Plasma Oscillations with Diffusion in Velocity Space

A. LENARD AND IRA B. BERNSTEIN
Project Matterhorn, Princeton University, Princeton, New Jersey
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A model of plasma oscillations in the presence of small-angle collisions is presented which admits of exact analytic solution. Certain features of the true collision terms are preserved. Namely, the effect of collisions is represented by a diffusion in velocity space, which makes the distribution function tend to the Maxwell distribution, and which conserves the number of particles. In the limit of infrequent collisions the results of Landau are recovered.

OF importance in the physics of fully ionized gases is the question of the effect of small-angle collisions on longitudinal plasma oscillations. A proper treatment of the problem requires solving for the electrons the linearized Boltzmann equation[1] with a Fokker-Planck collison term[2] representing electron-electron and electron-ion encounters. This has not proven mathematically feasible, and it is of interest to investigate a qualitatively similar problem which is amenable to exact solution.

Those features of the Fokker-Planck terms which one would like to preserve are the following: the property of conserving the number of electrons; the property of representing a diffusion in velocity space; the property of yielding the Maxwell distribution for the equilibrium state. An equation which achieves this[3] is

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}}-\frac{e}{m}\boldsymbol{\mathcal{E}}\cdot\frac{\partial f_0}{\partial \mathbf{v}}=\beta\frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{v}f_1+v_0^2\frac{\partial f_1}{\partial \mathbf{v}}\right], \qquad (1)$$

where $f=f_0+f_1$ is the joint distribution in position and velocity divided by the equilibrium density N, and $\partial/\partial\mathbf{v}$ means the gradient in velocity space;

$$f_0=(2\pi v_0^2)^{-3/2}\exp[-\tfrac{1}{2}(v/v_0)^2], \qquad (2)$$

e is the magnitude of the charge on the electron, m is the electron mass, $\boldsymbol{\mathcal{E}}$ the electric field, β an effective collision frequency,[4] and v_0 the root mean square speed corresponding to the equilibrium distribution f_0. The perturbed distribution function f_1 is assumed to be very much smaller in magnitude than f_0, of the same order of smallness as $\boldsymbol{\mathcal{E}}$. As usual in the theory of longitudinal plasma oscillations, Eq. (1) must be solved jointly with the appropriate Maxwell equations which, neglecting induction effects, are

$$\frac{\partial}{\partial \mathbf{r}}\times\boldsymbol{\mathcal{E}}=0, \qquad (3)$$

$$\frac{\partial}{\partial \mathbf{r}}\cdot\boldsymbol{\mathcal{E}}=-4\pi eN\int d^3v\, f_1. \qquad (4)$$

It is convenient to solve this system of equations by means of a Laplace transform with respect to time, and Fourier transforms with respect to velocity and position. That is, one defines

$$F(\mathbf{k},\boldsymbol{\sigma},s)=\int_0^\infty dt\, e^{-st}\int_{-\infty}^{\infty} d^3v\, e^{i\boldsymbol{\sigma}\cdot\mathbf{v}}\int_{-\infty}^{\infty} d^3k e^{i\mathbf{k}\cdot\mathbf{r}} f_1(\mathbf{r},\mathbf{v},t), \qquad (5)$$

$$\mathbf{E}(\mathbf{k},s)=\int_0^\infty dt\, e^{-st}\int_{-\infty}^{\infty} d^3k\, \boldsymbol{\mathcal{E}}(\mathbf{r},t). \qquad (6)$$

Equations (1), (3), and (4) then read on transformation

$$sF-G-\mathbf{k}\cdot\frac{\partial F}{\partial \boldsymbol{\sigma}}-\frac{e}{m}i\boldsymbol{\sigma}\cdot\mathbf{E}\exp(-\tfrac{1}{2}\sigma^2v_0^2) =\beta i\boldsymbol{\sigma}\cdot\left[i\frac{\partial F}{\partial \boldsymbol{\sigma}}+iv_0^2\boldsymbol{\sigma}F\right], \qquad (7)$$

$$i\mathbf{k}\times\mathbf{E}=0, \qquad (8)$$

$$i\mathbf{k}\cdot\mathbf{E}=-4\pi eNF(\mathbf{k},0,s)\equiv-4\pi eNF_0, \qquad (9)$$

where

$$G=G(\mathbf{k},\boldsymbol{\sigma})=\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}}\int d^3\sigma\, e^{i\boldsymbol{\sigma}\cdot\mathbf{v}} f_1(\mathbf{r},\mathbf{v},0). \qquad (10)$$

It follows from Eqs. (8) and (9) that

$$\mathbf{E}=-4\pi NeF_0\mathbf{k}/k^2, \qquad (11)$$

whence Eq. (7) can be written

$$(\beta\boldsymbol{\sigma}-\mathbf{k})\cdot(\partial F/\partial\boldsymbol{\sigma})+(s+\beta v_0^2\sigma^2)F =G-(\omega_p^2/k^2)\mathbf{k}\cdot\boldsymbol{\sigma}F_0\exp(-\tfrac{1}{2}v_0^2\sigma^2). \qquad (12)$$

In solving Eq. (12) it is convenient to choose units such that $v_0=1$, $\beta=1$, to define the plasma frequency $\omega_p=(4\pi Ne^2/m)^{\frac{1}{2}}$, and to set

$$F(\mathbf{k},\boldsymbol{\sigma},s)=\phi(\mathbf{k},\boldsymbol{\sigma},s)\exp(-\tfrac{1}{2}\sigma^2-\boldsymbol{\sigma}\cdot\mathbf{k}), \qquad (13)$$

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[1] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).
[2] Rosenbluth, McDonald, and Judd, Phys. Rev. **107**, 1 (1957).
[3] The greatest defect of this model of small-angle collisions is that the "diffusion coefficients" **v** and v_0^2 do not fall off with increasing velocity, as do those given by Fokker-Planck equation. This particular form for the Fokker-Planck terms is essentially the same as that used in the theory of Brownian motion [S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943), Chap. II].
[4] The order of magnitude of β can be obtained by a comparison with the true Fokker-Planck equation of reference 2. One gets approximately $\beta\cong4\pi e^4N/m^2v_0^3$.

whence Eq. (12) reads

$$(\boldsymbol{\sigma}-\mathbf{k})\cdot(\partial\phi/\partial\boldsymbol{\sigma})+(s+k^2)\phi = G\exp(\tfrac{1}{2}\sigma^2+\boldsymbol{\sigma}\cdot\mathbf{k})-(\omega_p^2/k^2)\mathbf{k}\cdot\boldsymbol{\sigma}\phi_0 e^{\boldsymbol{\sigma}\cdot\mathbf{k}}. \quad (14)$$

It simplifies matters to write ϕ as the sum of two scalars which will later admit of simple physical interpretation. To this end, choose a Cartesian coordinate system with its z axis along $\mathbf{k}$, and define $\psi(k,\sigma_z,s)$ and $\chi(\mathbf{k},\boldsymbol{\sigma},s)$ such that

$$(\sigma_z-k)(\partial\psi/\partial\sigma_z)+(s+k^2)\psi=A = G(\mathbf{k},(0,0,\sigma_z))\exp(\tfrac{1}{2}\sigma_z^2+k\sigma_z) - (\omega_p^2/k)\sigma_z e^{k\sigma_z}\psi(k,0,s), \quad (15\text{A})$$

$$(\boldsymbol{\sigma}-\mathbf{k})\cdot(\partial\chi/\partial\boldsymbol{\sigma})+(s+k^2)\chi=B = G(\mathbf{k},\boldsymbol{\sigma})\exp(\tfrac{1}{2}\sigma^2+\mathbf{k}\cdot\boldsymbol{\sigma}) - G(\mathbf{k},(0,0,\sigma_z))\exp(\tfrac{1}{2}\sigma_z^2+k\sigma_z). \quad (15\text{B})$$

Equation (15B) can be readily integrated by introducing the variable $\boldsymbol{\rho}=\boldsymbol{\sigma}-\mathbf{k}$. Observe that F is the Fourier transform of a distribution function which for large v behaves like $\exp(-\text{const } v^2)$. It must thus be an entire function of $\boldsymbol{\sigma}$, with corresponding properties in $\boldsymbol{\rho}$. Hence Eq. (15B) reads

$$\rho(\partial\chi/\partial\rho)+S\chi=B, \quad (16)$$

where, for convenience, we define $S=s+k^2$. The solution of Eq. (16) is

$$\chi=(e^{2\pi iS}-1)^{-1}\rho^{-S}\int_{c1} d\rho'\,\rho'^{(S-1)}B(\rho'), \quad (17)$$

where the contour of integration is chosen as indicated in Fig. 1.

Observe that $\chi(\mathbf{k},(0,0,\sigma_z),s)=0$, since the factor B in the integrand of Eq. (17) vanishes identically for this choice of $\boldsymbol{\sigma}$. Thus $\phi(\mathbf{k},0,s)=\psi(k,0,s)$ (essentially the perturbed electron density appropriately transformed) and the solution of Eq. (13) can be written

$$\phi=\psi+\chi. \quad (18)$$

Note that Eq. (15B) is effectively Eq. (13) with the coupling to the electric field deleted. It thus describes the relaxation due to collisions of an initially nonequilibrium distribution. Also observe that Eq. (15A) is essentially Eq. (13) with the σ_x and σ_y dependence deleted. It thus describes the behavior of a one-dimensional distribution of electrons in the presence of collisions and electrical coupling.

Observe that if the function B is entire, as is physically reasonable since one anticipates that

$$f(\mathbf{v})\cong\exp(-v^2/2),$$

then χ as given by Eq. (17) is an analytic function of s except for simple poles at the points $S=s+k^2=0,\,-1,\,-2,\,\cdots$. Those points $S=+1,\,+2,\,+3,\,\cdots$ for which the function $e^{2\pi iS}-1$ vanishes are not poles, for at these values of $S=s+k^2$ the numerator also vanishes. Thus

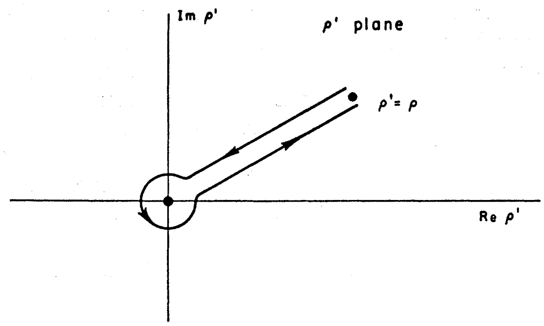

FIG. 1. Path of integration c_1 in the complex ρ' plane.

if one inverts the Laplace transforms in terms of these poles in S or s, there results for the contribution of χ to the double Fourier transform of f_1 terms of time dependence

$$\exp\{-[(k^2v_0^2/\beta)+n\beta]t\},\quad n=0,\,1,\,2,\,\cdots, \quad (19)$$

where we have restored the units. Now $1/kv_0$ is the time required for an average particle to move the distance $1/k$ characteristic of the spatial variation of the kth Fourier component, while $1/\beta$ is the time characteristic of relaxation via collisions. Thus expression (19) is in accord with the physical picture that a distribution of particle tends to become smooth via convection, owing to the proper motion of the particles, in competition with relaxation via collisions.

Return to Eq. (14) and suppress the subscript z. The solution can then be written

$$\psi=(e^{2\pi iS}-1)^{-1}(\sigma-k)^{-S}\int_{c2} d\sigma'(\sigma'-k)^{S-1}A(\sigma'), \quad (20)$$

where the contour c_2 is taken in the σ' plane in analogy with the contour c_1 in the ρ' plane of Fig. 1. Equation (20) can then be employed to solve for $\psi(k,0,S)$ which occurs as a coefficient in A. The result is

$$\psi(k,0,s)=C(S)/D(S), \quad (21)$$

where $S=s+k^2$ and

$$C(S)=(e^{2\pi iS}-1)^{-1}(-k)^{-S}\int_{c2} d\sigma'(\sigma'-k)^{S-1}G[k,(0,0,\sigma')], \quad (22)$$

$$D(S)=1+\omega_p^2(e^{2\pi iS}-1)^{-1}(-k)^{-S} \times\int_{c2} d\sigma'(\sigma'-k)^{S-1}(\sigma'/k)e^{k\sigma'}. \quad (23)$$

Observe that $\psi(k,0,s)$ has no poles in s at the points $S=s+k^2=0,\,\pm1,\,\pm2,\,\cdots$, since both C and D have simple poles simultaneously. Moreover, when Eq. (21) is substituted in Eq. (20) to solve for $\psi(k,\sigma,s)$, it is readily demonstrated that $\psi(k,\sigma,s)$ also has no poles at the points $S=0,\,\pm1,\,\pm2,\,\cdots$. Rather, $\psi(k,\sigma,s)$ is an analytic function of S everywhere except at those points where $D(S)=0$. There it has simple poles.

Let us now derive certain general properties of the roots of the secular equation (dispersion relation) $D(S)=0$. First it is possible to show that there are no roots for which $\operatorname{Re} s>0$. In order to effect this latter demonstration, observe that inversion of the Laplace transform of f_1 in terms of its poles is equivalent to an expansion of the solution of Eq. (1) in normal modes of space-time behavior $e^{+\lambda t+i\mathbf{k}\cdot\mathbf{r}}$, if we identify the eigenvalues λ with the roots s of $D=0$. Thus if one writes in units in which $v_0=1$, $\beta=1$

$$f_1(\mathbf{r},\mathbf{v},t)=\exp(\lambda t+i\mathbf{k}\cdot\mathbf{r}-\tfrac{1}{2}v^2)g(\mathbf{v}), \quad (24)$$

then it follows from Eqs. (1), (2), (3), and (4) that g satisfies

$$(\lambda+i\mathbf{k}\cdot\mathbf{v})\exp(-\tfrac{1}{2}v^2)g+(\omega_p^2/k^2)(2\pi)^{-3/2}i\mathbf{k}\cdot\mathbf{v}\times\exp(-\tfrac{1}{2}v^2)\int d^3v\exp(-\tfrac{1}{2}v^2)g=\frac{\partial}{\partial\mathbf{v}}\cdot\left[\exp(-\tfrac{1}{2}v^2)\frac{\partial g}{\partial\mathbf{v}}\right]. \quad (25)$$

Integrate Eq. (25) over all $\mathbf{v}$. There results

$$\lambda\int d^3v\exp(-\tfrac{1}{2}v^2)g=-i\mathbf{k}\cdot\int d^3v\,\mathbf{v}\exp(-\tfrac{1}{2}v^2)g. \quad (26)$$

Multiply Eq. (26) by g^* and integrate over all $\mathbf{v}$. There results, on employment of Eq. (26),

$$\lambda=-\frac{\int d^3v\exp(-\tfrac{1}{2}v^2)|\partial g/\partial\mathbf{v}|^2+i\mathbf{k}\cdot\int d^3v\,\mathbf{v}\exp(-\tfrac{1}{2}v^2)|g|^2}{\int d^3v\,|g|^2+\dfrac{\omega_p^2}{k^2}(2\pi)^{-3/2}\left|\int d^3v\exp(-\tfrac{1}{2}v^2)g\right|^2}. \quad (27)$$

Clearly $\operatorname{Re}\lambda\leqslant 0$, as was to be proved.

In order to proceed it is convenient to derive a different representation of $D(S)$ than that of Eq. (23). Namely if $\operatorname{Re} s>k^2$, Eq. (23) can be written in terms of an integral over the upper lip of the branch cut, namely

$$D(S)=1+\omega_p^2\int_0^1 dx\,x^{S-1}(1-x)\exp[k^2(1-x)], \quad (28)$$

where we have set $1-\sigma'/k=x$. But

$$\int_0^1 dx\,x^S\exp[k^2(1-x)]=-\frac{\exp[k^2(1-x)]}{k^2}x^S\Big|_0^1+\frac{S}{k^2}\int_0^1 dx\,x^{S-1}\exp[k^2(1-x)]=-\frac{1}{k^2}+\frac{S}{k^2}\int_0^1 dx\,x^{S-1}\exp[k^2(1-x)]. \quad (29)$$

Thus Eq. (28) can be written

$$D(S)=1+\frac{\omega_p^2}{k^2}-\frac{\omega_p^2}{k^2}s\int_0^1 dx\,x^{S-1}\exp[k^2(1-x)]. \quad (30)$$

If one expands $\exp(-k^2x)$ in its power series and integrates term by term, there results

$$\frac{k^2}{\omega_p^2}D(S)=1+\frac{k^2}{\omega_p^2}-s\exp(k^2)\sum_{n=0}^{\infty}\frac{(-k^2)^n}{n!}\frac{1}{s+k^2+n}. \quad (31)$$

The series above and all its derivatives converge uniformly for all values of s except about the points $s+k^2=0, -1, -2, -3, \cdots$, where the function $D(S)$ has simple poles. Thus, if one restores the dimensions, the equation $D(S)=0$ reads

$$1+\frac{k^2v_0^2}{\omega_p^2}=\frac{s}{\beta}\exp(k^2v_0^2/\beta^2)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{k^2v_0^2}{\beta^2}\right)^n\times\frac{1}{s/\beta+k^2v_0^2/\beta^2+n}=Q+1. \quad (32)$$

Observe from Eq. (32) that as $\omega_p^2\to 0$, the other parameters being kept fixed, the left-hand side approaches ∞. Thus roots can occur only in the vicinity of the poles of the left-hand side, namely when $s/\beta=-(k^2v_0^2/\beta^2)-n$; $n=0, 1, 2, \cdots$. Thus, as expected,

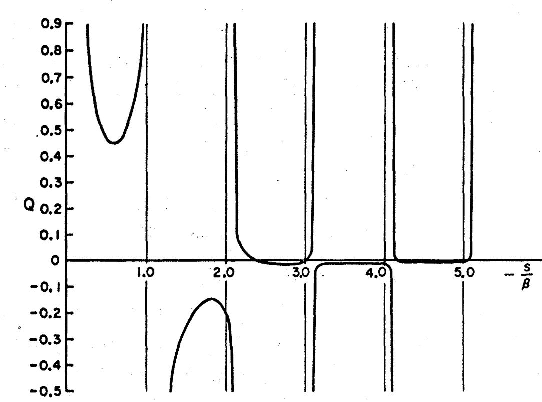

FIG. 2. Plot of Q vs $-s/\beta$ for $k^2v_0^2/\beta^2=0.1$.

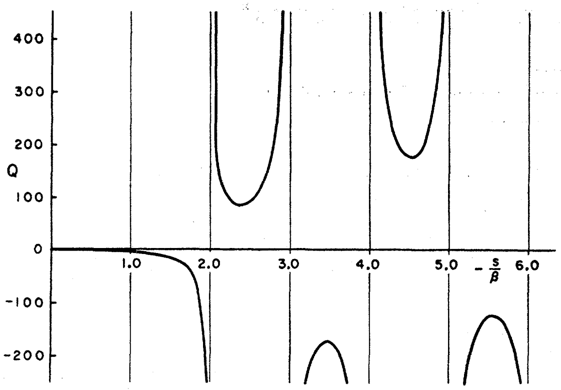

FIG. 3. Plot of Q *vs* $-s/\beta$ for $k^2v_0^2/\beta^2=2.0$.

the roots reduce to those found in the case of no coupling with the electric field.

In order to see what happens as one increases ω_p^2, consider Fig. 2 or 3 where Q is plotted *vs* $-s/\beta$ real for some representative values of $k^2v_0^2/\beta^2$. Any real roots are given by the intersection of the Q curves with the horizontal line $Q=k^2v_0^2/\omega_p^2$. For small ω_p^2, the roots are clearly given by the asymptotes $s/\beta+k^2v_0^2/\beta^2=0, -1, -2, \cdots$. As ω_p^2 increases, one acquires a pair of complex conjugate roots whenever $k^2v_0^2/\omega_p^2$ drops below a positive minimum of Q. It is physically reasonable to believe that for small $k^2v_0^2/\beta^2$ (frequent collisions) the roots found in this way are all the roots.

When $k^2v_0^2/\beta^2\approx 0$, it is possible to find readily the root with the smallest real part, for in this case Eq. (28) can be approximated by

$$0=1+\frac{\omega_p^2}{\beta^2}\int_0^1 dx(1-x)x^{s/\beta-1}$$

$$=1+\frac{\omega_p^2}{\beta^2}\left[\frac{1}{s/\beta}-\frac{1}{s/\beta+1}\right], \tag{33}$$

whose solutions are

$$\frac{s}{\beta}=-\frac{1}{2}\left\{1\pm\left(1-\frac{4\omega_p^2}{\beta^2}\right)^{\frac{1}{2}}\right\}. \tag{34}$$

Equation (34) shows clearly the coalescence of the two real roots and their bifurcation into two complex conjugate roots near $\omega_p^2=\frac{1}{4}\beta^2$.

In the limit of weak collisions, $\beta\rightarrow 0$, one expects to recover the result of Landau.[1] This is indeed so as can be seen by making in Eq. (28) the transformation $x=e^{-t\beta}$ and restoring the dimensions. There results for the dispersion relation

$$0=1+\frac{\omega_p^2}{\beta}\int_0^\infty dt\,\exp[-(s+k^2v_0^2/\beta)t + (k^2v_0^2/\beta^2)(1-e^{-\beta t})](1-e^{-\beta t})$$

$$=1+\omega_p^2\int_0^\infty dt\,\exp[-st-\tfrac{1}{2}k^2v_0^2t^2+O(\beta)][t+O(\beta)]. \tag{35}$$

When $\mathrm{Re}\,s>0$, in the limit $\beta=0$, Eq. (35) above can be written

$$0=1-\omega_p^2\frac{\partial}{\partial s}\int_0^\infty dt\,e^{-st}\int_{-\infty}^\infty dv(2\pi)^{-1/2} \times\exp[-\tfrac{1}{2}(v/v_0)^2-ikvt]$$

$$=1+\frac{\omega_p^2}{ik}\int_{-\infty}^\infty \frac{dv}{s+ikv}\frac{\partial}{\partial v}\{(2\pi)^{-1/2}\exp[-\tfrac{1}{2}(v/v_0)^2]\}. \tag{36}$$

Equation (36), apart from notation, is Landau's result.

The collisional correction to the usual long-wavelength plasma oscillation result can be readily obtained by integrating Eq. (28) by parts in such a way as to develop a series in descending powers of s. There results from the first of Eqs. (35)

$$0=1+\frac{\omega_p^2}{\beta^2}\left\{\frac{\beta^2}{s^2}-\frac{\beta^3}{s^3}+\frac{\beta^4-3k^2v_0^2\beta^2}{s^4}-\cdots\right\}, \tag{37}$$

whence, solving by successive approximation, one obtains

$$s=\pm i\omega_p-\beta\pm 3ik^2v_0^2/\omega_p. \tag{38}$$

Equation (38) is the usual result modified by a collisional damping term.

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