Below we give the variational principles analogous to (6′) and (7):

$$4\pi f(\mathbf{k},\mathbf{k}_0)=-\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}U(r)d\mathbf{r}-\frac{\int e^{-i\mathbf{k}\cdot\mathbf{r}}U(r)\phi(\mathbf{r},\mathbf{k}_0)d\mathbf{r}\int e^{i\mathbf{k}_0\cdot\mathbf{r}}U(r)\phi(\mathbf{r},-\mathbf{k})d\mathbf{r}}{\int\phi(\mathbf{r},-\mathbf{k})[\nabla^2+k^2-U(r)]\phi(\mathbf{r},\mathbf{k}_0)d\mathbf{r}},\tag{8}$$

and[2]

$$4\pi f(\mathbf{k},\mathbf{k}_0)=-\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}U(r)d\mathbf{r}+\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}_0\cdot\mathbf{r}'}d\mathbf{r}d\mathbf{r}'$$
$$+\frac{\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{-i\mathbf{k}\cdot\mathbf{r}}\phi(\mathbf{r}',\mathbf{k}_0)d\mathbf{r}d\mathbf{r}'\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{i\mathbf{k}_0\cdot\mathbf{r}}\phi(\mathbf{r}',-\mathbf{k})d\mathbf{r}d\mathbf{r}'}{\int\phi(\mathbf{r},\mathbf{k}_0)\phi(\mathbf{r},-\mathbf{k})U(r)d\mathbf{r}-\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')\phi(\mathbf{r},\mathbf{k}_0)\phi(\mathbf{r}',-\mathbf{k})d\mathbf{r}d\mathbf{r}'}.\tag{9}$$

To compare these methods we present the results for $\tan\delta_0$ ($l=0$) for $U(r)=-3e^{-r}$ and $k=1$ in Table I.

As a second example we consider the Yukawa potential

$$U(r)=-1.5e^{-r}/r \quad\text{and}\quad k=0.8.$$

With the trial function (λ=variation parameter):

$$rS_0=[1-e^{-r}+\lambda(e^{-r}-e^{-2r})]\cos kr,$$

the variation principle (6′) yields

$$\tan\delta_0=1.109086.$$

Numerical integration gives[3]

$$\tan\delta_0=1.109103.$$

[3] P.-O. Löwdin and A. Sjölander, Arkiv Fysik 3, 11, 155 (1951).

PHYSICAL REVIEW VOLUME 112, NUMBER 5 DECEMBER 1, 1958

Conductivity of Plasmas to Microwaves*

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If one assumes that the effect of an alternating electric field on an electron in a slightly ionized gas is on the average an addition, in the direction of the field, of a velocity component, the magnitude of which is dependent on the speed of the electron, then it is possible to derive very simply the expression for conductivity obtained earlier by Margenau. In addition an alternative expression is proposed which is of interest because it avoids difficulties with negative conductivities encountered by Margenau's formula.

INTRODUCTION

PREVIOUSLY[1,2] an expression was derived for the conductivity of a plasma to microwaves in which the electron distribution function is not restricted to be Maxwellian, and the collision frequency is not necessarily constant. In certain cases this expression leads to negative conductivities. If one assumes that the effect of an alternating electric field on an electron in a slightly ionized gas is, on the average, an addition, in the direction of the field, of a velocity component, the magnitude of which is dependent on the speed of the electron, then it is possible to derive the conductivity formula very simply. In addition this method suggests an alternative expression which in most cases agrees closely with the earlier result, but which avoids the difficulties of negative conductivities. The present method differs from the former in two respects. First, instead of expanding the distribution function in spherical harmonics about the axis of the electric field, we allow the field to cause a change in the velocity component along its axis. Secondly, in the absence of the orthogonality properties of the spherical harmonics, we exploit the parity of different parts of the distribution

* Supported by the Office of Naval Research.
[1] H. Margenau, Phys. Rev. 69, 508 (1946).
[2] H. Margenau, Phys. Rev. 109, 6 (1958).

function, which makes the calculation even simpler than before.

THE BOLTZMANN TRANSPORT EQUATION

Consider a plasma in an oscillating electric field of strength $E\cos\omega t$. If the x axis is taken in the direction of the field, the Boltzmann transport equation may be written

$$\gamma\cos\omega t\frac{\partial f(v_x,v_y,v_z,t)}{\partial v_x}+\frac{\partial f(v_x,v_y,v_z,t)}{\partial t}=\frac{Df(v_x,v_y,v_z,t)}{Dt},\quad(1)$$

where $f(v_x,v_y,v_z,t)$ is the electron distribution function, which we will consider to be normalized to one, Df/Dt is the change in f due to agencies other than the electric field and $\gamma=eE/m$. Let

$$\frac{Df}{Dt}=\frac{D_cf}{Dt}+\frac{D_af}{Dt},\quad(2)$$

where D_cf/Dt is the change in f due to collisions, and D_af/Dt is the change in f due to agencies other than collisions and other than the electric field. We now assume that D_af/Dt is a function of the absolute value of the velocity and not of its direction, *viz.*,

$$D_af/Dt=G(v),\quad(3)$$

where $G(v)$ is some unknown function of v.

Let $W(\mathbf{v},\mathbf{v}')d\mathbf{v}'dt$ be the probability that an electron with velocity $\mathbf{v}$ will undergo a collision in time dt resulting in a final velocity between $\mathbf{v}'$ and $\mathbf{v}'+d\mathbf{v}'$. We then obtain for the change in f due to collisions the following expression:

$$\frac{D_cf}{Dt}=\int f(\mathbf{v}',t)W(\mathbf{v}',\mathbf{v})d\mathbf{v}'-f(\mathbf{v},t)\int W(\mathbf{v},\mathbf{v}')d\mathbf{v}'.\quad(4)$$

The first term on the right side of (4) when multiplied by $nd\mathbf{v}$ represents the rate at which electrons are entering the velocity range $\mathbf{v}$ to $\mathbf{v}+d\mathbf{v}$ due to collisions. To calculate the average velocity of an electron immediately after collision we would multiply this expression by $\mathbf{v}$, integrate over all of velocity space, and divide by the average collision rate. We shall assume that this term is an even function with respect to v_x, i.e.,

$$\int f(\mathbf{v}',t)W(\mathbf{v}',\mathbf{v})d\mathbf{v}'=H(v_x,v_y,v_z,t)=H(-v_x,v_y,v_z,t).\quad(5)$$

The validity of this assumption is examined in the appendix. If (5) is true, then the average x component of the velocity of an electron immediately after collision is zero. The converse is of course not necessarily true. The integral in the second term on the right side of Eq. (4) is just the collision frequency $\nu(\mathbf{v})$. We shall assume that this is a function of v only and not of direction. We then have for this term

$$f(\mathbf{v},t)\int W(\mathbf{v},\mathbf{v}')d\mathbf{v}'=f(\mathbf{v},t)\nu(v).\quad(6)$$

Combining Eqs. (1) to (6), we obtain for the Boltzmann transport equation

$$\gamma\cos\omega t\frac{\partial f}{\partial v_x}+\frac{\partial f}{\partial t}=G+H-f\nu(v),\quad(7)$$

where $G+H$ is some unknown even function with respect to v_x.

When

$$f_+=\tfrac{1}{2}[f(v_x)+f(-v_x)],\quad(8)$$

and

$$f_-=\tfrac{1}{2}[f(v_x)-f(-v_x)]\quad(9)$$

are substituted in (7), the Boltzmann equation takes the form

$$\gamma\cos\omega t\frac{\partial}{\partial v_x}[f_++f_-]+\frac{\partial}{\partial t}[f_++f_-]=G+H-\nu(v)[f_++f_-].\quad(10)$$

On replacing v_x by $-v_x$ and subtracting the resulting equation from (10), one finds

$$\gamma\cos\omega t\frac{\partial f_+}{\partial v_x}+\frac{\partial f_-}{\partial t}=-\nu(v)f_-.\quad(11)$$

BASIC ASSUMPTIONS

Ignoring diffusion, the assumption will be made that for arbitrary a and b

$$\int_a^b f(v_x,v_y,v_z,t)dv_x=\int_{a-g(a,v_y,v_z,t)}^{b-g(b,v_y,v_z,t)}f_0(v_x,v_y,v_z,t)dv_x,\quad(12)$$

where $f_0(v_x,v_y,v_z)=f_0(v)$ is the equilibrium distribution function and $g=g(v,t)$. This is equivalent to assuming that all of the electrons which were in a given velocity range in the equilibrium distribution have been shifted by the electric field to a new range in which the x component of velocity differs from the original x component of velocity by an amount g, and the y and z components of velocity are unchanged. The physical argument for this assumption is that the effect on an electron of the field in the v_x direction and of collisions is on the average an addition of a velocity component g in the v_x direction. The magnitude of this component would be a function of the time only if there were no collisions. However collisions influence g, and since the collision frequency is a function of v only, it is plausible to assume $g=g(v,t)$.

Equation (12) may also be written

$$f(v_x,v_y,v_z,t)=f_0(v_x-g,\,v_y,\,v_z)(1-\partial g/\partial v_x).\quad(13)$$

This can be seen by replacing v_x on the right side of (12) by $v_x' - g(v_x')$. The occurrence of the factor $(1-\partial g/\partial v_x)$ in (13) arises from the fact that, although all of the electrons in a given region in the equilibrium distribution are shifted to a new region, the volumes of these two regions are in general not equal, and consequently the density in velocity space, represented by the distribution function, changes in the shift.

If $|\partial g/\partial v_x| \ll 1$, then

$$f(v_x, v_y, v_z, t) \approx f_0(v_x - g, v_y, v_z), \quad (14)$$

a result which recalls Langevin's[3] original approach to the mobility problem. In this approximation Eqs. (8) and (9) become

$$f_+ = \tfrac{1}{2}[f_0(v_x - g) + f_0(v_x + g)], \quad (15)$$

$$f_- = \tfrac{1}{2}[f_0(v_x - g) - f_0(v_x + g)]. \quad (16)$$

If f_+ and f_- are expanded in powers of g and one assumes rapid convergence of the resulting series, then

$$f_+ \approx f_0, \quad (17)$$

and

$$f_- \approx -(\partial f_0/\partial v_x)g, \quad (18)$$

Substituting (17) and (18) in (11) and solving the resulting differential equation for g, we find

$$g = \frac{\gamma\nu}{\omega^2 + \nu^2}\cos\omega t + \frac{\gamma\omega}{\omega^2 + \nu^2}\sin\omega t. \quad (19)$$

CURRENT DENSITY

To calculate the current density, we use the relation

$$J = ne\int\int\int_{-\infty}^{+\infty} v_x f(v_x, v_y, v_z, t)dv_x dv_y dv_z$$

$$= ne\int\int\int_{-\infty}^{+\infty} v_x f_- dv_x dv_y dv_z. \quad (20)$$

However there are several expressions one can use for the distribution function f. If we let

$$f = f_+ + f_- \approx f_0 - (\partial f_0/\partial v_x)g, \quad (21)$$

we obtain Margenau's formula

$$J = -\tfrac{1}{3}ne\int g\frac{\partial f_0}{\partial v}v d\mathbf{v} = -\tfrac{4}{3}\pi ne\int gv^3\frac{\partial f_0}{\partial v}dv. \quad (22)$$

The use of (14) in (20) followed by a translation in which $g(v_x)$ is approximated by $g(v_x - g)$ leads to

$$J = 4\pi ne\int [g + \tfrac{1}{3}v\partial g/\partial v]v^2 f_0 dv. \quad (23)$$

Equations (22) and (23) are identical since (23) can be obtained from (22) by an integration by parts.

However, if we use (13) instead of (14) in (20), then by a translation of axes, on assuming $g(v_x)$ approximately equal to $g(v_x - g)$,

$$J = 4\pi ne\int gv^2 f_0 dv. \quad (24)$$

In deriving this, one regards (17) and (18) as valid approximations for determining g, but not necessarily good approximations for determining $\partial g/\partial v_x$. In fact it would be surprising if good values of $\partial g/\partial v_x$ resulted from this method, since approximations (17) and (18) are obtained by neglecting terms in $\partial g/\partial v_x$. The accuracy of (24) depends on the accuracy of g only, whereas the accuracy of (23) depends on the accuracy of g and $\partial g/\partial v_x$; hence one would expect (24) to give more reliable values for the current than (23), provided of course that (13) is judged to be a physically more reasonable assumption than (14) for the form of the distribution function.

It is interesting to note that (24) could have been obtained by a simple physical argument from the Lorentz equation. The Lorentz equation yields the drift velocity $\dot{x}$ for an electron with collision frequency ν through the differential equation

$$m\ddot{x} + \nu m\dot{x} = eE\cos\omega t. \quad (25)$$

Solving for $\dot{x}$ and identifying $\dot{x}$ with g, we obtain the result (19). Multiplying by e to obtain the current due to a single electron with collision frequency ν, and then summing over all of the electrons, taking into account their velocity distribution, we obtain directly (24).

If $|\tfrac{1}{3}v\partial g/\partial v| \ll g$, expressions (23) and (24) are approximately equal. This would be the case if ν were constant.

The nature of the approximations involved in the derivation of (23) and (24) makes it difficult to appraise the validity of each on purely mathematical grounds. We have therefore compared them for special forms of f_0 and $\nu(v)$. Certainly (23) is in error when it predicts negative conductivities. This happens, for instance, if one uses the experimental values of the collision frequency for electrons in air and lets $f_0 = 4\pi v'^2\delta(v - v')$. Here Eq. (24) gives of course positive values for all v, and it differs from (23) by 50% even at velocities where both expressions are positive. To be sure, this is an extreme example which may be only of academic interest, especially since for a delta function (18) is not a good approximation to (16). On the other hand, if one uses the same collision frequencies together with a Maxwellian distribution, then (23) and (24) agree within the accuracy of the saddle point method of evaluating the integrals.[2]

APPENDIX

In this Appendix we shall examine conditions under which H in Eq. (5) may be expected to be an even

[3] P. Langevin, Ann. chim. et phys. 8, 245 (1905).

function with respect to v_x. Using (8) and (9), we write H in the form

$$H(\mathbf{v})=\int f_+(\mathbf{v}')W(\mathbf{v}',\mathbf{v})d\mathbf{v}'+\int f_-(\mathbf{v}')W(\mathbf{v}',\mathbf{v})d\mathbf{v}'. \quad (26)$$

If, as is generally true,

$$W(\mathbf{v}',\mathbf{v})=W(v',v,\mathbf{v}'\cdot\mathbf{v}), \quad (27)$$

then the first term on the right side of Eq. (26) is an even function with respect to v_x because

$$\int_{-\infty}^{\infty} f_+(v_x')W(v_x',v_x)dv_x' = \int_{\infty}^{\infty} f_+(-v_x')W(-v_x',\, v_x)dv_x' = \int_{-\infty}^{\infty} f_+(v_x')W(v_x',\, -v_x)dv_x'. \quad (28)$$

Here use has been made of (8) and (27). In similar fashion, the second term on the right side of (26) is shown to be odd with respect to v_x. Equation (26) may now be written

$$H=H_+ + H_-, \quad (29)$$

where

$$H_+=\int f_+(\mathbf{v}')W(\mathbf{v},'\mathbf{v})d\mathbf{v}', \quad (30)$$

and

$$H_-=\int f_-(\mathbf{v}')W(\mathbf{v}',\mathbf{v})d\mathbf{v}'. \quad (31)$$

H_- will vanish if

$$f_-(v_y)=f_-(-v_y), \quad (32)$$

$$f_-(v_z)=f_-(-v_z), \quad (33)$$

$$W(\mathbf{v}',\mathbf{v})=W(\mathbf{v}',\, -\mathbf{v}). \quad (34)$$

The sufficiency of these conditions is clear from the following:

$$H_-=\int\int_{-\infty}^{\infty}\int f_-(\mathbf{v}')W(\mathbf{v}',\mathbf{v})d\mathbf{v}'$$

$$=\int\int_{-\infty}^{\infty}\int f_-(-\mathbf{v}')W(-\mathbf{v}',\mathbf{v})d\mathbf{v}'$$

$$=-\int\int_{-\infty}^{\infty}\int f_-(\mathbf{v}')W(\mathbf{v}',\mathbf{v})d\mathbf{v}'=-H_-=0. \quad (35)$$

Assumptions (32) and (33) follow from (18). Assumption (34) together with (27) imply that the probability of scattering at a deflection angle θ is equivalent to the probability of scattering at an angle $\pi-\theta$. This would be true in the case of scattering from rigid elastic spheres.

If H_- does not vanish, then it must be included as an added term on the right side of Eq. (11) whence it makes a contribution to Eq. (20). For the case in which the electrons are considered to be scattered elastically from rigid scatterers, inclusion of the term H_- in Eq. (11) will result in the collision frequency in (19) being replaced by the collision frequency for momentum transfer. In general H_- may be expected to be negligible when the average x component of velocity of an electron immediately after collision is much less in absolute value than the average x component of velocity before collision.

ACKNOWLEDGMENT

Grateful acknowledgment is made of benefits derived from discussions with Professor P. Rosen.