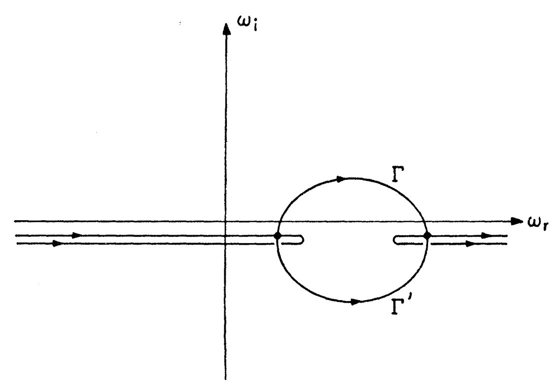

FIG. 19. Displacement of diagram characterizing amplifying waves due to small losses.

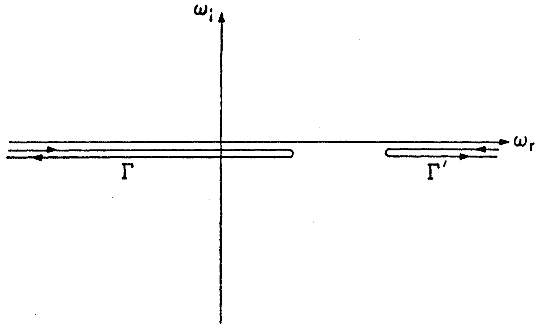

FIG. 20. Displacement of diagram characterizing evanescent waves due to small losses.

may be seen from Fig. 21, in which the region of the ω_r-ω_i plane which contributes to the integral (3.1) is shown shaded. However, if we now consider the function $f(\omega)$ defined by (3.2), with $N=2$, we find that the integral (3.1) may not be expressed as an integral over real values of k, since the area of the ω_r-ω_i plane contributing to the integral is now that shown in Fig. 22, which is not crossed by the curve Γ. Hence a system characterized by such a curve Γ would have the curious property that certain types of disturbance are amplified and propagated, whereas other types of disturbance must be classed as nonpropagating. Since the k_r-k_i diagrams would be similar to Figs. 21 and 22, we should conclude that the nonpropagating disturbances represent nonconvective instability. Hence, if Ω_i/Ω_r tends to a nonzero value as $k\rightarrow\pm\infty$, or if, equivalently, K_i/K_r tends to a nonzero limit as $\omega\rightarrow\pm\infty$, we should conclude that the system is "essentially" unstable, since disturbances represented by functions of the type (3.2) will, if N is sufficiently large, be associated with nonconvective instability.

It may happen that, in setting up a model for a given physical system, one will arrive at dispersion relations which lead to the curious situation noted in the preceding paragraph. However, it seems likely that this will happen only for models which must be classed as inexact or incomplete. The writer believes that it is unphysical to assume that Ω_i/Ω_r tends to any limit other than zero as $k\rightarrow\pm\infty$. The rationale of this conjecture

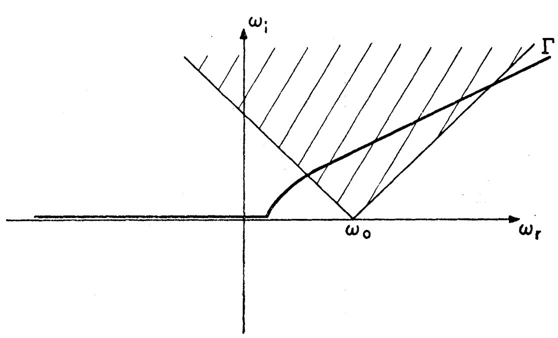

FIG. 21. Example of time-like wave packet which is space-like for given asymptotic behavior of the dispersion relation.

is as follows. The complex media which we are describing are composed of a number of discrete "carriers," such as the electromagnetic field, electron beams, etc. Amplification or instability can occur only by virtue of cooperative interaction between these carriers. Now consider a disturbance applied very suddenly to any one of these carriers. At the moment of this disturbance, the other carriers will be unaffected due to the inertia of their components, and possibly also to the finite velocity of propagation of signals. Hence the cooperative interaction necessary for growth of this disturbance will not follow immediately, but after a time determined by the finite inertia and finite signal velocities of the system. This suggests that, for very high frequencies, the dispersion relation separates into a number of terms which characterize each carrier of the system, independently of the other carriers. The point which we wish to make is well demonstrated by the dispersion relation representing the interaction, by electrostatic

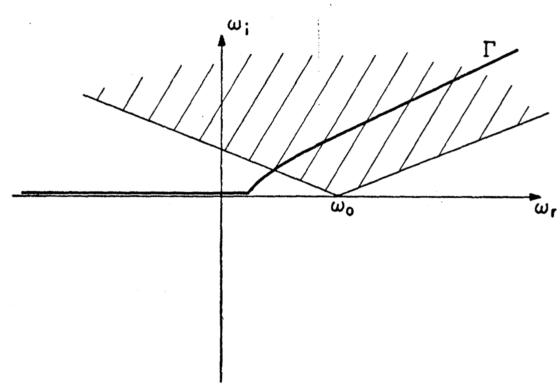

FIG. 22. Example of time-like packet which is not space-like for same asymptotic behavior of dispersion relation.

forces, of a number of superposed electron streams[29]:

$$\sum_i \left[\omega_i^2/(\omega-v_ik)^2\right]=1, \tag{6.1}$$

in which v_i is the velocity of the ith stream, and ω_i is the plasma frequency of the stream. We see at once that, as $k\rightarrow\infty$, the roots of (6.1) tend to $\omega=v_ik$, taken twice over for each stream. The point is not well exemplified by mechanical models, such as those considered in Sec. 5, since mechanical models usually involve such unphysical assumptions as infinite forces of constraint.

This point deserves further study but, in this communication, we merely note that if our conjecture is correct, the classification of waves into "amplifying" and "nonamplifying" (or, more pertinently, the classification of instability into "convective" and "nonconvective") will be independent of the nature of the wave packet.

In this paper we have noted that, in a system which supports growing waves, the behavior of a wave packet

[29] G. Ecker, Z. Physik 140, 274 (1955).

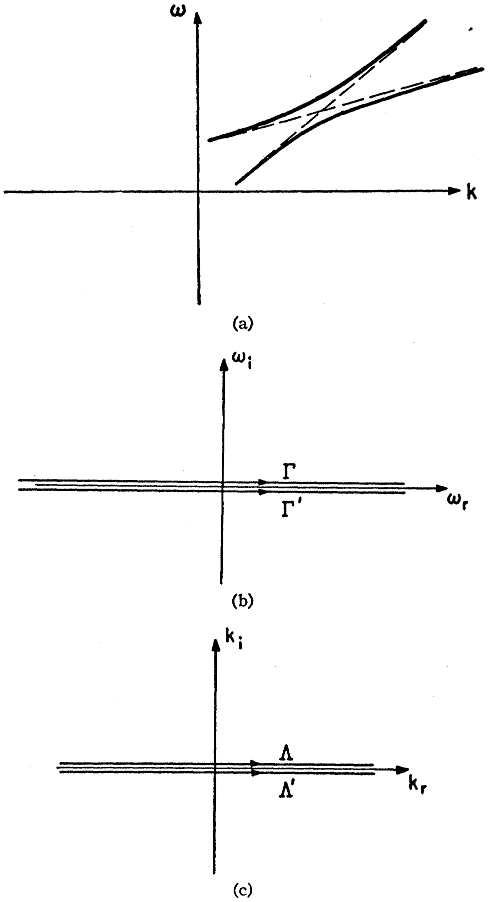

FIG. 23. Mode coupling leads to simple propagating waves.

will depend, to a large extent, upon the particular spectrum of that packet. However, there are certain important remarks which may be made which are independent of the nature of the spectrum, which underlies our classification of growing waves and of instabilities. Although we do not expect to be able to assign a definite velocity of propagation to a growing signal, we do expect that one may at least determine, from the dispersion relation, the direction in which a signal is propagated. This information should be contained in the diagram of Fig. 7, or in that of Fig. 10. It seems clear that the direction of propagation of disturbances is determined by the sense in which the curve Γ (or Λ) is traced as k (or ω) runs through the range of values from $-\infty$ to ∞. If we consider that the coupling between the carriers responsible for the amplification is removed, then the curves Γ and Λ will lie upon the real axes of the appropriate planes. The sense in which these curves are traced is then determined by the sign of $d\omega/dk$, which is now the group velocity. If we now assume that the original coupling between carriers is restored by degrees, the sense (i.e. the sense of traversal) of the curves Γ and Λ will be unaffected, and so will the sense of propagation of disturbances. Hence the identification which we know to hold in the absence of amplification must still hold when there is amplification.

It is proposed that, for simplicity, we return to the consideration of lossless systems which support amplifying waves only in a finite part of the spectrum.

The reader has doubtless conjectured that the classification of a mode as "amplifying" in Sec. 3 is synonymous with its classification as a "convective instability" according to Sec. 4. This identification is in full accord with our interpretation of the nature of "amplifying waves." In order to pursue this point, it would be necessary to consider the topology of the two-surfaces represented by the dispersion relation in the four-dimensional space with coordinates k_r, k_i, ω_r, ω_i. However, the relationship between the classifications of Sec. 3 and Sec. 4 is clarified by the consideration of the possible modes which may arise due to the coupling of just two carriers, or due to the coupling of one mode from each of two propagating systems.[13,30–32] The consideration of this simplified problem will also enable us to relate the diagrams drawn in Secs. 3 and 4 with

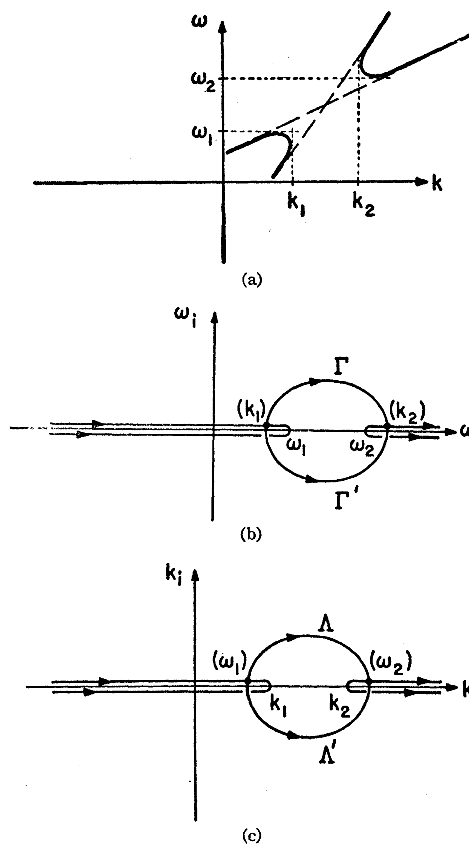

FIG. 24. Mode coupling leads to amplifying waves (convective instability).

[30] J. R. Pierce, J. Appl. Phys. **25**, 179 (1954).
[31] J. R. Pierce and P. K. Tien, Proc. Inst. Radio Engrs. **42**, 1389 (1954).
[32] H. Heffner, Proc. Inst. Radio Engrs. **43**, 210 (1955).

the more familiar diagram relating ω and k when both may be taken to be real.

We first consider the coupling between two modes in which, for the uncoupled state, the group velocities of the two modes are in the same sense. The uncoupled modes are indicated by dashed lines in Fig. 23(a). When these modes are coupled, the curves characterizing the modes of the coupled system may be either of the form shown in Fig. 23(a), or of the form shown in Fig. 24(a). The case shown in Fig. 23(a) is that the coupled system supports two simple waves: there are no amplifying or evanescent bands, and there is no type of instability. The curves Γ therefore lie along the real ω axis as shown in Fig. 23(b), and the curves Λ lie upon the real k axis as shown in Fig. 23(c).

Now consider the more interesting case shown in Fig. 24(a). For values of k between k_1 and k_2, ω is complex; for values of ω between ω_1 and ω_2, k is complex. It is clear that the curves Γ and Λ cannot be of the forms shown in Figs. 8 and 11; the curves will be of the types shown in Figs. 7 and 10, as shown in Figs. 24(b) and (c).

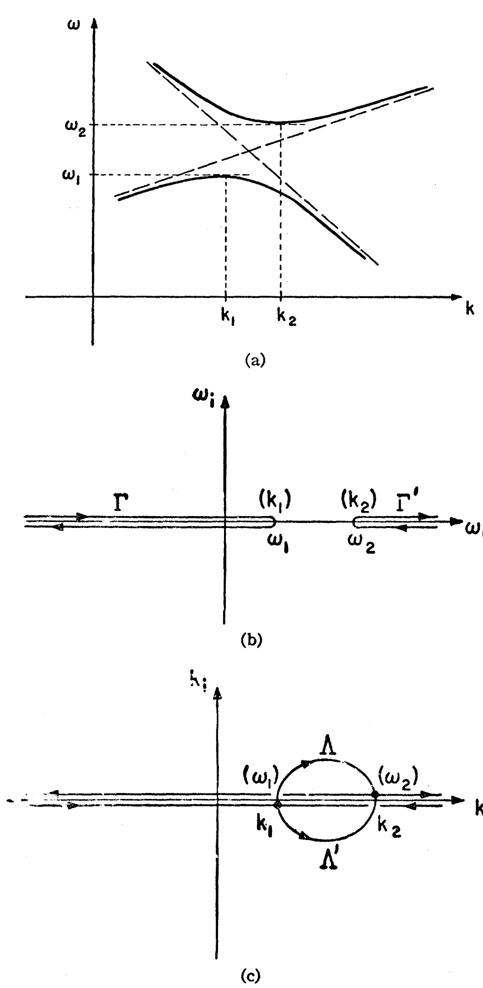

FIG. 25. Mode coupling leads to evanescent wave.

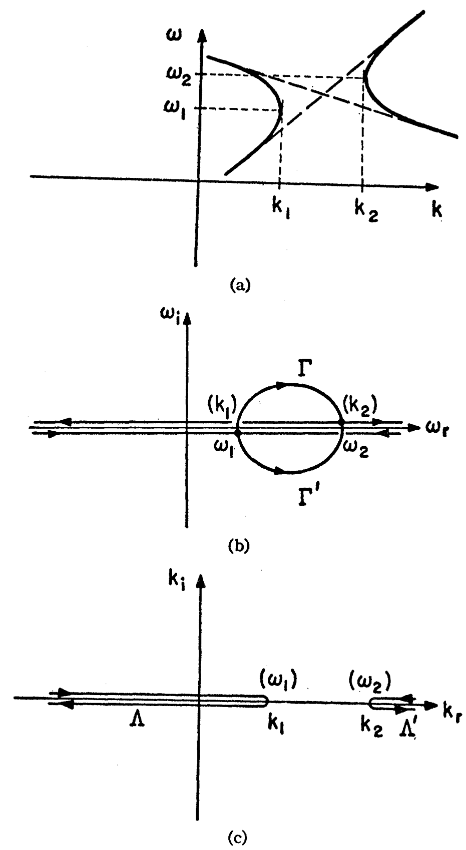

FIG. 26. Mode coupling leads to nonconvective instability.

We see that this model confirms our identification of the terms "amplifying wave" and "convective instability."

Let us now consider the coupling of two modes, the group velocities of which are opposite in sign. The modes may, upon coupling, take either the form shown in Fig. 25(a) or that shown in Fig. 26(a). If the curves are as shown in Fig. 25(a), all real values of k correspond to real values of ω, so that there is no instability. On the other hand, some real values of ω correspond to complex values of k. The appropriate curves Γ and Λ are readily seen to be as shown in Figs. 25(b) and (c). Since Fig. 25(b) is of the type shown in Fig. 8, it is clear that the band ω_1 to ω_2 is evanescent. The loops appearing in Fig. 25(c) are reminiscent of Fig. 10, but it should be remembered that, for all real values of k, ω is real so that the question of instability does not arise.

Let us now consider the diagram of Fig. 26(a). In this case, k is real for all real values of ω, so that there will be neither amplifying nor evanescent waves. On the other hand, ω is complex if k is within the band k_1 to k_2, so that the system is unstable. The Γ, Λ diagrams are seen to be as shown in Figs. 26(b) and (c). Since there are no growing waves, Fig. 26(b) is of no interest. Figure 26(c) is of the type shown in Fig. 11, so that the instability represented by Fig. 26(a) is nonconvective.

We have seen that the types of behavior which may occur when two simple modes are coupled is restricted by the relative signs of the group velocities of the uncoupled modes. If the group velocities are such that the uncoupled carriers are propagating in the same sense, then coupling results either in two simple waves or in "convective instability," which may otherwise be interpreted as the existence of an amplifying wave. If, on the other hand, the group velocities represent propagation of the uncoupled carriers in opposite senses, then coupling results either in a band of evanescent waves, or in a band of wave numbers associated with nonconvective instability. Hence we may construct amplifiers only by coupling modes, the group velocities of which are in the same sense. On the other hand, by coupling modes, the group velocities of which are in opposite senses, we are constructing a system which may form the basis of an oscillator.

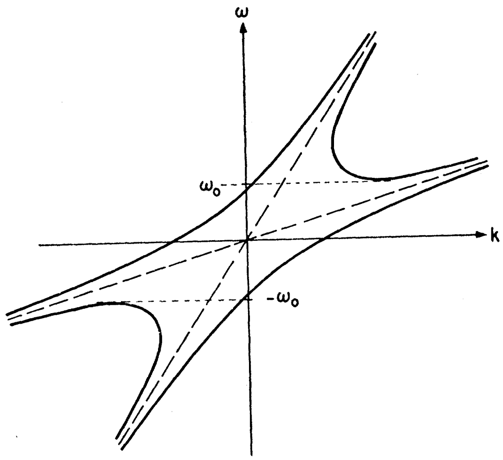

FIG. 27. Dispersion relation for two-stream amplifier: stream velocities in same direction.

In constructing a traveling-wave tube,[27] we arrange for the interaction of an electron beam and a circuit wave which propagate in the same sense, and so arrive at the situation characterized by Fig. 24(a). If we arrange for the interaction of an electron beam with a circuit wave, the group velocity of which is in the opposite direction to the velocity of the beam, the resulting system is characterized by a diagram of the type shown in Fig. 26(a): this represents the mechanism of the backward-wave oscillator.[27,33–35]

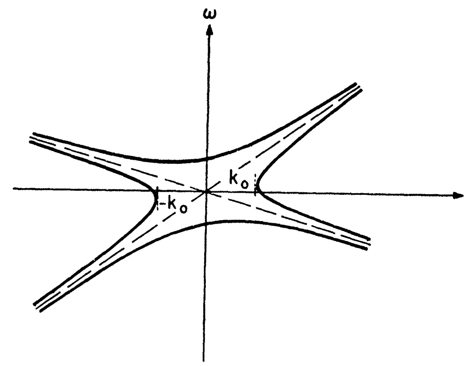

FIG. 28. Dispersion relation for two-stream "amplifier": stream velocities in opposite directions.

Let us, in conclusion, consider the ω-k diagram representative of the interaction between two superposed electron streams. The dispersion relation for this double-stream system is the particular case of (6.1):

$$\omega_1^2/(\omega-v_1k)^2+\omega_2^2/(\omega-v_2k)^2=1. \tag{6.2}$$

If v_1 and v_2 are in the same direction, the diagram is as shown in Fig. 27. We see that two of the four modes represent simple waves; the other two modes represent amplifying waves for frequencies below ω_0. If the velocities v_1 and v_2 are opposite in sense, the diagram is as shown in Fig. 28. The modes which were simple now lead to a band of evanescent waves, a point which has been noted by Gould.[20] However, a more important characteristic of this system is that the pair of modes which before represented convective instability now represent nonconvective instability for all wave numbers below k_0. This indicates that a two-stream system, in which the streams are moving in opposite directions, will disrupt or exhibit large-amplitude oscillations if the region of interaction exceeds the critical value π/k_0.

[33] R. Kompfner and N. T. Williams, Proc. Inst. Radio Engrs. **41**, 1602 (1953).
[34] H. Heffner, Proc. Inst. Radio Engrs. **42**, 930 (1954).
[35] H. R. Johnson, Proc. Inst. Radio Engrs. **43**, 684 (1955).