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Microwave Propagation in Hot Magneto-Plasmas*

J. E. DRUMMOND†
Poulter Laboratories, Stanford Research Institute, Menlo Park, California
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The indices of refraction of circularly polarized microwave signals propagating along the magnetic field direction within a high-temperature ionized gas are computed by use of a new conductivity tensor. It is found that phase changes and Landau diffusion damping resulting from electron drift become important for certain ranges of the parameters. These effects are enhanced by the resonance that occurs near the electron-cyclotron frequency. The refractive indices depend sensitively upon electron density and temperature.

1. INTRODUCTION

THE purpose of this paper is to apply a new conductivity tensor[1] to the solution of a special problem of microwave propagation in a high-temperature plasma in a magnetic field. The importance of this study lies in the rather large temperature dependence that is found. In some cases the usual results for low-temperature plasmas[2] have to be severely modified for the high-temperature cases.

The special case considered here is for propagation along the axis of a uniform, static magnetic field. The case of propagation normal to this axis has been given in I. This led to predictions of bands of abnormal transmission which depend upon electron density and temperature. This results from phase changes due to the partial spanning of a wavelength by the electron cyclotron orbits. The sensitivity of the effect to small values of electron temperature was limited by the fact that no resonance in refractive index occurs in the zero-temperature theory for this propagation direction. However, for the present case, such a resonance does occur for right-hand polarized waves near the electron-cyclotron frequency. As will be shown, this makes the effects of thermal electron drift through the very short axial wavelengths become important at just a few electron volts of mean kinetic energy.

The previous and present choices of propagation axes greatly simplify the analysis by making z, x and z, y elements of the conductivity tensor given in I equal to zero, and hence making the formalism of the usual theory[2] applicable even though the remaining tensor elements differ from the usual ones. For the special case chosen the indices of refraction are to be computed retaining only the first-order temperature effects given by the conductivity tensor in I.

2. BASIS OF THE CALCULATIONS

The basis of the calculations in the present paper is the conductivity tensor derived in I. For a static magnetic field $\mathbf{H}_0$ directed along the z axis, the conductivity tensor is

$$\boldsymbol{\sigma}=\begin{bmatrix} C_{\parallel}/\mathfrak{D} & -C_{\perp}/\mathfrak{D} & \sigma_{xz} \\ C_{\perp}/\mathfrak{D} & C_{\parallel}/\mathfrak{D} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \tag{2.1}$$

where

$$\mathfrak{D}(\omega_c)=1+\frac{16\pi^2 i\omega e^2}{mc^2\omega_c^2}\int_0^\infty e^{(i\omega-\nu)s}\bar{F}_c(s)(1-\cos\omega_c s)ds, \tag{2.2}$$

$$C_{\parallel}(\omega)=\frac{-4\pi e^2}{m}\int_0^\infty\left[\frac{\bar{F}_c(s)}{\omega_c^2}\left(\frac{\omega^2-c^2k_{\parallel}^2}{c^2}-\frac{\omega^2-c^2k_{\parallel}^2}{c^2}\cos\omega_c s-\frac{k_\perp^2}{4}+\frac{k_\perp^2}{4}\cos 2\omega_c s\right)-\bar{f}_c(s)\cos\omega_c s\right]e^{(i\omega-\nu)s}ds, \tag{2.3}$$

$$C_\perp(\omega)=\frac{-4\pi e^2}{m}\int_0^\infty\left[\frac{\bar{F}_c(s)}{\omega_c^2}\frac{k_\perp^2}{2}(\sin\omega_c s-\tfrac{1}{2}\sin 2\omega_c s)+\bar{f}_c(s)\sin\omega_c s\right]e^{(i\omega-\nu)s}ds, \tag{2.4}$$

$$\sigma_{xz}=-\sigma_{zx}=\frac{-2\pi e^2\displaystyle\int_0^\infty s(1-\cos\omega_c s)\bar{F}_c(s)e^{(i\omega-\nu)s}ds}{m\omega_c\left[1+\dfrac{8\pi^2 i\omega e^2}{m\omega_c c^2}\displaystyle\int_0^\infty s\sin\omega_c s\bar{F}_c(s)e^{(i\omega-\nu)s}ds\right]}\frac{\partial^2}{\partial y\partial z}, \tag{2.5}$$

* This work performed under the first Poulter Research Fellowship.
† Now at Boeing Scientific Research Laboratories, Office of the Director of Research, Seattle, Washington.
[1] J. E. Drummond, Phys. Rev. **110**, 293 (1958). Hereinafter referred to as I.
[2] See the review article by Lyman Mower, Sylvania Laboratory Report MPL-1, 1956 (unpublished).

$$\sigma_{yz}=-\sigma_{zy}=\frac{2\pi e^2\int_0^\infty s(1-\cos\omega_c s)\bar{F}_c(s)e^{(i\omega-\nu)s}ds}{m\omega_c\left[1+\dfrac{8\pi^2 i\omega e^2}{m\omega_c c^2}\int_0^\infty s\sin\omega_c s\bar{F}_c(s)e^{(i\omega-\nu)s}ds\right]}\frac{\partial^2}{\partial x\partial z}, \tag{2.6}$$

$$\sigma_{zz}=\frac{-2\pi e^2}{m}\int_0^\infty e^{(i\omega-\nu)s}\left[\left(\frac{\omega^2}{c^2}-k_\perp^2-k_{\shortparallel}^2\right)s^2\bar{F}_c(s)-2\bar{f}_c(s)\right]ds\bigg/\left[1+\frac{8\pi^2 i\omega e^2}{mc^2}\int_0^\infty e^{(i\omega-\nu)s}s^2\bar{F}_c(s)ds\right], \tag{2.7}$$

with $\bar{f}_c(s)$ and $\bar{F}_c(s)$ combined Fourier cosine and Hankel transforms of the symmetrical unperturbed electron distribution function $f_0(v^2)$,

$$\bar{f}_c(s)=\int_0^\infty\int_0^\infty f_0(v^2)v_\perp J_0\left(\frac{2k_\perp v_\perp}{\omega_c}\sin(\tfrac{1}{2}\omega_c s)\right)dv_\perp\cos(k_{\shortparallel}v_{\shortparallel}s)dv_{\shortparallel}, \tag{2.8}$$

$$\bar{F}_c(s)=\frac{-\omega_c}{k_\perp\sin(\frac{1}{2}\omega_c s)}\int_0^\infty\int_0^\infty f_0(v^2)v_\perp^2 J_1\left(2\frac{k_\perp v_\perp}{\omega_c}\sin(\tfrac{1}{2}\omega_c s)\right)dv_\perp\cos(k_{\shortparallel}v_{\shortparallel}s)dv_{\shortparallel}. \tag{2.9}$$

In these equations the usual symbolism is used. The symbols e and m stand, respectively, for the charge and mass of an electron, c is the speed of light in vacuum, ω_c the electron cyclotron frequency (eH_0/mc), and ν the frequency of momentum transfer collisions between an electron and neutral particles. This frequency is assumed to be very small compared to ω. The symbols $k_\perp$ and $k_{\shortparallel}$ are the wave numbers of the electric field "normal mode" parallel and perpendicular to the z axis,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathbf{E}_\omega(\mathbf{r})+k_\perp^2\mathbf{E}_\omega(\mathbf{r})=0, \tag{2.10}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{E}_\omega(\mathbf{r})+k_{\shortparallel}^2\mathbf{E}_\omega(\mathbf{r})=0. \tag{2.11}$$

Detailed assumptions and conditions underlying the conductivity tensor are given in I. In addition the following restrictions will be imposed for convenience in the present paper,

$$k_\perp^2=0, \tag{2.12}$$

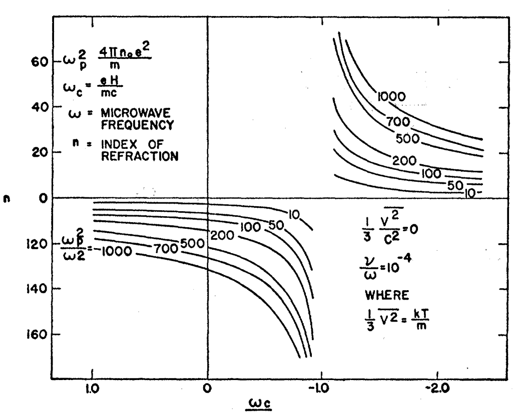

Fig. 1. Index of refraction for circularly polarized microwave signals propagating along the magnetic field in plasmas at 0°K.[3]

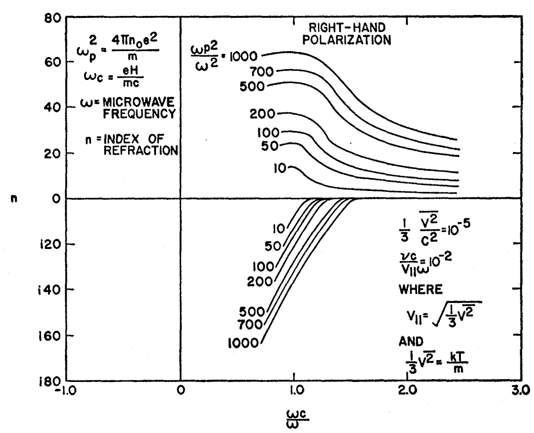

Fig. 2. Index of refraction for circularly polarized microwave signals propagating along the magnetic field in plasmas at 5.8×10^{4}°K.[3]

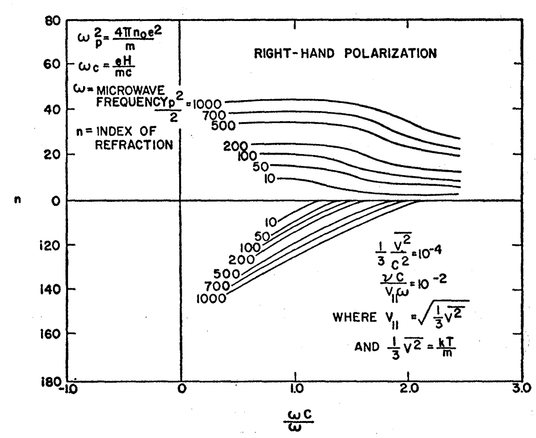

Fig. 3. Index of refraction for circularly polarized microwave signals propagating along the magnetic field in plasmas at 5.8×10^{5}°K.[3]

[3] After Drummond, "A microwave thermometer for millions of degrees," *Proceedings of the Conference on Extremely High Temperatures, March, 1958*, edited by H. Fisher and L. Mansur (by permission of John Wiley and Sons, Inc., New York).

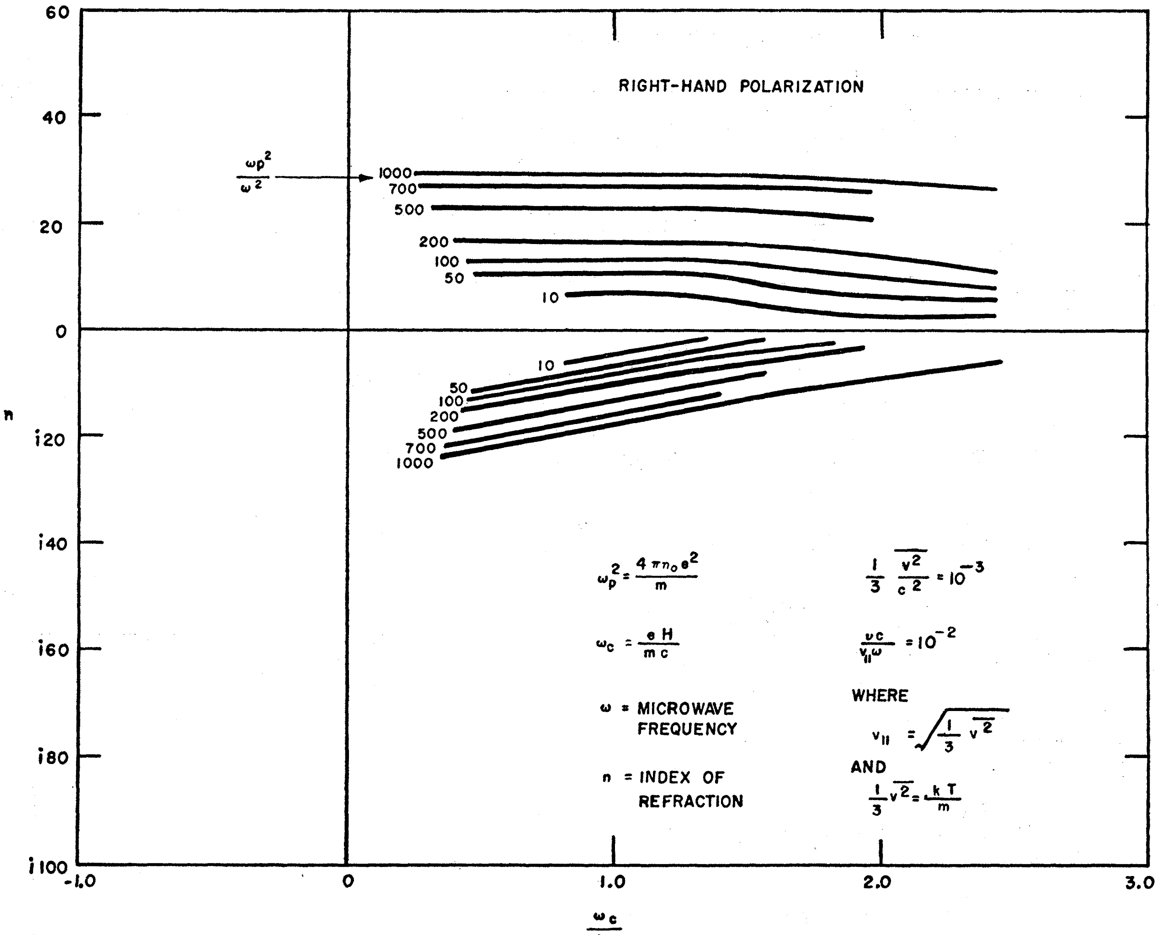

FIG. 4. Index of refraction for circularly polarized microwave signals propagating along the magnetic field in plasmas at 5.8×10^{6}°K.[3]

$$k_{\parallel}^2KT/m\omega^2 \lesssim 0.1, \tag{2.13}$$

$$0.1 > \nu/\omega > 0, \tag{2.14}$$

$$KT/mc^2 \lesssim 0.01, \tag{2.15}$$

$$f_0(v^2) = n_0(m/2\pi KT)^{\frac{3}{2}} \exp(-mv^2/2KT). \tag{2.16}$$

Equation (2.13) permits the retention of only first-order terms in $k_{\parallel}^2KT/m\omega^2$ except for $\omega \approx -\omega_c$ as will be discussed below.

After the conductivity tensor is obtained for these conditions the indices of refraction will be plotted using this result with selected small values of the collision frequency ν.

3. SPECIALIZED CONDUCTIVITY TENSOR

With the help of condition (2.16) the integrations in Eqs. (2.8)–(2.9) can be performed.[4]

$$\bar{f}_c(s) = \frac{n_0}{4\pi} \exp\left[-\frac{KT}{2m}\left(k_{\parallel}^2 s^2 + \frac{4k_{\perp}^2}{\omega_c^2}\sin^2(\omega_c s/2)\right)\right]; \tag{3.1}$$

therefore, by Eq. (2.12),

$$f_c(s) = \frac{n_0}{4\pi} \exp\left(-\frac{KT}{2m}k_{\parallel}^2 s^2\right). \tag{3.2}$$

$$\bar{F}_c(s) = \frac{-n_0}{2\pi}\frac{KT}{m} \exp\left[-\frac{KT}{2m}\left(k_{\parallel}^2 s^2 + \frac{4k_{\perp}^2}{\omega_c^2}\sin^2(\omega_c s/2)\right)\right]; \tag{3.3}$$

therefore, by Eq. (2.12),

$$= \frac{-n_0}{2\pi}\frac{KT}{m} \exp\left(-\frac{KT}{2m}k_{\parallel}^2 s^2\right). \tag{3.4}$$

Asymptotic expansions in inverse powers of $m\omega^2/k_{\parallel}^2KT$ for the integrals in Eqs. (2.2)–(2.6) may now be obtained by means of successive integration by parts.[5] This expansion has been checked by numerical integration and found valid in the low-frequency limit

[4] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), p. 394.

[5] A. Erdelyi, *Asymptotic Expansions* (Dover Publications, Inc., New York, 1956), pp. 26–29, 35.

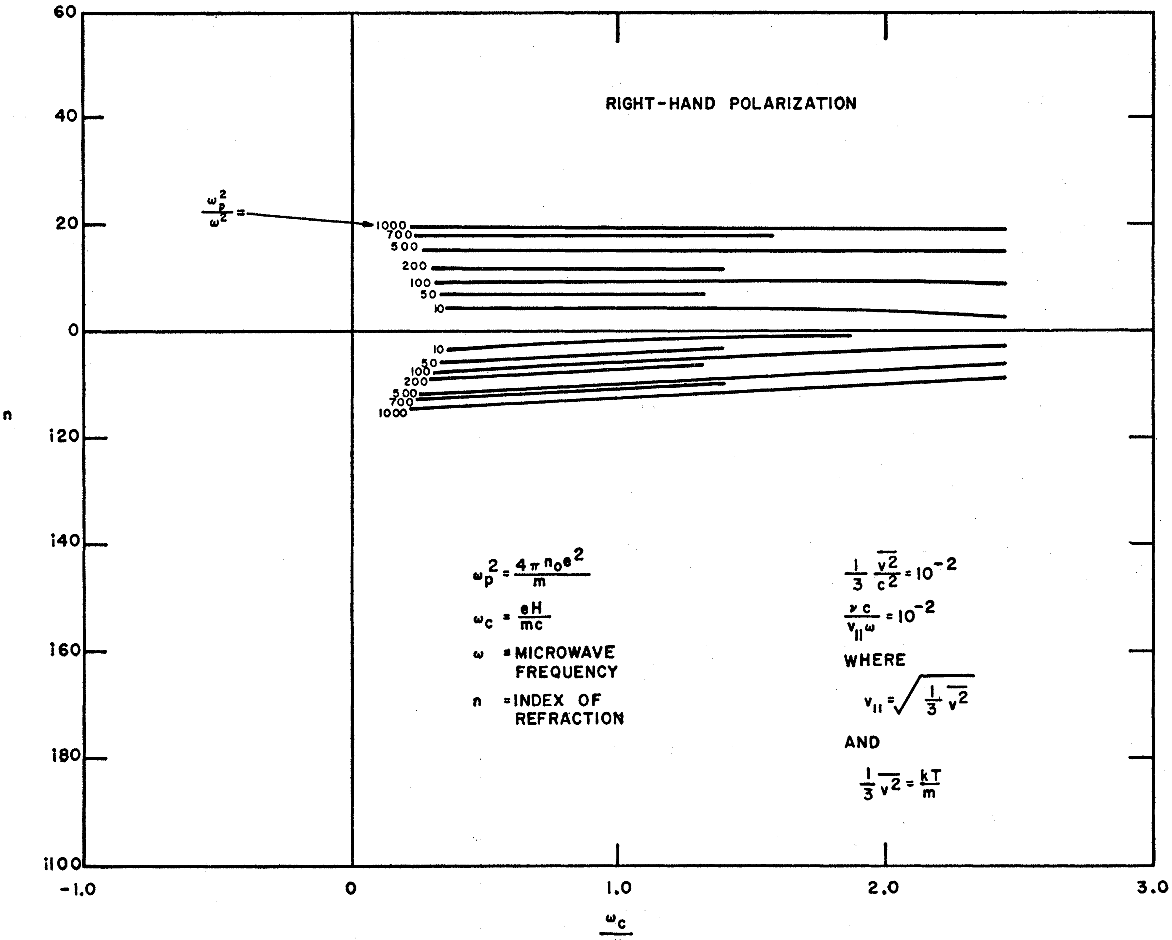

FIG. 5. Index of refraction for circularly polarized microwave signals propagating along the magnetic field in plasmas at 5.8×10^{7}°K.

and up to a little less than $\omega=|\omega_c|$ for right-hand polarization. Near this resonance, severe divergences occurred and the numerical integration alone sufficed. Beyond this point, a power series expansion was checked by the numerical integration and found valid. These two expansions and useful forms of the integrals are given by Born.[6] When the asymptotic expansion is valid, it leads to a particularly simple form of the conductivity tensor.

$$\mathfrak{D}(\omega_c)=1-\frac{2\omega\omega_p^2KT}{mc^2(\omega+i\nu)[(\omega+i\nu)^2-\omega_c^2]}, \tag{3.5}$$

$$\sigma_{yy}=\sigma_{xx}=\frac{(i\omega-\nu)\omega_p^2}{4\pi\mathfrak{D}(\omega_c)[(\omega+i\nu)^2-\omega_c^2]}\times\left\{1+\frac{k_{\parallel}^2KT}{m}\left[\frac{3\omega_c^2+(\omega+i\nu)^2}{[(\omega+i\nu)^2-\omega_c^2]^2}+\frac{2}{(\omega+i\nu)^2}\right]\right\}, \tag{3.6}$$

$$\sigma_{xy}=-\sigma_{yx}=\frac{-\omega_c\omega_p^2}{4\pi\mathfrak{D}(\omega_c)[(\omega+i\nu)^2-\omega_c^2]}\times\left\{1+\frac{k_{\parallel}^2KT[\omega_c^2+3(\omega+i\nu)^2]}{m[(\omega+i\nu)^2-\omega_c^2]^2}\right\}, \tag{3.7}$$

$$0=\sigma_{xz}=-\sigma_{zx}, \tag{3.8}$$

$$0=\sigma_{yz}=-\sigma_{zy}, \tag{3.9}$$

$$\sigma_{zz}=\frac{(i\omega-\nu)\omega_p^2}{4\pi(\omega+i\nu)^2\mathfrak{D}(0)}\left\{1+\frac{KT}{m}\left[\frac{3k_{\parallel}^2}{(\omega+i\nu)^2}\right]\right\}. \tag{3.10}$$

Using Eqs. (3.5)–(3.10) in the conductivity tensor (2.1), one notes that all the elements σ_{ij} except σ_{zz} contain a common denominator, $\mathfrak{D}(\omega_c)$. Thus, for polarizations of **E** normal to $\mathbf{H}_0$, the denominator $\mathfrak{D}(\omega_c)$ may be considered as affecting the average electron density, n_0. Likewise the correction terms in $C_{\parallel}(\omega)$ and $C_{\perp}(\omega)$ involving $k_{\perp}^2$ and $k_{\parallel}^2$ can be incorporated as changes in the effective values of ω_p^2 and ω_c. This can be done most simply for the special propagation

[6] M. Born, *Optik* (Verlag Julius Springer, Berlin, 1933), pp. 482–486.

conditions considered in the manner shown in the next section. For propagation either directly along, or directly across the magnetic field $\sigma_{xz}=\sigma_{zx}$ and $\sigma_{yz}=\sigma_{zy}$ are zero and the basic equations[7] for the propagation constants are unchanged in form.

4. INDICES OF REFRACTION

Two different indices of refraction occur corresponding to left- or right-hand circular polarization. The square of the propagation constant is given in Mower's review article.[2] In Gaussian units and for right-hand polarization it is

$$k_{\parallel}^2=\frac{\omega^2}{c^2}+\frac{4\pi i\omega}{c^2}(\sigma_{xx}+i\sigma_{xy}). \tag{4.1}$$

The result for left-hand polarization can, of course, be obtained from this by replacing ω_c by $-\omega_c$.

Substituting σ_{xx} and σ_{xy} from Eqs. (3.6)–(3.7) into Eq. (4.1) and neglecting $k_\perp$ and ν, one obtains for the index of refraction, n, the following formula:

$$n^2\equiv\left(\frac{k_{\parallel}c}{\omega}\right)^2=\frac{1+\omega_c'-\omega_p'^2-2v'^2\omega_p'^2/(1-\omega_c')}{1+\omega_c'+v'^2\omega_p'^2/(1+\omega_c')^2}, \tag{4.2}$$

where the following dimensionless variables have been used:

$$\omega_c'\equiv\frac{eH_0}{mc\omega}=-\left|\frac{eH_0}{mc\omega}\right|, \tag{4.3}$$

$$\omega_p'\equiv\frac{e}{\omega}\left(\frac{4\pi n_0}{m}\right)^{\frac{1}{2}}, \tag{4.4}$$

$$v'\equiv\frac{1}{c}\left(\frac{KT}{m}\right)^{\frac{1}{2}}. \tag{4.5}$$

Equation (4.2) is an explicit expression for the index of refraction in terms of the three independent variables, ω_c', ω_p', and v'. It is easily tabulated by a digital computer. As mentioned earlier it is valid for $\omega_c\gtrsim 1.5\omega$. This was used to obtain the right-hand parts of the graphical representations given in Figs. 1–5.[3] In the region of the resonance, the integral for the conductivity components was evaluated numerically and used in an iteration procedure which converged fairly rapidly for both the real and imaginary parts of $n=k_{\parallel}c/\omega$ in Eq. (4.1). To the left of the resonance a power series expansion was used to evaluate the integral in the iteration procedure.

It will be noticed that the onset of attenuation is considerably altered by temperature. An important thing to note is that over several orders of magnitude the ratio of the microwave to oscillation frequencies at which the attenuation begins depends almost entirely upon β_e, the ratio of electron kinetic energy density to magnetic energy density. This dependence is shown in Fig. 6 where

$$\beta_e=8\pi n_0KT/H_0^2. \tag{4.6}$$

[7] Reference 2, Eqs. (2.16) and (2.24).

5. CONCLUSIONS

Indices of refraction for circularly polarized microwaves propagating along the magnetic field in a plasma have been calculated. Because of the resonance in refractive index that occurs when the frequency of a right-hand polarized wave is approximately equal to the electron gyro-frequency, the wavelength becomes very small in this case. Thus the thermal drift of electrons along the magnetic lines of force introduces significant transit-time phase changes in the response of the medium to the wave. This alters the real part of the refractive index. The statistical spread in the thermal drift times causes mixing of the phases and hence entropy production which removes energy from the organized oscillations and heats the electrons. This power transfer is indicated by the existance of the imaginary as well as the real parts to the index of refraction in the neighborhood of $|\omega_c/\omega|\approx 1$. Of course thermodynamics indicates the limit of the inverse process for the material radiating energy into this band.

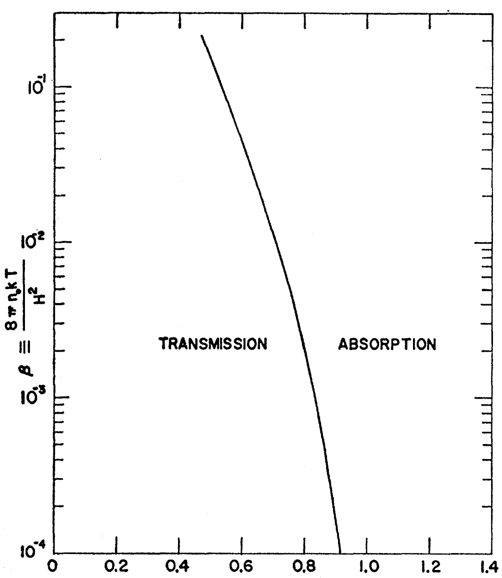

FIG. 6. Frequency at which attenuation begins for right-hand circularly polarized waves propagating along the magnetic field in a plasma.

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